*A First Course in Sobolev Spaces*, Second edition,
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For the original text I use the color Red, for corrections the color Blue, and for improvements and additions the color Cyan. Names in brackets refer to the persons who called the error to my attention (to the best of my recollection) or suggested improvements and additions.[1]

Gagliardo-Nirenberg file student

Support of traces

## CHAPTER 2:

p. 45 Line 5 from below: Let $Z$ be the subspace should be replaced by Let $Z$ be the subset .

p. 45 The last formula

$$v^{-1}(\{0\}) = \left\{x \in [0,1] : \lim_{n\to\infty} n(u(x+\tfrac{1}{n}) - u(x)) = 0\right\}$$
$$= \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \left\{x \in [0,1] : n|u(x+\tfrac{1}{n}) - u(x)| < \tfrac{1}{k}\right\}.$$

should be replaced by

$$v^{-1}(\{0\}) = \left\{x \in [0,1] : \lim_{n\to\infty} n(u(x+\tfrac{1}{n}) - u(x)) = 0\right\}$$
$$= \bigcap_{k=1}^{\infty} \bigcap_{\ell=1}^{\infty} \bigcup_{n=\ell}^{\infty} \left\{x \in [0,1] : n|u(x+\tfrac{1}{n}) - u(x)| < \tfrac{1}{k}\right\}.$$

[Harvey-Arnold Scott].

p. 46 Line 3: We claim that $Z$ is a Banach space should be replaced by We claim that $Z$ is a complete metric space .

p.46 There is a gap in Weil's original paper, which I missed. In Step 2 the functions $v$ and $w$ belong to $Z$ but there is no reason for $v+w$ to belong to $Z$. By Exercise 1.40 the space $Z$ contains a nonzero function $w \in Z$ with $w(x_0) \neq 0$. should be replaced by Let $A$ be a countable subset of $v^{-1}(\{0\})$ dense in $[0,1]$. By Exercise 1.40 the space $Z$ contains a nonzero function $w_0 \in Z$ with $w_0(x_0) \neq 0$. Let $B$ be a countable subset of $w_0^{-1}(\{0\})$ dense in $[0,1]$. By Exercises 2.39A and 2.39B there exists a $C^1$ function $\varphi : [0,1] \to [0,1]$ with $\varphi' > 0$ such that $\varphi(A) \subseteq B$ and $\varphi(x_0) \in [0,1] \setminus v^{-1}(\{0\})$. Define $w := (w_0 \circ \varphi)\varphi'$. Then $v + w$ belongs to $Z$ since $A \subseteq (v+w)^{-1}(\{0\})$. The added exercises are the following **Exercise 2.39A**: Let $A := \{a_n : n \in \mathbb{N}\}$

[1] The style of this file is inspired by http://www.hss.caltech.edu/ kcb/IDA-Errata.pdf

and $B := \{b_n : n \in \mathbb{N}\}$ be two sets of $(0,1)$, with $a_n \neq a_m$ and $b_n \neq b_m$ for $n \neq m$ and with $B$ dense in $(0,1)$. Consider the functions

$$g_1(x) := x(x-1), \quad g_{n+1}(x) := \prod_{i=1}^{n} x(x-1)(x-a_i),$$

for $n \in \mathbb{N}$, and

$$f_0(x) := x, \quad f_n(x) := x + \sum_{i=1}^{n} c_i g_i(x).$$

Prove that the constants $c_i$ can be chosen inductively in such a way that $|c_i|(\|g_i\|_\infty + \|g_i'\|_\infty) < \frac{1}{2^{i+1}}$ and that $f_n(a_i) \in B$ for all $n$ and all $i = 1, \dots, n$. Deduce that the function

$$\varphi(x) := x + \sum_{i=1}^{\infty} c_i g_i(x)$$

is of class $C^1$, maps $[0,1]$ onto $[0,1]$, that $\varphi' > 0$ and that $\varphi(A) \subseteq B$.

