Supplementary Material

Quasi-Static State Analysis of Differential, Difference, Integral, and Gradient Systems

Courant Lecture Notes #21, American Mathematical Society

Frank Hoppensteadt

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The supplementary material provided here expands certain calculations in the lecture notes; in particular, the first part of Section 1.5 has been rewritten. Notational errors pointed out by Professor Eric Benoît, who reviewed the notes for the Mathematical Reviews [MR (MR2724833)], and those found after publication, are also corrected in this document. Additions to this file may be submitted to frank.hoppensteadt@nyu.edu. Thanks to Dr. Benoît and others for their feedback.

page 7 line 5: Should read

... , then $y = 0$ is stable, meaning ...

line 14: Should be

$$\ldots \lim_{\omega\to\infty} \frac{1}{t}\int_0^t U^{\mathsf{T}}(\omega t')BU(\omega t')\,dt'$$

Note: The spectral decomposition of J gives

$$U(\omega t) = \exp(\omega J t) = \exp(\imath\,\omega\,t)P_1 + \exp(-\imath\,\omega\,t)P_2,$$

and since

$$U^{\mathsf{T}}(\omega t) = \exp(-\omega J t),$$

we have

$$U^{\mathsf{T}}(\omega t)BU(\omega t) = P_1BP_2 + P_2BP_1 + \exp(2\imath\omega t)P_1BP_1 + \exp(-2\imath\omega t)P_2BP_2.$$

It follows that

$$\frac{1}{t}\int_0^t U(-\omega t')BU(\omega t')\,dt'$$

$$= P_1BP_2 + P_2BP_1 + \frac{1}{t}\int_0^t (\exp(2\imath\omega t')P_1BP_1 + \exp(-2\imath\omega t')P_2BP_2)\,dt'.$$

Direct calculation shows that the last integral approaches 0 as $\omega \to \infty$ (or as $t \to \infty$), so

$$\bar{B} = P_1BP_2 + P_2BP_1.$$

page 10 line 2 and line 5: Should be

$$\frac{dr}{dt} = \frac{\varepsilon r}{\omega} f(t) \ldots$$

page 11 line 4: Replace the phrase

where the frequencies ω_n satisfy $\omega_0 = 0 < \omega_1 < \omega_2 < \ldots$.

by the phrase

for some sequence of frequencies $\{\omega_n\}$. ...

page 13 line 5: Should be

$$\ldots \int_0^t f_x U(t,s) \int_0^s g(s')\, ds'\, ds.$$

page 15 line -3: (1.15): Should be

$$\frac{d\phi}{dt} = \varepsilon f(t) \cos^2(\omega t + \phi),$$

page 16 line 10: Should be

$$\frac{dR_0}{ds} = bR_0 \lim_{T\to\infty} \frac{1}{4T} \int_0^T \left(\sin(\nu t + 2\omega t + 2\Phi_0) - \sin(\nu t - 2\omega t - 2\Phi_0)\right)\, dt$$

line 14: Should be

$$\frac{dR_0}{ds} = R_0(s) \frac{b \sin(2\Phi_0)}{4}, \qquad \frac{d\Phi_0}{ds} = \frac{b \cos(2\Phi_0)}{4},$$

lines 15 - 17: Replace the sentence

Therefore, if $b > 0$ ... bounded or not.

by

Therefore, $\Phi_0 \to \operatorname{sgn}(b)\pi/4$ as $s \to \infty$. The amplitude may grow without bound (for this linear problem), but the phase variable Φ_0 will be captured.

page 20 line 7: Should be ... (1.22), ...

page 25 line 11: Note: ‘it’ refers to equation (1.32).

page 29 line 3: Should be

$$\ldots \tau_2 \frac{dv_2}{dt} \ldots$$

line 6: Replace β by $-\beta$ in both equations in system (1.41).

line 8: Should be

$$\ldots \phi_1 - \phi_2 = -\arcsin\left(\frac{2\mu}{\beta}(\Delta^2 - 1)\Delta\right).$$

page 30 line 5: Should be

$$\frac{d\Phi}{dt} = \omega - \cos(\Phi + \omega t)\left(\frac{f}{\omega r} + ...\right).$$

line -7: Replace μ by $-\mu$.

