Errata

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Errata

Changes appear in yellow. Line $k+$ (resp., line $k-$) denotes the kth line from the top (resp., the bottom) of a page. My thanks go to the following individuals who have contributed to this list: Tobias Wöhrer, Simon Becker, Dennis Cutraro, Mateusz Piorkowski, Laura Kanzler, Mateus Sampaio, Laura Shou, Noema Nicolussi, Andreas Geyer-Schulz, Rene Allerstorfer, Manuel Culqui Rodriguez, Fritz Gesztesy, Marcel Griesemer, Michael Hofacker, Maxim Zinchenko, Jannik Pitt.

Page 16. First line: for $a \in \ell^p(\mathbb{N})$, $b \in \ell^q(\mathbb{N})$.

Page 25. Proof of Theorem 0.25: and we can choose $m_2 = \sqrt{\sum_j \|u_j\|_1^2}$.

Page 34. Proof of Lemma 0.36: (if $K_2(x,.)f(.) \notin L^p(Y, d\nu)$, the inequality is trivially true).

Page 36. Add the following at the end of Lemma 0.39: Moreover, if u and f both have compact support, then $f_k \in C_c^\infty(\mathbb{R}^n)$.

Page 36. Proof of Lemma 0.41: ... $\varphi_n \in C_c^\infty(\mathbb{R}^n)$ with support inside some open ball X which converges ... continuous functions φ_n with support in X which converges to g ...

Page 54. Proof of Lemma 1.11: (ii) follows from $\langle\varphi, A^{**}\psi\rangle = \langle A^*\varphi, \psi\rangle = \langle \varphi, A\psi\rangle$.

Page 60. Last sentence in the proof of Theorem 1.16: Since $f - \varepsilon < f_{z_l}$ for all z_l we have $f - \varepsilon < f_\varepsilon$ and we have found a required function.

Page 61. Problem 1.23: Show that the span of $\{(t-z)^{-1} | z \in U\}$ is dense in $C_\infty(\mathbb{R})$.

Page 66. Line after (2.15): measurable function $A : \mathbb{R}^d \to \mathbb{C}$.

Page 72. Line 18+: Clearly we have $\overline{\alpha A} = \alpha\overline{A}$ for $\alpha \in \mathbb{C}\backslash\{0\}$ and $\overline{A+B} = \overline{A} + B$ provided A is closable and B is bounded (Problem 2.8).

Page 75. Proof of Lemma 2.7:

$$\begin{aligned} \|(A-z)\psi\|^2 &= \|(A-x)\psi - \mathrm{i}y\psi\|^2 \\ &= \|(A-x)\psi\|^2 + y^2\|\psi\|^2 \geq y^2\|\psi\|^2, \end{aligned} \tag{2.46}$$

Page 76. Problem 2.8: Suppose that if A is closable and B is bounded. Show that $\overline{\alpha A} = \alpha\overline{A}$ for $\alpha \in \mathbb{C}\backslash\{0\}$ and $\overline{A+B} = \overline{A} + B$.

Page 78. Proof of Lemma 2.11:

$$\mathfrak{D}(\tilde{A}) = \{\psi \in \mathfrak{H}_A | \exists\tilde{\psi} \in \mathfrak{H} : \langle\varphi, \psi\rangle_A = \langle\varphi, \tilde{\psi}\rangle, \forall\varphi \in \mathfrak{D}(A)\} = \mathfrak{H}_A \cap \mathfrak{D}(A^*)$$

as $\mathfrak{D}(A) \subset \mathfrak{H}_A$ is dense and $\langle\varphi, \psi\rangle_A = \langle(A+1)\varphi, \psi\rangle$ for $\varphi \in \mathfrak{D}(A)$, $\psi \in \mathfrak{H}_A$.

Page 82. Proof of Lemma 2.15:

$$\begin{aligned} 2|\operatorname{Re}\langle\varphi, A\psi\rangle| &= \frac{1}{2}|q(\psi+\varphi) - q(\psi-\varphi)| \leq \frac{\|q\|}{2}(\|\psi+\varphi\|^2 + \|\psi-\varphi\|^2) \\ &= \|q\|(\|\psi\|^2 + \|\varphi\|^2) \end{aligned}$$

Page 87. Proof of Theorem 2.19:

$$f'(\lambda) = -\|(A - E + \lambda)^{-1}\varphi\|^2 \leq -f(\lambda)^2$$

Page 88. Problem 2.18: Then so does $A + B$ if $\|B\| < \|A^{-1}\|^{-1}$.

