ERRATA FOR *INTRODUCTION TO TROPICAL GEOMETRY*

DIANE MACLAGAN AND BERND STURMFELS

Thanks to all who have pointed out errors and typos after the book was published. These include Jarek Buczyński, Justin Chen, Jan Draisma, Christian Haase, Joachim Jelisiejew, Joe Kileel, Sara Lamboglia, Hannah Markwig, Henryk Michalewski, Lisa Sauermann, and Cynthia Vinzant.

1. Chapter 2

(1) p47. Line 8. "Thus $\text{val}(c_j^{l+1}) > 0$ for $0 \leq j \leq r_{l+1}$,". This replaces l by j in the range.

(2) p66. Statement of Lemma 2.4.2. Replace the last sentence by "Further, if $\mathbf{w} \in \Gamma_{\text{val}}^{n+1}$ then whenever $g \in \text{in}_{\mathbf{w}}(I)$, $g = \text{in}_{\mathbf{w}}(f)$ for some $f \in I$." In the proof of Lemma 2.4.2, after the sentence ending "let $W_{\mathbf{u}} = \text{trop}(f_{\mathbf{u}})(\mathbf{w}) + \mathbf{w} \cdot \mathbf{u}$.", add the sentence "Note that $W_{\mathbf{u}} \in \Gamma_{\text{val}}$."

(3) p69. Second paragraph of the proof of Lemma 2.4.8. Replace the sentence "This is possible by Lemma 2.4.2" by "To see that this is possible, note that we can write $x^{\mathbf{u}} = \sum a_{\mathbf{v}} \text{in}_{\mathbf{w}}(f_{\mathbf{v}})$ where $a_{\mathbf{v}} \in \Bbbk^*$ and $f_{\mathbf{v}} \in I$ for all $\mathbf{v}$. If $\text{supp}(\text{in}_{\mathbf{w}}(f_{\mathbf{v}})) \cap \text{supp}(\text{in}_{\mathbf{w}}(f_{\mathbf{v}'})) \neq \emptyset$, $\text{trop}(f_{\mathbf{v}})(\mathbf{w}) - \text{trop}(f_{\mathbf{v}'})(\mathbf{w}) \in \Gamma_{\text{val}}$, so as in the proof of Lemma 2.4.2 we can write $a_{\mathbf{v}} \text{in}_{\mathbf{w}}(f_{\mathbf{v}}) + a_{\mathbf{v}'} \text{in}_{\mathbf{w}}(f_{\mathbf{v}'}) = \text{in}_{\mathbf{w}}(f')$ for some $f' \in I$. Thus we can write $x^{\mathbf{u}} = \sum a_{\mathbf{v}} \text{in}_{\mathbf{w}}(f_{\mathbf{v}})$ where the supports of the $\text{in}_{\mathbf{w}}(f_{\mathbf{v}})$ do not intersect. This means that there is no cancellation in this expression, so there must be only one $f_{\mathbf{v}}$, and $\text{in}_{\mathbf{w}}(f_{\mathbf{v}}) = x^{\mathbf{u}}$."

(4) p77. Equation (2.5.2). The subscript "$\mathbf{u} \in I$" should be "$x^{\mathbf{u}} \in I$".

(5) p77. Proof of Theorem 2.5.7. Replace $J \subset M_d$ with $J \subseteq M_d$ (twice). Similarly for J' and J''.

(6) p77. Proof of Theorem 2.5.7. Replace the sentence " Since σ_d is a maximal cell, this minimum is achieved at only one term, indexed by $J \subseteq \mathcal{M}_d$." by " Since σ_d is a maximal cell, this minimum is achieved at only one term. We claim that this term

Date: November 2016.

is indexed by a unique $J \subseteq \mathcal{M}_d$. Indeed, suppose that there are J, J' achieving the minimum with $\prod_{x^{\mathbf{u}} \in J} x^{\mathbf{u}} = \prod_{x^{\mathbf{u}} \in J'} x^{\mathbf{u}}$ and $\mathrm{val}(\det(A_d^J)) = \mathrm{val}(\det(A_d^{J'}))$. Write B for the submatrix of A_d with columns indexed by the monomials in J andJ'. Fix $x^{\mathbf{v}} \in J$, and consider the Plücker relation $\mathcal{P}_{J\setminus x^{\mathbf{v}}, J' \cup x^{\mathbf{v}}}$ evaluated at the minors of the matrix B. Since $\mathrm{val}(\mathcal{P}_{J\setminus x^{\mathbf{v}}, J' \cup x^{\mathbf{v}}}) \neq \min_{x^{\mathbf{v}'} \in J' \cup x^{\mathbf{v}}} (\mathrm{val}(\det(A_d^{J \setminus x^{\mathbf{v}} \cup x^{\mathbf{v}'}}) \det(A_d^{J \setminus x^{\mathbf{v}} \cup x^{\mathbf{v}'}})))$, Lemma 2.1.1 implies that the minimum must be achieved at least twice, so there is $x^{\mathbf{v}'} \in J'$ with