page 36 Replace Section 1.5, pages 36, 37, 38 through line 6 on page 39 by the following paragraphs:

1.5. Pendulum Arrays in an Oscillatory Environment

Canonical models models of various neural circuits have been derived and effectively used. When focussing on frequency aspects of neural activity, it is convenient to use a model that is cast in terms of phase variables, such as the voltage controlled oscillator neuron model [**18, 20, 23**]. We consider this in the form

$$\tau \frac{d^2\theta}{dt^2} + \frac{d\theta}{dt} + A \sin 2\pi\theta = \omega, \tag{1.55}$$

which also describes motions of a mechanical pendulum [**19**]. Here τ is a time constant, A represents homeostatic mechanisms and ω represents the system's center frequency. The factor of 2π is included to interpret the angle variable θ in terms of cycles rather than radians. This facilitates the simulations since natural units there are hertz (Hz).

The output frequency of the device is $v = d\theta/dt$, and it can be improved by passing it through a filter. An equivalent first order system for this is

$$\begin{aligned} &\frac{d\theta}{dt} = v, \qquad \tau\frac{dv}{dt} = \omega - v - A(t)\sin 2\pi\theta \\ &\tau_1 \frac{du}{dt} + u = v \end{aligned} \tag{1.56}$$

The new variable u represents the output of a low–pass filter having time constant τ_1 and being driven by v. From the last equation we have

$$u(t) = \frac{1}{\tau_1}\int_0^t e^{(t'-t)/\tau_1} v(t')\,dt'.$$

On the other hand, integrating the first equation shows that the output frequency (ρ in Hz) of the circuit is proportional to the average of v since

$$\rho \equiv \lim_{t\to\infty}\frac{\theta}{t} = \lim_{t\to\infty}\frac{1}{t}\int_0^t v(t')\,dt'.$$

Both averaging and filtering remove the oscillatory parts of v, so we conclude that

$$u(t) \approx \rho$$

for sufficiently large time.

For example, consider an oscillatory signal v having a generalized Fourier series, say

$$v(t) = \sum_{n=0}^{\infty} c_n e^{2\pi \imath \omega_n t},$$

where the frequencies $0 < \omega_1 < \omega_2 < \omega_3 < \ldots$ satisfy the natural conditions that

$$\sum_{n=1}^{\infty} \left| \frac{c_n}{\omega_n} \right| < \infty \text{ and } \sum_{n=1}^{\infty} |c_n \omega_n| < \infty.$$

Then

$$u(t) = \frac{1}{\tau_1} \int_0^t e^{(t'-t)/\tau_1} v(t')\, dt' = c_0 + O\left(\left| \frac{1}{1 + \imath \omega_1 \tau_1} \right| \right) + O\left(e^{-t/\tau_1} \right)$$

On the other hand,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{n=0}^{\infty} c_n e^{2\pi \imath \omega_n t'}\, dt' = c_0 \approx u(t).$$

This heuristic calculation is apparent in the simulation in Section 1.5.1.

Background oscillations may be described by taking A to be a trigonometric polynomial

$$A(t) = a_0 + \sum_{j=1}^{N} a_j \cos 2\pi(\Omega_j t + \psi_j),$$

as in the theory of weakly connected neural networks [**23**]. The background frequencies are $\Omega_1 < \Omega_2 < \ldots < \Omega_N$, and $\psi_1, \psi_2, \ldots, \psi_N$, are the phase deviations of the various modes. Here a_0 corresponds to a direct current bias in the background. The model becomes

$$\begin{aligned} \frac{d\theta}{dt} &= v \\ \tau \frac{dv}{dt} &= \omega - v - \left(a_0 + \sum_{j=1}^{N} a_j \cos 2\pi(\Omega_j t + \psi_j) \right) \sin 2\pi\theta \\ \tau_1 \frac{du}{dt} &= v - u \end{aligned} \tag{1.57}$$

along with given initial conditions $(\theta(0), v(0), u(0))$.