Page 93. Paragraph after Lemma 2.28: A conjugate linear map $C : \mathfrak{H} \to \mathfrak{H}$ is called a **conjugation** if it satisfies $C^2 = \mathbb{I}$ and $\langle C\psi, C\varphi\rangle = \langle\varphi, \psi\rangle$.

Page 135. Problem 4.11:

$$\chi_\Omega(A) = -\frac{1}{2\pi\mathrm{i}}\int_\Gamma R_A(z)dz,$$

Page 138.

$$\||A|\psi\|^2 = \langle\psi, |A|^2\psi\rangle = \langle\psi, A^*A\psi\rangle = \|A\psi\|^2, \quad \psi \in \mathfrak{D}(|A|) = \mathfrak{D}(A), \tag{4.31}$$

Page 139:

$$U^*U = P_{\mathrm{Ker}(A)^\perp}, \qquad UU^* = P_{\mathrm{Ker}(A^*)^\perp}, \tag{4.34}$$

Page 139: Last line of Theorem 4.10: $\mathrm{Ker}(U)^\perp = \mathrm{Ker}(A)^\perp$

Page 141: (ii) We have

$$\inf_{\psi\in U(\varphi_1,\dots,\varphi_{n-1})} \langle\psi, A\psi\rangle \ge E_n, \tag{4.40}$$

since A restricted to $\mathrm{span}\{\varphi_1,\dots,\varphi_{n-1}\}^\perp$ is bounded from below by E_n (which is immediate from the spectral theorem).

Page 141: Corollary 4.13: Suppose A and B are self-adjoint operators with $\mathfrak{D}(A) = \mathfrak{D}(B)$ and $A \ge B$ (i.e., $A - B \ge 0$).

Page 143: Proof of Theorem 4.16: Thus $\langle\psi,(A-\lambda_2)(A-E)\psi\rangle = \|A\psi\|^2 + \lambda_2 E \ge 0$ and ...

Page 146: Proof of Lemma 5.2:

$$\frac{d}{dt}\varphi(t) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\big(U(-t-\varepsilon) - U(-t)\big)\big(\psi(t) - \varepsilon \mathrm{i} A\psi(t) + o(\varepsilon)\big) + U(-t)\big(\psi(t+\varepsilon)-\psi(t)\big)\Big) = 0.$$

Page 148: Top of page: Let $\psi \in \mathfrak{D}(A)$ and abbreviate $\psi(t) = (U(t) - V(t))\psi$. Then

$$\lim_{\varepsilon\to 0}\frac{\psi(t+\varepsilon)-\psi(t)}{\varepsilon} = -\mathrm{i}\overline{A}\psi(t)$$

and hence $\frac{d}{dt}\|\psi(t)\|^2 = -2\,\mathrm{Re}\langle\psi(t), \mathrm{i}\overline{A}\psi(t)\rangle = 0$.

Page 152: Proof of Theorem 5.7: Since $K(A-\mathrm{i})^{-1}$ is compact by assumption,

Page 154: Proof of Theorem 5.9: We will assume that K is compact.

Page 155. Problem 5.7:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \langle\varphi, \mathrm{e}^{\mathrm{i}t(A-\lambda_0)}\psi\rangle dt = \langle\varphi, P_A(\{\lambda_0\})\psi\rangle$$

Page 155. Problem 5.9:

$$\mathfrak{H}_{rc} = \{\psi\in\mathfrak{H} | \lim_{t\to\infty}\langle\psi, \mathrm{e}^{-\mathrm{i}tA}\psi\rangle = 0\} \supseteq \mathfrak{H}_{ac}, \tag{5.27}$$

Page 159. Theorem 6.4:

$$\gamma - \max\Big(a|\gamma| + b, \frac{b}{1-a}\Big). \tag{6.4}$$

Page 159. Proof of Theorem 6.4; last sentence: The explicit bound (6.4) follows since this condition implies $\|BR_A(-\lambda)\| < 1$ by virtue of (6.2) from the proof of the previous lemma.

