MAPPING DEGREE THEORY

Graduate Studies in Mathematics 108

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TYPOS & COMMENTS

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1. Ch. II §4, II.3.4, proof of the Sard-Brown Theorem for smooth mappings (c), p. 65, l. -4 *should read:*

for any $x \in C_i \cap K$ and $y \in K$.

2. Ch. II §4, II.4.1, Step II of proof, p. 69, l. 14-15 *should read:*

because $M \setminus V_k \subset M \setminus V_{k+1}$. On the other hand, for every $x \in M$, say $x \in V_k$, we have

3. Ch. II §4, II.4.3, proof (c), p. 73, l. 14,15 *should read:*

$$E \supset \{(x, u) \in \nu M : \|x - z\| < \tfrac{1}{2}\varepsilon,\ \|u\| < \tfrac{1}{6}\varepsilon\},$$

and the latter set is an open neighborhood of $(z, 0)$.

4. Ch. II §5, II.5.1, p. 75, l. -7 *should read:*

the inverse image of every point of N is compact.

5. Ch. II §5, II.5.3, proof, p. 78, l. -2,-1 *should read:*

$$\begin{aligned}\|f(x) - H_t(x)\| &= \big\|f(x) - \big((1-t)f(x) + tg(x)\big)\big\| \\ &= t\|f(x) - g(x)\| < \varepsilon(x) = \operatorname{dist}(f(x), \mathbb{R}^q \setminus U),\end{aligned}$$

6. Ch. II §6, II.6.5, proof, p. 83, l. 5 *should read:*

$$G(\theta(2t-1), F_1(x)) \text{ for } t \geq \tfrac{1}{2}.$$

(instead of: $F_1(G(\theta(2t-1), x))$.)

7. Ch. II §6, II.6.7(1), p. 83, l. -16 *should read:*

... every diffeomorphism that preserves orientation is increasing,

8. Ch. II §6, II.6.7(1), p. 83, l. -14 *should read:*

... they must be numbered in exactly the same order.

9. Ch. III §3, Exercises and problems, Number 5, p. 113, l. 13. *should read:*

mappings $f : \mathbb{S}^m \to \mathbb{S}^m$ and $g : \mathbb{S}^n \to \mathbb{S}^n$, their

10. Ch. III §6, III.6.2, proof, p. 125, l. -3. *should read:*

Then we have $f_p^{-1}(u) = M \cap L$, and

(no need of: , if p is far enough from our compact hypersurface M,.)

11. Ch. III §7, III.7.3, statement, p. 134, l. -3, and Index, p.242, second column, l-9, *should read:*

Perron-Frobenius Theorem

12. Ch. IV, summary, p. 137, l. 13, and Index, p.239, first column, l. -19, *should read:*

Borsuk-Hirsch Theorem

13. Ch. IV §5, title and intro, p. 160, l. -8 to l. -2. To mention properly the authors of each result, it *should read:*

5. The Borsuk-Hirsch Theorem

Our primary goal here is to obtain the important theorem in the title, and afterwards deduce several purely topological results, including the Invariance of Domain Theorem.

Recall that a subset E of a Euclidean space is called *symmetricSymmetric set* if $x \mapsto -x$ induces a homeomorphism on E. Then a mapping f defined on E is *even* if $f(x) = f(-x)$ and *odd* if $f(x) = -f(-x)$. The Borsuk-Hirsch Theorem concerns the degree of these mappings. Borsuk showed in 1933 that odd mappings have odd degree and G. Hirsch showed in 1946 that even mappings have even degree. The results have very many equivalent versions, often called Borsuk-Ulam Theorems, because Borsuk mentioned Ulam as a source of inspiration.

14. Ch. IV §5, IV.5.1. (2), p. 161, l. 4-5, *should read:*

Let $D \subset \mathbb{R}^n$ be a symmetric bounded open set, and let $X = \overline{D} \setminus D$. We suppose that $0 \notin \overline{D}$. Now

15. Ch. IV §5, IV.5.1. (1), proof, p. 162, l. 8-9, *should read:*

$$\|\bar{f}(x)\| \geq \|h''(x)\| - \|h'''(x)\| \geq \tfrac{1}{2}\varepsilon - \|h'''(x)\| > \tfrac{1}{2}\varepsilon - \tfrac{1}{2}\varepsilon = 0.$$