$$\mathrm{val}\left(\det(A_d^J)\right) + \mathrm{val}\left(\det(A_d^J)\right) \geq \mathrm{val}\left(\det(A_d^{J\setminus x^{\mathbf{v}} \cup x^{\mathbf{v}'}})\right) + \mathrm{val}\left(\det(A_d^{J' \setminus x^{\mathbf{v}'} \cup x^{\mathbf{v}}})\right).$$

Thus the minimum is also achieved at $J \setminus x^{\mathbf{v}} \cup x^{\mathbf{v}'}$ and $J' \setminus x^{\mathbf{v}'} \cup x^{\mathbf{v}}$. However neither $\prod_{x^{\mathbf{u}} \in J \setminus x^{\mathbf{v}} \cup x^{\mathbf{v}'}} x^{\mathbf{u}}$ nor $\prod_{x^{\mathbf{u}} \in J' \setminus x^{\mathbf{v}'} \cup x^{\mathbf{v}}} x^{\mathbf{u}}$ equals $\prod_{x^{\mathbf{u}} \in J} x^{\mathbf{u}}$, which contradicts the assumption that the minimum in $\mathrm{trop}(g_d)$ is achieved only at a single term. We thus conclude that the choice of J achieving the minimum is unique."

(7) p82. Lemma 2.6.2. Change the statement of part (1) to "If $x^{\mathbf{u}} \in \mathrm{in}_{\mathbf{w}}(I)$ then $x^{\mathbf{u}} = \mathrm{in}_{\mathbf{w}}(h)$ for some $h \in I$. Furthermore, if $\mathbf{w} \in \Gamma_{\mathrm{val}}^n$ then every $g \in \mathrm{in}_{\mathbf{w}}(I)$ has the form $g = \mathrm{in}_{\mathbf{w}}(h)$ for some $h \in I$." At the end of the first paragraph, add the sentence "The case $g = x^{\mathbf{u}}$ follows similarly using the argument of the proof of Lemma 2.4.8.".

2. Chapter 3

(1) p125. Paragraph after the proof of Theorem 3.4.12. Replace the last sentence by "Then $\mathrm{trop}(X) \cap \mathrm{trop}(Y)$ is the union of a line segment and two rays, while $\mathrm{trop}(X \cap Y) = \{A, B\}$ consists only of the two endpoints of the line segment.".

(2) p137. Second paragraph of Example 3.6.6. In the sentence "For $\mathbf{v} = (1, 1)$ the fan $\mathrm{star}_{\Sigma_1}(\sigma)$ intersects $\mathbf{v} + \mathrm{star}_{\Sigma_2}(\sigma)$ in two points, $(1, 0)$ and $(0, 1)$.", replace $(1, 0)$ by $(2, 0)$ and $(0, 1)$ by $(0, 2)$.

(3) p148. Proof of Corollary 3.6.16. Line 2. Change $G(n-d, n+1)$ to $G(n + 1 - d, n + 1)$. Also on line 4 it should be $L \in G(n + 1 - d, n + 1)$, and on line 4 of p149, and line 6 of p149.

3. Chapter 4

(1) p166. Proof of Proposition 4.2.10. Replace from the last sentence of the third paragraph ("We may assume that C is a circuit ...") by: "Choose $i \in C$ achieving the minimum. As

$C\backslash\{i\}$ is independent, there is a basis B' of M with $C \subset B' \cup \{i\}$. By the stronger form of the basis exchange property there is $j \in B' \setminus B$ with both $B \cup \{j\} \setminus \{i\}$ and $B' \cup \{i\} \setminus \{j\}$ bases of M. Since the subset of C of minimal weight is contained in B, and $j \notin B$, we have $w_j > w_i$. But this means that $\sum_{r \in B} w_r < \sum_{r \in B \cup \{j\} \backslash \{i\}} w_r$, contradicting the assumption that B had maximal weight."

