Lie Superalgebras and Enveloping Algebras.

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The most significant corrections are in red. Several minor errors or omissions have now been detected in the statement and proof of Lemma 9.4.3. For this reason we give the entire corrected Lemma in blue at the end.

page 35 Lines 5-6, Replace

"The automorphism of $\mathfrak{o}(2m)$ induces a map on the Cartan subalgebra fixing ..." by

"The outer automorphism of $\mathfrak{o}(2m)$ induces a map on the dual of the Cartan subalgebra fixing ..."

page 48 In Lemma 3.4.10 replace "Suppose that $X = A, B, C$ or D, and $\sigma \in \text{Shff}_C(I_0, I_1)$." by "Suppose that $X = A, B, C$ or D and $\sigma \in \text{Shff}(I_0, I_1)$."

page 48 Lemma 3.4.10 should appear immediately before Theorem 3.4.8.

page 67 Replace the first part of the hint for Exercise 3.7.21, which refers to the proof of Proposition 3.6.11 (a), with the following:

Suppose that the Borel subalgebras $\mathfrak{b}_I$ and $\mathfrak{b}_J$ are adjacent, that is $\mathfrak{b}_I \cap \mathfrak{b}_J$ has codimension one in both $\mathfrak{b}_I$ and $\mathfrak{b}_J$. Then $\dim \mathfrak{b}_I = \dim \mathfrak{b}_J$ and it follows that $|I| = |J|$. Assume that $d(\mathfrak{b}_I) = d(\mathfrak{b}_J) + 1$, and let $\sigma = \sigma_I$. Recall (3.6.7) and use the one-line notation for σ preceding (3.6.8). Then if $1 \le i, j \le m$ and $\alpha = \epsilon_{\sigma(i)} - \epsilon_{\sigma(-j)}$ is an odd root we have $\mathfrak{g}^\alpha \subseteq \mathfrak{b}_I$ if and only if $\mathfrak{g}^\alpha \subseteq \mathfrak{b}_J$. It follows that there is an odd root $\alpha = \epsilon_{\sigma(k)} - \epsilon_{\sigma(\ell)}$ with $1 \le k < \ell \le m$ such that $\mathfrak{g}^\alpha \subseteq \mathfrak{b}_I$ and $\mathfrak{g}^{-\alpha} \subseteq \mathfrak{b}_J$. Since α is a simple root of $\mathfrak{b}_I$ we must have $k \in I, \ell = k+1 \notin I$, and it follows that $k+1 \in J$ and $I \setminus \{k\} = J \setminus \{k+1\}$.

Proposition 3.6.11 (a) is correct as stated. The erratum in the hint for the Exercise was pointed out by Shushma Rani.

page 101 Replace Equation (5.2.2) with

$$\mathfrak{n}^\pm = \bigoplus_{\substack{\alpha \in Q^+ \\ \alpha \neq 0}} \mathfrak{g}^{\pm\alpha}.$$

page 191 Line -5 Replace $F^\lambda = F_{\widetilde{M}(\lambda)}$ by $F^\lambda(\ ,\ ) = (\ ,\ )_{\widetilde{M}(\lambda)}$.

page 213 Theorem 9.2.10 is proved in arxiv 1311.0570, in the case that Π contains no non-isotropic odd root. Whether the statement about the degree of Šapovalov elements is valid in Theorem 9.2.10 if Π contains a non-isotropic odd root is open. However in this case a different bound is given in the same paper.

page 215 Line 4, replace "Thus by the results quoted above" with "Thus by Theorem 9.3.1."

page 218 Line 5, replace $\widetilde{M}(\lambda)^{-m\gamma}$ with $\widetilde{M}(\lambda)^{\lambda-m\gamma}$.

page 220 In Lemma 9.4.3, replace (a) with (a1), and add the hypothesis

(a2) if α is odd non-isotropic, then $q = 2$ and p is odd.

page 220 In the third sentence of the proof of Lemma 9.4.2, replace
Suppose that $e_\alpha \in \mathfrak{g}^\alpha$ and $e_{-\alpha} \in \mathfrak{g}^{-\alpha}$ are chosen such that $[e_\alpha, e_{-\alpha}] = h_\alpha$.

with the text

Set $t_\alpha = 2h_\alpha/(\alpha,\alpha)$. Suppose that $e_\alpha \in \mathfrak{g}^\alpha$ and $e_{-\alpha} \in \mathfrak{g}^{-\alpha}$ are chosen such that $[e_\alpha, e_{-\alpha}] = t_\alpha$. By construction the elements $f = e_{-\alpha}$, $h = t_\alpha$ and $e = e_\alpha$ satisfy the defining relations for $\mathfrak{sl}(2)$ as in Equation (A.4.2). Then $v_\lambda = e^p_{-\alpha} v_\mu$ generates a submodule of $\widetilde{M}(\mu)$ which may be identified with $\widetilde{M}(\lambda)$. (By Theorem 9.3.2 $\widetilde{M}(\lambda)$ is uniquely embedded in $\widetilde{M}(\mu)$).

page 220 Three lines after (9.4.3) replace $h_\alpha v = (p + mq)v$ by $t_\alpha v = (p + mq - 1)v$. Also in the last displayed equation in the proof h_α should be replaced by t_α.