Page 161. Lemma 6.8:

$$s_n(K) = \min_{\psi_1,\dots,\psi_{n-1}} \sup_{\psi\in U(\psi_1,\dots,\psi_{n-1})} \|K\psi\|, \tag{6.9}$$

Page 162. Proof of Lemma 6.9: last formula

$$\gamma_n = \|K - K_n\| = \sup_{\|\psi\|=1} \|K(\psi - \sum_{j=1}^{n} \langle\varphi_j, \psi\rangle \varphi_j)\|$$

Page 164. Proof of Lemma 6.10: Conversely, choose $\varphi_i = \hat{\phi}_i$

Page 176. Theorem 6.25: add for $\lambda > \frac{b}{a} - \gamma$. after (6.44)

Page 176. Line before equation (6.45): Furthermore, we can define $C_q(\lambda)$ for all $-z \in \rho(A)$, using

Page 179. Problem 6.18: Suppose A is self-adjoint, $\lambda \in \mathbb{R}$, and R is bounded. Show that $R = R_A(\lambda)$ if and only if $\langle (A-\lambda)\varphi, R\psi\rangle = \langle\varphi, \psi\rangle$ for all $\varphi \in \mathfrak{D}(A)$, $\psi \in \mathfrak{H}$.

Page 180. Corollary 6.32: Then this holds for all z in the interior of Γ.

Page 195. Line 2+: Clearly $H^{r+1}(\mathbb{R}^n) \subset H^r(\mathbb{R}^n)$

Page 200. Discussion after Lemma 7.20: $|\psi(x,t)|^2 d^n x = |\hat{\psi}(\frac{x}{2t})|^2 \frac{d^n x}{(2t)^n}$

Page 209. Last line of the proof of Theorem 8.2: $0 = (z + z^*)\|\hat{A}\psi\|^2$

Page 209.

$$\psi(x) = \left(\frac{\lambda}{\pi}\right)^{n/4} \mathrm{e}^{-\frac{\lambda}{2}|x-x_0|^2 + \mathrm{i}p_0 x}, \tag{8.13}$$

Page 211. First equation in the proof of Lemma 8.3:

$$\frac{1}{\sqrt{2\pi}} \int \overline{\varphi(x)} \mathrm{e}^{-\frac{x^2}{2}} \sum_{j=0}^{k} \frac{(\mathrm{i}tx)^j}{j!} dx = 0$$

Page 212. Theorem 8.4: There exists an orthonormal basis of simultaneous eigenvectors for the operators L^2 and L_3.

Page 222. Proof of Lemma 9.5: Choosing $f_1 = v$, $f_2 = f$, $f_3 = v^*$, $f_4 = f^*$, we infer (9.15).

Page 222. Problem 9.1: and $f(d) = \gamma$, $(pf')(d) = \delta$.

Page 222. Problem 9.3: Let $\phi \in L^1_{loc}(I)$ be real-valued.

Page 222. Problem 9.4: Add the assumption that a is regular. Otherwise one can also start the integration at an arbitrary point in (a, b).

Page 223. Replace the last sentence by: Moreover, the following set is a core for A

$$\mathfrak{D}_1 = \{f \in \mathfrak{D}(\tau) | \ \exists x_0 \in I : \forall x \in (a, x_0),\ V_x(f) = 0, \\ \exists x_1 \in I : \forall x \in (x_1, b),\ W_x(f) = 0\}, \tag{9.21}$$

where we set $V_x(f) = W_x(v, f)$, $W_x(f) = W_x(w, f)$ if τ is l.c. at a, b and $V_x(f) = f(x)$, $W_x(f) = f(x)$ if τ is l.p. at a, b, respectively.

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$$W_a(v, f) = 0 \quad \Leftrightarrow \quad \cos(\alpha) BC_a^2(f) + \sin(\alpha) BC_a^1(f) = 0, \tag{9.23}$$

where $\tan(\alpha) = -\frac{BC_a^2(v)}{BC_a^1(v)}$.

Page 228. Theorem 9.10: Delete "(which are simple)". And the following claim about simplicity of eigenvalues only applies to separated boundary conditions as in Theorem 9.6.

Page 231.

$$(Uf)(\lambda) = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\sqrt{\lambda}x} f(x)\, dx \\ \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}x} f(x)\, dx \end{pmatrix}, \quad \lambda \in \sigma(H_0) = [0, \infty). \tag{9.37}$$

Page 233. Proof of Lemma 9.13:

$$\sum_j \int_{\mathbb{R}} F_j(\lambda)^* \int_a^b u_j(\lambda, x) g(x) r(x)\, dx\, d\mu_j(\lambda) = \int_a^b (U^{-1}\underline{F})(x)^* g(x) r(x)\, dx.$$

Interchanging integrals on the left-hand side

Page 233. Delete the last sentence: ~~Note that since we can replace $u_j(\lambda, x)$ by $\gamma_j(\lambda) u_j(\lambda, x)$ where $|\gamma_j(\lambda)| = 1$, it is no restriction to assume that $u_j(\lambda, x)$ is real-valued.~~

Page 250. Second line in Section 9.7: on $(a, b) = \mathbb{R}$.