We focus on one mode of oscillation, say (a_J, Ω_J) for some $J \in \overline{1, N}$, and set $\phi = \theta - \Omega_J t - \psi_J$. The system becomes

$$\frac{d\phi}{dt} = v - \Omega_J$$

$$\tau \frac{dv}{dt} = \omega - v - \frac{a_J}{2}\sin 2\pi\phi + f(t,\phi) \qquad (1.58)$$
$$\tau_1 \frac{du}{dt} = v - u$$

where $f(t,\phi) = (a_J/2)\sin 2\pi\phi - A(t)\sin 2\pi(\phi + \Omega_J t + \psi_J)$ represents the oscillatory part of $A(t)\sin 2\pi(\phi + \Omega_J t + \psi_J)$. In particular, the function f is a trigonometric polynomial in t having no constant term, and so it averages to zero over t uniformly in ϕ.

We will study an accelerated version of this system by replacing $f(t,\phi)$ by $f(t/\varepsilon,\phi)$ where ε is a small, artificial parameter that we will use for analysis. This scaling enables us to perform a rigorous perturbation analysis of the problem (1.58) in the following theorem, which motivates the computer simulations in the next section.

Theorem 1.4 *Suppose that* $2|(\omega - \Omega_J)/a_J| < 1$ *and that* $(\phi(t), v(t), u(t))$ *is a solution of (1.58). Under the conditions above, if* $u(0)$ *and* $v(0)$ *are near* Ω_J*, then*

$$u(t) \to \Omega_J + O(\varepsilon)$$

as $t \to \infty$ *for sufficiently small* ε*.*

Proof : Define $U = u - \Omega_J$ and $V = v - \Omega_J$, with the result that

$$\begin{aligned}
\frac{d\phi}{dt} &= V \\
\tau \frac{dV}{dt} &= \omega - \Omega_J - V - \frac{a_J}{2}\sin 2\pi\phi + f(t/\varepsilon,\phi) \\
\tau_1 \frac{dU}{dt} &= V - U
\end{aligned}$$

Averaging this system over the fast time t/ε gives a system for the first approximation $(\bar\phi, \bar V, \bar U)$:

$$\begin{aligned}
\frac{d\bar\phi}{dt} &= \bar V \\
\tau \frac{d\bar V}{dt} &= \omega - \Omega_J - \bar V - \frac{a_J}{2}\sin 2\pi\bar\phi \\
\tau_1 \frac{d\bar U}{dt} &= \bar V - \bar U.
\end{aligned}$$

There is a static state for this system:

$$\bar U^* = \bar V^* = 0, \quad \bar\phi^* = \frac{1}{2\pi}\arcsin\left(\frac{2(\omega - \Omega_J)}{a_J}\right).$$

The coefficient matrix of the linearization at this static state is

$$\begin{pmatrix} 0 & 1 & 0 \\ -\beta/\tau & -1/\tau & 0 \\ 0 & 1/\tau_1 & -1/\tau_1 \end{pmatrix}$$

where $\beta = a_J \pi \cos 2\pi \bar{\phi}^*$. The eigenvalues of this matrix are

$$\lambda = -\frac{1}{\tau_1}, \frac{1}{2}\left(-\frac{1}{\tau} \pm \sqrt{\frac{1}{\tau^2} - \frac{4\beta}{\tau}}\right).$$

which have negative real parts. It follows that the static state of the averaged system is exponentially asymptotically stable. This implies that the state is stable under persistent disturbances for the full perturbed problem. According to the mean-stable averaging theorem, Theorem 1.2, if $(\phi(0), v(0) - \Omega_J, u(0) - \Omega_J)$ is in the domain of attraction of this stable state, then we have that

$$|u(t) - \Omega_J - \bar{U}(t)| = O(\varepsilon)$$

for sufficiently small ε uniformly for $0 < t_0 \leq t < \infty$ for any large t_0. Moreover,

$$|u(t) - \Omega_J| \leq |u(t) - \Omega_J - \bar{U}(t)| + |\bar{U}(t)| \to O(\varepsilon)$$

as $t \to \infty$, since $\bar{U}(t) \to 0$ as $t \to \infty$. This completes the proof of the theorem.