**Exercise 2.39B**: Let $A$ and $B$ as in the previous set and let $C$ be another dense set in $(0,1)$. Given $x_0 \in (0,1) \setminus A$, prove that the function $\varphi$ can be constructed in such a way that $\varphi(x_0) \in C$.These exercises are adapted from the papers P. Franklin, *Analytic transformations of everywhere dense point sets*, Trans. Amer. Math. Soc. 27 (1925) 91-100 and J. Blažek, E. Borák, and J. Malý, *On Köpcke and Pompeiu functions*, Časopis pro pěstování matematiky 103.1 (1978), 53-61.

p.46 The following exercise, taken from the paper K.C. Ciesielski, *Monsters in calculus*, Amer. Math. Monthly 125(8) (2018), 739-744; gives another explicit example of a nowhere monotone, differentiable function with bounded derivative. **Exercise 2.41A** Prove that there exists a set $E = \{r_n : n \in \mathbb{N}\}$ dense in $\mathbb{R}$ with $|r_n| \leq n$ for every $n$. Define $u$ as in Exercise 1.40 with now $x \in \mathbb{R}$ and prove that $u : \mathbb{R} \to \mathbb{R}$ is still continuous, strictly increasing and has a derivative at every point, with $u' = \infty$ in $E$. Let $v$ be the inverse function of $u$, let $Z := \{x \in \mathbb{R} : v'(x) = 0\}$ and let $D \subset \mathbb{R} \setminus Z$ be a a countable set dense in $\mathbb{R}$. Prove that the set

$$G := \bigcap_{x \in D} ((-x + Z) \cap (x - Z))$$

is a $G_\delta$. Let $t \in G_\delta$ and define $w(x) := v(x+t) - v(x)$, $x \in \mathbb{R}$. Prove that $w$ is nowhere monotone.

## CHAPTER 3:

p. 73 **Proof of Theorem 2.40, continued.** Another way to prove that $u'$ is not Riemann integrable over any interval $[a, b]$ is to reason by contradiction, assume that it is, and use the fundamental theorem of calculus for Riemann integrals and the following exercise to conclude that $u$ must be constant. **Exercise 3.15** Prove that if $v : [a, b] \to \mathbb{R}$ is a Riemann integrable function which is zero $\mathcal{L}^1$-a.e. in $[a, b]$, then the Riemann integral $\int_a^b v(x)\, dx = 0$.

p. 83 I would like to thank Max Lipton and Xinrui Zhao for providing a simple solution to Exercise 3.32(ii), please see the file Lipton-Zhao-exercise-3-32.pdf. For a different example, see also the file exercise-3-32.pdf.

## CHAPTER 5:

p. 153 **Lemma 5.48.** While the statement of the lemma is correct, it is much simpler to assume in the statement that $\mathbb{R}^2 \setminus \Gamma$ is not connected (as in the original paper of Maheara), which is in the form in which the lemma is used in the theorem anyway. In the proof of the lemma the sentence "since $\partial U \subset \Gamma$, there must be at least another connected component" is far from trivial (thanks to Zach Sussman for pointing this out). To prove this fact one could proceed as follows. Indeed, let $x_0 \in \Gamma \setminus \partial U$. Then there exists $B(x_0, r) \subset \mathbb{R}^2 \setminus \overline{U}$, since otherwise taking $r = \frac{1}{n}$ we could find a sequence $\{x_n\}_n$ of points in $\overline{U}$ with $\|x_n - x_0\| < \frac{1}{n}$. Since $\overline{U}$ is closed, this would imply that $x_0 \in \overline{U}$, which is a contradiction. Take $0 < \delta < r$. Then $\overline{B(x_0, \delta)} \subseteq \mathbb{R}^2 \setminus \overline{U}$. So now there are two cases. **Case 1**: There is a point $x_1 \in \overline{B(x_0, \delta)}$ which does not belong to $\Gamma$, which means that $x_1 \in \mathbb{R}^2 \setminus (\Gamma \cup U)$. Hence, there is another connected component $V$ of $\mathbb{R}^2 \setminus \Gamma$. **Case 2**: $\overline{B(x_0, \delta)} \subset \Gamma$. Since $\Gamma$ is the range of a continuous closed simple curve $\gamma$ in $\mathbb{R}^2$, we can find $\varphi : [a, b] \to \mathbb{R}^2$ continuous and injective with $\overline{B(x_0, \delta)} \subseteq \varphi([a, b])$. Then $\varphi^{-1} : \varphi([a, b]) \to [a, b]$ is continuous and so $\varphi^{-1}(\overline{B(x_0, \delta)})$ is a connected compact set of $[a, b]$, namely a closed interval $[c, d]$. Removing an interior point $e$ in $[c, d]$ would disconnect $[c, d]$, while removing a point in $\overline{B(x_0, \delta)}$ would not. This is a contradiction. So Case 2 cannot happen.