Page 252. Proof of Lemma 9.35: where $M_n = \sup_{|m| \ge n} \int_m^{m+1} |q(x)| dx$.

Page 255. First line: the zeros of ψ_n interlace the zeros of ψ_{n+1}.

Page 256. Problem 9.18: Change the hint according to:
(Hint: Let $\varphi_\varepsilon(x) = \exp(-\varepsilon^2 x^2)$ and investigate $\langle \varphi_\varepsilon, H\varphi_\varepsilon \rangle$.)

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(10.23)

$$A\Phi = \tau\Phi, \quad \mathfrak{D}(A) = \{\Phi \in L^2(0,2\pi)| \;\; \Phi \in AC^1[0,2\pi], \Phi'' \in L^2(0,2\pi), \\ \Phi(0) = \Phi(2\pi), \Phi'(0) = \Phi'(2\pi)\}.$$

Page 268. Line 3+: Note that the $L_j^{(k)}(r)$ are polynomials of degree j which

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$$F(z) = \int_Y f(z,y)\, d\mu(y) \tag{A.55}$$

Page 330. Proof of Lemma A.35:

$$\mu(x-) \le \liminf \mu_n(x) \le \limsup \mu_n(x) \le \mu(x+)$$

Page 330. Problem A.32 can be deleted as the claim is part of Lemma A.36.

Page 333. Problem A.34. This claim is clearly wrong (take a function which is constant on an interval). It should be deleted.

Addendum

Page 81. Proof of Theorem 2.14: Since the rest is not so straightforward, here is a complete proof:

Proof. Since $\mathfrak{H}_q$ is dense, $\tilde{\psi}$ and hence A is a well-defined operator. Moreover, replacing q by $q(.) - \gamma\|.\|^2$ and A by $A - \gamma$, it is no restriction to assume $\gamma = 0$. Next it will be convenient to look at the definition from a somewhat more abstract point of view: We have a conjugate linear continuous embedding $j : \mathfrak{H} \to \mathfrak{H}_q^*$, $\psi \mapsto \langle\psi, .\rangle$ (here $\mathfrak{H}_q$ is equipped with $\|.\|_q$) with $\mathrm{Ran}(j)$ dense. Indeed, if $\mathrm{Ran}(j)$ were not dense, there would be some nonzero $\varphi \in \mathfrak{H}_q^{**} \cong \mathfrak{H}_q$ (the identification given by the Riesz lemma via evaluation) such that $\varphi(j(\psi)) = j(\psi)(\varphi) = \langle\psi,\varphi\rangle = 0$ for all $\psi \in \mathfrak{H}$ implying the contradiction $\varphi = 0$.

Next, there is a conjugate linear isometric isomorphism $\hat{A} : \mathfrak{H}_q \to \mathfrak{H}_q^*$, $\psi \mapsto s(\psi, .) + \langle\psi, .\rangle$ (Riesz lemma) and our operator A is given by $j^{-1}\hat{A} - \mathbb{I}$. Moreover, $\mathfrak{D}(A) = \hat{A}^{-1}\,\mathrm{Ran}(j)$ is dense in $\mathfrak{H}_q$ and hence also in $\mathfrak{H}$. By construction, $q_A(\psi) = q(\psi)$ for $\psi \in \mathfrak{D}(A)$, which shows that A is nonnegative and as in the proof of Lemma 2.11, it follows that $\mathrm{Ran}(A+1) = \mathfrak{H}$. Thus A is self-adjoint. Finally, note that the fact that $\mathfrak{D}(A)$ is dense in $\mathfrak{H}_q$ implies $\mathfrak{H}_A = \mathfrak{H}_q$.

Concerning uniqueness let $\tilde{A}$ be another self-adjoint operator with the same properties. Then equality of the associated quadratic forms (and hence of the sesquilinear forms) on $\mathfrak{Q}$ implies $\langle A\psi, \varphi\rangle = \langle\psi, \tilde{A}\varphi\rangle$ for $\psi \in \mathfrak{D}(A)$, $\varphi \in \mathfrak{D}(\tilde{A})$. But this shows $\psi \in \mathfrak{D}(\tilde{A}^*) = \mathfrak{D}(\tilde{A})$ and $\tilde{A}\psi = \tilde{A}^*\psi = A\psi$ and vice versa. □