An interesting modification of this system is made by feeding the output u back into the system. Consider system (1.58), but with u added back

$$\begin{aligned} \frac{d\phi}{dt} &= v - \Omega_J \\ \tau \frac{dv}{dt} &= \omega + u - v - \frac{a_J}{2} \sin 2\pi\phi + f(t/\varepsilon, \phi) \\ \tau_1 \frac{du}{dt} &= v - u \end{aligned} \tag{1.58'}$$

where the data are as before. Averaging over the fast time t/ε gives

$$\begin{aligned} \frac{d\bar{\phi}}{dt} &= \bar{v} - \Omega_J \\ \tau \frac{d\bar{v}}{dt} &= \omega + \bar{u} - \bar{v} - \frac{a_J}{2} \sin 2\pi\bar{\phi} \\ \tau_1 \frac{d\bar{u}}{dt} &= \bar{v} - \bar{u} \end{aligned} \tag{1.57}$$

This system has a static state

$$\bar{v}^* = \Omega_J, \bar{u}^* = \Omega_J, \bar{\phi}^* = \frac{1}{2\pi} \arcsin\left(\frac{2\omega}{a_J}\right)$$

provided the (new) locking condition

$$\left|\frac{2\omega}{a_J}\right| < 1$$

is satisfied. A result for this problem is

Theorem 1.4′ *Suppose that* $2|\omega/a_J| < 1$ *and that* $(\phi(t), v(t), u(t))$ *is a solution of (1.58'). Under the conditions above, if* $u(0)$ *and* $v(0)$ *are near* Ω_J*, then*

$$u(t) \to \Omega_J + O(\varepsilon)$$

as $t \to \infty$ *for sufficiently small* ε*.*

The proof of this result is essentially the same as for Theorem 1.4, and it is not presented here. The difference between these results is that in Theorem 1.4, the eventual output of the system is determined by the value of the center frequency ω being near one of the driving frequencies Ω_J, while in Theorem 1.4', the eventual output is determined solely by the location of $(v(0), u(0))$ provided ω is sufficiently small. The former case requires that the system parameter ω be changed to switch the output u from one frequency to another. In the latter case all of the Ω_j's appear as static states, each having a domain of attraction, and these domains partition (not exhaustively) the vu-space. An initial state $u(0)$ will evolve into one of the driving frequencies, but if the state u is re-initialized, another output frequency will result. These ideas are developed further and applied to memory models in neuroscience in [**21**].

1.5.1 Example:. ....

page 39 line 21: Note:

ε here refers to Theorem 1.4.

page 43 line 4: Replace

$$\frac{r_1}{r_2} \text{ by } \frac{r_2}{r_1}$$

line 6: Replace

$$\frac{r_2}{r_1} \text{ by } \frac{r_1}{r_2}$$

line -6: Should be

$$\ldots \frac{dr}{dt} = rf(r\cos\theta \ldots$$

page 44 line -4: Should be

$$\ldots \exp(-At)B\exp(At)\,y.$$

page 46 line -4: Should be

$$\ldots \omega + \varepsilon \sum_{n=1}^{N} \ldots$$

page 51 line -5: Add

Let $U(s)$ for $0 \le s \le \varepsilon\mathbf{n}$ be determined ...

(Implicit in this is the assumption that U exists over this interval.)

page 55 line -6: Note the initial condition for U_0 is $U_0(0) = I$. The line should read

$$\mathbf{p}_n = \mathbf{p}_0 U_0(\varepsilon n) + O(\varepsilon) \ldots$$

page 55 line -4: Delete the last two sentences

Thus, $B = D + \tilde{D}$ ... elements.

page 62 line 5: Should be

$$\ldots \omega nh + \bar{U}_0(nh\varepsilon) + \ldots$$

page 70 line -12: Should be

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{2}\frac{\partial}{\partial x}\left(g^2(x)p\right) - f(x)p\right)$$

page 71 line 2: Should be

$$\ldots = t \int_Y h(y)\rho(dy)$$

line -10: Should be

$$\ldots \int_0^\infty [P(t, y, B) - \rho(B)]\, dt$$

page 73 line 16: Note: Here B is any measurable subset of Y.