## CHAPTER 6:

p. 168 In equation (6.22) $\|\lambda\|(E) = \{\sum_{i=1}^n \|\lambda(E_i)\|\}$ should be replaced by $\|\lambda\|(E) = \sup \{\sum_{i=1}^n \|\lambda(E_i)\|\}$ . [Sebastián Muñoz Thon]

## CHAPTER 7:

p. 192 In Corollary 7.20 Let $\Omega \subseteq \mathbb{R}$ be an open bounded set should be replaced by Let $\Omega \subset \mathbb{R}$ be an open bounded set with finitely many components . [Thomas Horstkamp]

p. 193 In Corollary 7.21 Let $\Omega \subseteq \mathbb{R}$ be an open bounded set should be replaced by Let $\Omega \subset \mathbb{R}$ be an open bounded set with finitely many components . Also, to prove that $\bar{u}$ belongs to $L^p(\Omega)$, one needs Poincare's inequality, so this corollary should be moved after Theorem 7.29. [Thomas Horstkamp]

p. 197 In Step 1 in the proof of Theorem 7.34, $I_x := I \cap [x - \ell/2, x + \ell/2]$ should be replaced by $I_x$ be an interval of length $\ell$ containing $x$ and contained in $I$ . Alternatively one should replace $\ell$ with $\frac{\ell}{2}$ in (7.9). [Andrea Braides]

p. 197 In Step 1 in the proof of Theorem 7.34, $\min_{I_x}$ should be replaced by $\min_{\overline{I_x}}$ . [Andrew Chen]

## CHAPTER 8:

p. 206 In Exercise 8.2, weakly measurable should be replaced by strongly measurable

p. 220-222 In the statements and proofs of Theorems 8.29, 8.33 and in Exercise 8.37, $BV_{\text{loc}}$ should be replaced by $BPV_{\text{loc}}$

## CHAPTER 9:

p. 241 In the statement of Theorem 9.6, Schwartz should be replaced by Schwarz. [Andrea Braides]

p. 275 In Step 2 assume that $\max_{\overline{R'}} f < b_N$. The rectangle $R := R' \times (0, b_N)$ should be replaced by $R := R' \times (0, c_N)$, where $c_N > \max_{\overline{R'}}(b_N - f)$ .

p. 277 In Step 3 assume that $\max_{\overline{R'}} f < b_N$.

p. 278 One line above (9.54), observe that if $\psi_k$ is zero outside some $Q_j$ and $\Omega \cap Q_j$ satisfies (9.53), we can modify the function $f_j$ outside $Q'(0, r_j)$ in such a way that $\max_{\overline{Q'(0,2r_j)}} f < 2r_j$ and then apply Step 3 in $\{y \in Q(0, 2r_j) : y_N > f_j(y')\}$.