page 237 Replace line 1 with "(c) Show that the supertrace form on $\mathfrak{h}$ satisfies"

page 238 Line 5, replace $v_{\widetilde{\lambda}}$ with v_λ (twice).

page 248 Line -11, replace

$$\mathcal{H}^0_{m,n} = \mathcal{H}(m,n)\backslash\mathcal{H}(m-1,n-1).$$

by

$$\mathcal{H}^0_{m,n} = \mathcal{H}(m,n)\backslash(\mathcal{H}(m,n-1) \cup \mathcal{H}(m-1,n)).$$

page 305 Line 5, replace "after Proposition 8.1.6" by "in Proposition 8.1.6".

page 345 Line 8, replace "where" by "were".

page 355 2 lines before 16.1.1, delete "edskip".

page 378 In the statement of Theorem 16.6.7, replace the Equation

$$\theta_{p+2,0}(d_2(u)) = -c \cup \theta_{p,0}(u)$$

by

$$\theta_{p+2,0}(d_2(u)) = -c \cup \theta_{p,1}(u).$$

This erratum was pointed out by Samir Kumar Hazra.

page 410 line 11, replace "Lemma 8.3.1 (c)" by "Lemma 8.3.2 (a)".

page 456 In Exercise A.4.3 (a) replace "$ef^\ell - f^\ell e = e^{\ell-1}(h - \ell + 1)$" by

"$ef^\ell - f^\ell e = f^{\ell-1}(h - \ell + 1)$."

page 486 In the Index, the Kac module is defined in section 8.2, not 8.1.

Lemma 9.4.3. *Suppose that $\alpha \in \Pi_0$, and set*

$$\mu = s_\alpha \cdot \lambda, \quad \gamma' = s_\alpha \gamma, \quad \kappa = m\gamma, \quad \kappa' = m\gamma'.$$

Assume that

(a) $p = (\mu + \rho, \alpha^\vee) \in \mathbb{N}\backslash\{0\}$ *and* $q = (\gamma, \alpha^\vee) \in \mathbb{N}.\backslash\{0\}$.

(b) *If α is odd non-isotropic, then $q = 2$ and p is odd.*

(c) $\theta' \in U(\mathfrak{n}^-)^{-\kappa'}$ *is such that* $v = \theta' v_\mu \in \widetilde{M}(\mu)$ *is a highest weight vector.*

Then there is a unique $\theta \in U(\mathfrak{n}^-)^{-\kappa}$ such that

$$e_{-\alpha}^{p+mq}\theta' = \theta e_{-\alpha}^p. \tag{0.1}$$

Proof. We assume that α is even. It is easy to modify the proof to deal with the case where α is a non-isotropic odd root. Set $t_\alpha = 2h_\alpha/(\alpha, \alpha)$. Suppose that $e_\alpha \in \mathfrak{g}^\alpha$ and $e_{-\alpha} \in \mathfrak{g}^{-\alpha}$ are chosen such that $[e_\alpha, e_{-\alpha}] = t_\alpha$. By construction the elements $f = e_{-\alpha}$, $h = t_\alpha$ and $e = e_\alpha$ satisfy the defining relations for $\mathfrak{sl}(2)$ as in Equation (A.4.2). Then $v_\lambda = e_{-\alpha}^p v_\mu$ generates a submodule of $\widetilde{M}(\mu)$ which may be identified with $\widetilde{M}(\lambda)$. (By Theorem 9.3.2 $\widetilde{M}(\lambda)$ is uniquely embedded in $\widetilde{M}(\mu)$). We claim that

$$e_{-\alpha}^{p+mq} v \in \widetilde{M}(\lambda). \tag{0.2}$$

By Lemma 9.4.2 there is a positive integer ℓ such that $e_{-\alpha}^\ell \theta' \in U(\mathfrak{n}^-)e_{-\alpha}^p$, and hence $e_{-\alpha}^\ell v \in \widetilde{M}(\lambda)$. We may assume that $\ell \geq (p+mq)$. Note that $v = \theta' v_\mu$ has weight $\mu - \kappa'$ and $(\mu - \kappa', \alpha^\vee) = p + mq - 1$, so $t_\alpha v = (p + mq - 1)v$. If $\ell > (p + mq)$, then by Exercise A.4.3, $\widetilde{M}(\lambda)$ contains

$$\begin{aligned} e_\alpha e_{-\alpha}^\ell v &= [e_\alpha, e_{-\alpha}^\ell]v = \ell e_{-\alpha}^{\ell-1}(t_\alpha - \ell + 1)v \\ &= \ell(p + mq - \ell)e_{-\alpha}^{\ell-1} v. \end{aligned}$$

This gives (0.2). Now since $\widetilde{M}(\lambda) = U(\mathfrak{n}^-)e_{-\alpha}^p v_\lambda$ is a free $U(\mathfrak{n}^-)-$module, there exists an element $\theta \in U(\mathfrak{n}^-)^{-\gamma}$ such that (0.1) holds. Since $e_{-\alpha}$ is not a zero divisor in $U(\mathfrak{n}^-)$, the element θ satisfying (0.1) is unique. □