16. Ch. IV, §5, p. 163, l. 1, *should read:*

Borsuk-Hirsch

17. Ch. IV, §5, IV.5.2, p. 163, l. 3, *should read:*

Theorem 5.2. *Let $D \subset \mathbb{R}^n$ be*

18. Ch. IV, §5, IV.5.2 (1), p. 163, l. 5, *should read:*

(1) (Borsuk) *Assume*

19. Ch. IV, §5, IV.5.2 (2), p. 163, l. 8, *should read:*

(2) (Hirsch) *Assume*

20. Ch. IV §5, p. 166, l. -1, *should read:*

The Borsuk-Hirsch statements have the following rephrasings:

21. Ch. IV, §5, IV.5.3, p. 166, l. 1, *should read:*

Theorem 5.3. *Let* $D \subset \mathbb{R}^n$ *be*

22. Ch. IV, §5, IV.5.3, statement, p. 166, l. 3-8, *should read:*

(1) (Borsuk) *If the winding number* $w(f,0)$ *is even, then there is some* $x \in X$ *such that*

$$\frac{f(x)}{\|f(x)\|} = \frac{f(-x)}{\|f(-x)\|}.$$

(2) (Hirsch) *If the winding number* $w(f,0)$ *is odd, then there is some* $x \in X$ *such that*

$$\frac{f(x)}{\|f(x)\|} = \frac{-f(-x)}{\|f(-x)\|}.$$

23. Ch. IV, §5, IV.5.3, proof, p. 166, l. 9-13, *should read:*

Proof. (1) Assume by way of contradiction that

$$\frac{f(x)}{\|f(x)\|} \neq \frac{-f(-x)}{\|f(-x)\|}$$

for all $x \in X$. Then by IV.5.2(1), p. 163, the winding number $w(f,0)$ must be odd, against the assumption.

Part (2) is proven similarly from IV.5.2(2), p. 163. □

24. Ch. IV §5, p. 166, l. 14-17, *should include:*

Two nice facts that now follow immediately are: (1) an even continuous mapping $f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ collapses some pair of antipodal points ($f(-x) = f(x)$ for some $x \in \mathbb{S}^{n-1}$), and (2) an odd continuous mapping $f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ maps some antipodal pair to another ($f(-x) = -f(x)$ for some $x \in \mathbb{S}^{n-1}$).

25.

Ch. IV §5, IV.5.5, proof, p. 167, l. -1, *should read:*

so that $a_\varepsilon = (0, \ldots, 0, \varepsilon)$ is close to 0, hence in the same connected component of $\mathbb{R}^n \setminus \tilde{g}(X)$. Then,

26. Ch. IV §5, IV.5.8 (2), p. 169, l. 11, *should read:*

$f : S \to T$ be a homeomorphim.

27. Ch. IV §5, Exercises and problems, Number 1, p. 170, l. -15,-14. The problem is correct, but the functions f_i in (1) are rather bounded. Thus to help the reader *it should read:*

... and odd continuous function $f_i : \mathbb{S}^m \to [-2, 2]$ ...

28. Ch. V §2, IV.2.3(2), p. 193, l. -1, *should read:*

which is $\equiv -a$ off a neighborhood of $f^{-1}(a)$.

(no need of a compact neighborhood.)

29. Ch. V §4, V.4.1, proof, p. 201, l. 9-10, *should read:*

p.100), which implies that g reverses orientation when restricted to $f^{-1}(a)$ and to $f^{-1}(b)$. Consequently $\deg(g \times g) = +1$, and we conclude $\deg(g)H(f) = H(f \circ g)$.

The point is that g *does the same on both inverse images,* be it preserving or reversing orientations. This is a general fact, because any diffeomorphism h of $\mathbb{S}^{2m-1}$ does on all inverses images what it does on $\mathbb{S}^{2m-1}$, as the following commutative diagram explains.

$$\begin{array}{ccc} \zeta_{\mathbb{S}^{2m-1},x} & \xrightarrow{\equiv} & \zeta_{f^{-1}(a),x} \oplus \zeta_{\mathbb{S}^m,a} \\ {\scriptstyle d_x h}\downarrow & & \downarrow{\scriptstyle d_x h| \,\oplus\, \mathrm{Id}} \\ \zeta_{\mathbb{S}^{2m-1},h(x)} & \xrightarrow{\equiv} & \zeta_{h(f^{-1}(a)),h(x)} \oplus \zeta_{\mathbb{S}^m,a} \end{array}$$