page 75 line -6: Replace

$b^2/2$

by

a variance b of your choice.

page 76 line 17: Replace t/ε by T/ε.

page 77 lines 1, 10: Replace t/ε by T/ε (two instances).

page 82 line -14: Add

... solutions of the following equation ...

page 93 line 17: Add

... around a minimum $y = y^*$.

page 96 Figure 4.2, Add to caption:

The horizontal axis is [-1.5,1.5]. The vertical axis is [-2.0,1.0].

page 97 line 2: Add

... the positive real root ...

page 100 line -8: Write

$$\ldots = \frac{\partial W}{\partial t} + \ldots$$

page 102 line -12: Add

... is not stable.

page 107 line -3: Should be

$$G = -\frac{dy_n}{dt}\varepsilon^{n+1} \ldots$$

page 111 line -11: Add

$$= K(0)f(t) - K\left(\frac{t}{\varepsilon}\right) f(0) - \int_0^t K \ldots$$

Note that for each $t > 0$, $K(t/\varepsilon) \to 0$ and the integral is $O(\varepsilon)$ as $\varepsilon \to 0$.

page 112 line 13: Note the Jacobian condition $\det(\Phi_x(t,x)) \neq 0$ is not needed. Hypothesis H.8 should read:

Hypothesis H.8. Suppose that there is a function $y = \Phi(t,x)$ such that $g(t,x,\Phi(t,x),0) = 0$ for $t_0 \leq t \leq T$ and x in B_R. Moreover, the function $\Phi \in C^2(I \times B_R)$.

page 113 line -13: Should be

$$\varepsilon\frac{dW}{dt} = \nabla_y W \cdot g + \varepsilon\frac{\partial W}{\partial t} + \varepsilon\nabla_x W \cdot f,$$

page 114 line -2: Note: 'invariant' means here that solutions lying within the layer defined by such a level set for V remain in the layer for all future time. In terms of Wazewski's method [**12**], all points on such a level set are strict ingress points.

page 115 line 7: Should be

$$\ldots = \left(\frac{x_0}{\sqrt{1 + 2\,t\,x_0^2}}, 0\right) + o(1)$$

page 120 line 19: Write

... it is derived from a simple binomial ...

page 127 lines -15, -14: The formula should read

$$y(t,\varepsilon) = a\exp(-\alpha t) + \frac{A}{\varepsilon}\int_0^t \exp\left(-\frac{t-t'}{\varepsilon}\right)\sin\omega t'\,dt'$$
$$+B\int_0^t \exp(-2(t-t'))\sin\omega t'\,dt'$$

page 136 Figure 6.3 Caption: Add

... to scale. Here $\rho_1 = R/\nu$ and $\rho_2 = R\mu$.

page 138 line -11: within system (6.9): Should be

$$\dots (V_{\text{iii}} - V_{\text{off}})/(R\mu) + I_{\text{off}}. \dots$$

page 147 line 15: Should be

$$\frac{dR}{d\theta} = \frac{\varepsilon}{\omega} \dots$$

page 149 line 15: Add

If $\omega(u, v) < 0$, then all eigenvalues of $B - T$ have negative real parts. If $\omega(u, v) > 0$, then $B - T$ ...

page 150 line 11: Add

... solved: For a constant K that depends on u_0, v_0 and ξ, the solution is

page 150 line -5, ff: Note: Here u' denotes the derivative of u with respect to its independent variable, say t. So, $u' = du/dt$ and $u'' = d^2u/dt^2$.

page 152 Figure 6.5 Caption: Add

... rapidly. The horizontal axis represents $x(t)$ modulo 2π and the vertical axis represents $y(t) + \pi$ modulo 2π.

Figure 6.6 Caption: Add

... QSM. The horizontal axis represents time t and the vertical axes represent $\cos x(t)$ (Top) and $\cos y(t)$ (Bottom).

page 155 line 3: Should be

$$\dots q = \int_{-\infty}^{t} I(t')\, dt', \dots$$

page 156 line -1: Should be

$$\dots = C \frac{dV}{dt} \dots$$

page 157 line 12: Should be

$$\tau \frac{d^2\theta}{dt^2} + \dots$$