## CHAPTER 10:

p. 290 In Definition 10.14 on the right-hand side of the formula in display $(-1)^{|\alpha|} \left(\frac{\partial^{|\alpha|}\phi}{\partial x^\alpha}\right)$ should be replaced by $(-1)^{|\alpha|} T \left(\frac{\partial^{|\alpha|}\phi}{\partial x^\alpha}\right)$ . [Andrea Braides]

p. 299 Before the statement of Theorem 10.35, one should add that the space $\mathcal{S}'(\mathbb{R}^N)$ is endowed with the weak star topology. [Noah Stevenson]

p. 301 In Example 10.38, the lines

$$\leq 2c_0\|\phi\|_\infty + \sum_{n=2}^{\infty} \int_{B(0,n)} \frac{(1+\|x\|)^{2k}}{(1+\|x\|)^{2k}} \|\phi\| \, d\mu$$
$$\leq 2c_0\|\phi\|_\infty + cc_0\|\phi\|_{0,2k} \sum_{n=1}^{\infty} \frac{(1+n)^k}{(1+n)^{2k}}.$$

should be replaced by

$$\leq 2c_0\|\phi\|_\infty + \sum_{n=2}^{\infty} \int_{B(0,n)\setminus B(0,n-1)} \frac{(1+\|x\|)^{3k}}{(1+\|x\|)^{3k}} \|\phi\| \, d\mu$$
$$\leq 2c_0\|\phi\|_\infty + cc_0\|\phi\|_{3k,0} \sum_{n=1}^{\infty} \frac{(1+n)^k}{n^{3k}}.$$

p. 305 In the first formula in display on the right-hand side $\int_{\mathbb{R}^N} \psi(x)\phi(x)\,dx$ should be replaced by $\int_{\mathbb{R}^N} u(x)\phi(x)\,dx$ . [Andrea Braides]

p. 309 In formula (10.25), and in the line below and on the last line, $x^\alpha$ should be replaced by $(2\pi i x)^\alpha$ .

**CHAPTER 11:**

p. 331 In the second to last line $v_n := \phi_n u$ should be replaced by $v_n := \psi_n u$ . [Andrea Braides]

p. 332 Line 2, define the function $u_{n,t} : U_n^t \to \mathbb{R}$ by $u_{n,t}(y', y_N) := u_n(y', y_N + t)$, where should be replaced by define the function $\bar{u}_{n,t} : U_n^t \to \mathbb{R}$ by $\bar{u}_{n,t}(y', y_N) := \bar{u}_n(y', y_N + t)$, where $\bar{u}_n$ is the function $u_n$ written in local coordinates $y$. Then after the definition of the set $U_n^t$ one should add Let $u_{n,t}$ be the function $\bar{u}_{n,t}$ written in background coordinates $x$.

p.332 One line above formula (11.14), $\operatorname{supp}\psi_n + B(0,t_n) \Subset U_n^{t_n}$ should be replaced by $\operatorname{supp}\psi_n + B(0,t_n) \Subset B(x_n, r_{x_n})$.

p. 332 One line below formula (11.14), such that $\phi_n(y_N) = 1$ if $y_N > f(y') - t_n/2$ and $\phi_n(y_N) = 0$ if $y_N \leq f(y') - 3t_n/4$ should be replaced by such that in local coordinates $y$, $\bar{\phi}_n(y_N) = 1$ if $y_N > f(y') - t_n/2$ and $\bar{\phi}_n(y_N) = 0$ if $y_N \leq f(y') - 3t_n/4$ .

p.346 One line below formula (11.35), for all $\phi \in \mathcal{D}(\mathbb{R}^N)$ should be removed. [Andrea Braides]

**CHAPTER 12:**

p. 360 An alternative way to prove Step 5 is to skip Step 4 and use the density Theorem 11.34 to assume that $u \in C_c^\infty(\mathbb{R}^N)$.