Page 118. Here is an amplification of Theorem 3.16:

Theorem 3.16. *For every self-adjoint operator A there is an ordered spectral basis $\{\psi_j\}_{j=1}^N$. Moreover, it can be chosen such that $d\mu_{\psi_j} = \chi_{\Omega_j} d\mu$, where μ is a maximal spectral measure and $\Omega_{j+1} \subseteq \Omega_j$. The dimension N is the spectral multiplicity of A.*

Proof. First of all observe that for every φ there is a maximal spectral vector ψ such that $\varphi \in \mathfrak{H}_\psi$. To see this start with a maximal spectral vector $\tilde{\psi}$. Then $d\mu_\varphi = f\, d\mu_{\tilde{\psi}}$ and we set $\Omega = \{\lambda | f(\lambda) > 0\}$. Then $P_A(\Omega)\varphi = \varphi$ since $\|P_A(\Omega)\varphi\|^2 = \int_\Omega d\mu_\varphi = \int_{\mathbb{R}} f\, d\mu_{\tilde{\psi}} = \|\varphi\|^2$. Now set $\psi = \varphi + P(\mathbb{R}\backslash\Omega)\tilde{\psi}$ and observe $d\mu_\psi = d\mu_\varphi + \chi_{\mathbb{R}\backslash\Omega} d\mu_{\tilde{\psi}} = (f + \chi_{\mathbb{R}\backslash\Omega}) d\mu_{\tilde{\psi}}$. Since $f + \chi_{\mathbb{R}\backslash\Omega} > 0$ we see that $d\mu_{\tilde{\psi}}$ is absolutely continuous with respect to $d\mu_\psi$ and hence ψ is a maximal spectral vector with $\varphi = P_A(\Omega)\psi \in \mathfrak{H}_\psi$ as required.

Now start with some total set $\{\tilde{\psi}_j\}$ and proceed as in Lemma 3.4 to obtain an ordered spectral basis $\{\psi_j\}$. Since $\mu_{\psi_{j+1}}$ is absolutely continuous with respect to μ_{ψ_j} all spectral measures are absolutely continuous with respect to $\mu = \mu_{\psi_1}$, that is, $d\mu_{\psi_j} = f_j d\mu$. Choosing $\Omega_j = \{\lambda | f_j(\lambda) > 0\}$ we can replace $\psi_j \to \chi_{\Omega_j}(A) f_j(A)^{-1/2} \psi_j$ such that $f_j \to \chi_{\Omega_j}$. Since $\mu_{\psi_{j+1}}$ is absolutely continuous with respect to μ_{ψ_j} we can even assume $\Omega_{j+1} \subseteq \Omega_j$.

Finally, we show that the spectral multiplicity of A is N. By the first part we can assume that A is multiplication by λ in $\bigoplus_{j=1}^N L^2(\mathbb{R}, \chi_{\Omega_j} d\mu)$. Let $\{\psi_j\}_{j=1}^n$ be a spectral basis with $n < N$. We will show that there is some vector in the orthogonal complement of $\bigoplus_j \mathfrak{H}_{\psi_j}$. Of course such a vector exists pointwise for every λ but it is not clear that the components can be chosen measurable. To see this we use a Gauss-type elimination: For this note that we can multiply every vector ψ_j with a non-vanishing function or add multiples of the other vectors to a given one without changing $\bigoplus_j \mathfrak{H}_{\psi_j}$. Hence we can first normalize the first component of every ψ_j to be a characteristic function. Moreover, by adding all other vectors to ψ_1 we can assume that its first component is positive on a maximal set $\tilde{\Omega}_1$. In fact, after another normalization we can assume that $\psi_{1,1} = \chi_{\tilde{\Omega}_1}$ and after subtracting multiples of ψ_1 from the remaining vectors we can assume $\psi_{j,1} = 0$ for $j \geq 2$. If $\mu_1(\mathbb{R}\backslash\tilde{\Omega}_1) > 0$ then $\varphi = (\chi_{\mathbb{R}\backslash\tilde{\Omega}_1}, 0, \dots)$ would be in the orthogonal complement and we are done. So assume $\chi_{\Omega_1} = 1$ and continue with the other components until they satisfy $\psi_{j,k} = \delta_{j,k}$ for $1 \leq j, k \leq n$. Then $\varphi = (-\psi_{1,n+1}, \dots, -\psi_{n,n+1}, 1, 0, \dots)$ is in the orthogonal complement contradicting our assumption that $\{\psi_j\}_{j=1}^n$ is a spectral basis. □