(2) p167. Proof of Theorem 4.2.12, line 11. Replace the sentence "Define a linear functional ..." by "Fix $i \in \sigma \backslash \sigma'$. Set $s = |\sigma \backslash \sigma'|$. We may assume that $s > 1$, as otherwise $\mathbf{e}_{\sigma'}$ is the desired vertex. Choose ϵ with $0 < \epsilon \ll 1$, and set $\phi(x) = \epsilon x_i + (s-\epsilon)/(s-1) \sum_{j \in \sigma \backslash \sigma', j \neq i} x_j + \sum_{j \in \sigma' \backslash \sigma} x_j + 2 \sum_{j \in \sigma \cap \sigma'} x_j$." In the next sentence, replace r by $r + |\sigma \cap \sigma'|$. Also, at the end of the proof, replace from "If $k \notin \sigma'$ ..." to the end of the proof by "Since $\phi(\sigma \backslash \{l\} \cup \{k\}) \geq \phi(\sigma)$, we must have $l = i$, and $k \in \sigma'$."

(3) p171. Definition 4.3.3. Replace "Assign a length $\ell_e \in \mathbb{R}$ to each edge e of τ." by "Assign a length $\ell_e \in \mathbb{R}$ to each edge e of τ, with $\ell_e \geq 0$ when e is not a pendant edge."

(4) p185. Second line of the proof of Lemma 4.3.16. "$\mathbf{w} \in \mathrm{Gr}_M$" should be "$\mathbf{w} \in \mathrm{trop}(\mathrm{Gr}_M)$".

(5) p189. In the first displayed equation of Example 4.4.9, replace Gr_M by $\mathrm{trop}(\mathrm{Gr}_M)$. In the last paragraph, replace "Next suppose $\mathbf{w} \in \mathrm{Gr}_M \backslash L$" by "Next suppose $\mathbf{w} \in \mathrm{trop}(\mathrm{Gr}_M) \backslash L$".

(6) p189. Paragraph beginning "First suppose that" line 4. Replace $-\mathbf{v}$ by $\mathbf{v}$.

(7) p190. Example 4.4.10 line 3. Replace $\mathrm{Gr}(3,6)$ by $\mathrm{trop}(G^0(3,6))$.

(8) p192, second paragraph of the proof of Proposition 4.5.1. Replace "$\sum_{i=1}^n a_i z_i (\partial g / \partial x_i)(\mathbf{z})$" by "$\sum_{i=0}^n a_i z_i (\partial g / \partial x_i)(\mathbf{z})$".

(9) p193, second line of equations in the proof of Proposition 4.5.1. Replace "$z^{\mathbf{u}-\mathbf{e}_i}$" by "$\mathbf{z}^{\mathbf{u}-\mathbf{e}_i}$".

(10) p193, second-to-last paragraph. Replace "The assumption that $\Delta_{c_\mathbf{u}}$ is unimodular" by "The assumption that $\Delta_{\mathrm{val}(c_\mathbf{u})}$ is unimodular".

(11) p193, last paragraph. Replace "indeed $\mathrm{val}(\mathbf{a} \cdot \mathbf{u}) = 0$ unless" by "indeed $\mathrm{val}(\mathbf{a} \cdot \mathbf{u}) \in \{0, \infty\}$ unless".

(12) p193. Replace line -3 by

$$\mathrm{val}(c_{\mathbf{u}'}) + \mathrm{val}(\mathbf{a} \cdot \mathbf{u}') + \mathbf{w} \cdot \mathbf{u}' = \mathrm{val}(c_{\mathbf{u}'}) + \mathbf{w} \cdot \mathbf{u}'.$$

(13) p214 Section 4.7. Exercise 2. Replace "$n = 6$" by "$n = 7$".

(14) p215 Exercise 9. Replace the last two sentences with "Compare the Bergman fan of X with the fan in Theorem 4.2.6."

(15) p215 Exercise 10, line 5. Replace "intersection of these flats" with "join of these flats".

4. Chapter 5

(1) p273. Proof of Proposition 5.5.11. At the end of the proof, add the sentence "It is necessary to divide all multiplicities by the degree of the map ϕ restricted to $X \times Y$."

5. Chapter 6

(1) p279. line 5. Replace "We recall this here in the case that the fan Σ is smooth." by "We recall this here in the case that the fan Σ is smooth and spans all of $N_{\mathbb{R}}$.
(2) p288, line 8. Replace "for the vector in $\mathbb{R}^n$" by "For the vector in $\overline{\mathbb{R}}^n$".
(3) p298. Statement of Proposition 6.4.4. Replace "Write $\text{trop}(\phi)\colon \mathbb{R}^n \to \mathbb{R}^m$" by "Write $\text{trop}(\phi)\colon \mathbb{R}^m \to \mathbb{R}^n$".
(4) p312. Second line of Example 6.5.7. Replace "T^n" by "T^3".
(5) p324. Remark 6.6.5. Replace "R-module" by "R-algebra" (twice).