p. 375 Line 2, centered at $k$ should be replaced by centered at $z$ . [Junichi Koganemaru]

p. 403 In the statement of Theorem 12.87 in the list of exceptional cases, one should add (iii) If $q = r = \infty$ and $1 < p < \infty$, then (12.66) fails for $\theta \in (0, 1)$.

p.404 In the statement of Proposition 12.88, $\nu < \infty$ should be replaced by $\nu \leq 1$ [Joseph Kalarickal]

## CHAPTER 13:

p. 412 In inequality (13.2), $\|\partial_N \mathcal{E}(u)\|_{L^p(\mathbb{R}^N)} = \|\partial_N u\|_{L^p(\Omega)}$ should be replaced by $\|\partial_N \mathcal{E}(u)\|_{L^p(\mathbb{R}^N)} = 2\|\partial_N u\|_{L^p(\Omega)}$ . [Andrea Braides]

p. 418 Line 2 from below, $x_t$ should be replaced by $g^t(x)$ . [Junichi Koganemaru]

p. 425 In formula (13.43), $\phi_n := \varphi_{\varepsilon/4} * \chi_{\Omega_n^{3\varepsilon/4}}$ should be replaced by $\phi_n := \varphi_{\varepsilon/8} * \chi_{\Omega_n^{3\varepsilon/8}}$ . In turn, in formula (13.45) and in the displayed formula just above that $\partial^\alpha \varphi_{\varepsilon/4} * \chi_{\Omega_n^{3\varepsilon/4}}$ should be replaced by $\partial^\alpha \varphi_{\varepsilon/8} * \chi_{\Omega_n^{3\varepsilon/8}}$ . [Enzo Vitillaro]

p. 432-433 In Exercise 13.26, Theorem 13.27, and Remark 13.28, when $m \geq 2$ in addition to $\Omega$ having finite measure one should also assume that

$$\int_\Omega \|x\|^{m-1} dx < \infty.$$

p. 431 Line 7 $|\alpha| < m - 1$ should be replaced by $|\alpha| \leq m - 1$ . [Junichi Koganemaru]

p. 432 In Theorem 13.27 and all the following theorems in this section, the sentence "In particular, if $m = 1$" is a bit misleading in the sense that I am not claiming that when $m = 1$ the constant polynomial $p_E(u)$ coincides with $u_E$, but only that in view of (3.61), the function $u - u_E$ has the property that $p_E(u - u_E) = 0$.

p. 431 Line 10 $\leq \cdots \leq \frac{d^{p(m-|\alpha|)}}{p^{m-|\alpha|}} \int_\Omega |\partial^{\alpha+e_N} u(x)|^p dx$ should be replaced by $\leq \cdots \leq \frac{d^{p(m-|\alpha|)}}{p^{m-|\alpha|}} \int_\Omega |\partial^{\alpha+(m-|\alpha|)e_N} u(x)|^p dx$ . [Junichi Koganemaru]

p. 433 Line 14 $\|\partial^\beta v_n\|_{L^p(\Omega)}$ should be replaced by $\|\partial^\beta v_{n_k}\|_{L^p(\Omega)}$ . [Junichi Koganemaru]

p. 448 Line 7. The inequality

$$\|u - u_{B_n}\|_{L^p(\Omega_n)} \le \|u - u_{\Omega_n}\|_{L^p(\Omega_n)} + |u_{\Omega_n} - u_{B_n}|(\mathcal{L}^N(\Omega_n))^{1/p} \\ \le 2\|u - u_{\Omega_n}\|_{L^p(\Omega_n)} \le 2c_n\|\nabla u\|_{L^p(\Omega_n)}.$$

should be replaced by

$$\|u - u_{B_n}\|_{L^p(\Omega_n)} \le \|u - u_{\Omega_n}\|_{L^p(\Omega_n)} + |u_{\Omega_n} - u_{B_n}|(\mathcal{L}^N(\Omega_n))^{1/p} \\ \le \left(1 + (\mathcal{L}^N(\Omega_n)/\mathcal{L}^N(B_n))^{1/p}\right)\|u - u_{\Omega_n}\|_{L^p(\Omega_n)} \\ \le 2(\mathcal{L}^N(\Omega_n)/\mathcal{L}^N(B_n))^{1/p}c_n\|\nabla u\|_{L^p(\Omega_n)}.$$

p. 448 Remark 13.49. The constant

$$c_* := 2\sum_{n=1}^{\ell} c_n + c_1(m,n,p)\sum_{n=1}^{\ell}\sum_{k=1}^{\ell_n-1} R_{k,n}(R_{k,n}/r_{k,n})^{(N-1)/p} \\ \times (\mathcal{L}^N(\Omega_n)/\mathcal{L}^N(B^*_{k,n}))^{1/p}$$

should be replaced by

$$c_* := 2\sum_{n=1}^{\ell} c_n \times (\mathcal{L}^N(\Omega_n)/\mathcal{L}^N(B_n))^{1/p} + c_1(m,n,p)\sum_{n=1}^{\ell}\sum_{k=1}^{\ell_n-1} R_{k,n}(R_{k,n}/r_{k,n})^{(N-1)/p} \\ \times (\mathcal{L}^N(\Omega_n)/\mathcal{L}^N(B^*_{k,n}))^{1/p}$$

## CHAPTER 16:

p. 518 In Exercise 16.4, $t^{-1}K(x,t) = K(x,t^{-1})$ should be replaced by $t^{-1}K(x,t;X_0,X_1) = K(x,t^{-1};X_1,X_0)$ . [Noah Stevenson]

## CHAPTER 18:

p. 599 In Exercise 18.11 $u, v \in W^{1,1}(\mathbb{R}^N_+)$ should be replaced by $u, v \in W^{1,1}(\mathbb{R}^N_+) \cap L^\infty(\mathbb{R}^N_+)$ . [Junichi Koganemaru]

p. 615 At some point I convinced myself that the inequality

$$|w|^\diamond_{B^{s,p}(\partial\Omega)} \le (2M)^{1/p'}\left(\sum_{n=1}^{\infty}(|w_n|^\diamond_{B^{s,p}(\partial\Omega)})^p\right)^{1/p}$$

in Exercise 18.39 was false, so let me prove it before I forget again. For every $x \in \partial\Omega$, let $I_x := \{n \in \mathbb{N} : w_n(x) \ne 0\}$. Note that the cardinality of $I_x$ is at most $M$. Given $x, y \in \partial\Omega$, we can write

$$w(x) - w(y) = \sum_{n \in I_x \cup I_y}(w_n(x) - w_n(y)).$$

By the discrete Hölder's inequality,

$$|w(x)-w(y)| \le \left(\sum_{n\in I_x\cup I_y} 1^{p'}\right)^{1/p'} \left(\sum_{n\in I_x\cup I_y} |w_n(x)-w_n(y)|^p\right)^{1/p}$$
$$\le (2M)^{1/p'} \left(\sum_{n\in I_x\cup I_y} |w_n(x)-w_n(y)|^p\right)^{1/p}.$$

Hence,

$$|w(x)-w(y)|^p \le (2M)^{p-1}\sum_{n=1}^{\infty} |w_n(x)-w_n(y)|^p,$$

and so

$$\int_{\partial\Omega}\int_{\partial\Omega\cap B(x,\varepsilon)} \frac{|w(x)-w(y)|^p}{\|x-y\|^{N-1+sp}} d\mathcal{H}^{N-1}(y)d\mathcal{H}^{N-1}(x)$$
$$\le (2M)^{p-1}\sum_{n=1}^{\infty}\int_{\partial\Omega}\int_{\partial\Omega\cap B(x,\varepsilon)} \frac{|w_n(x)-w(y)|^p}{\|x-y\|^{N-1+sp}} d\mathcal{H}^{N-1}(y)d\mathcal{H}^{N-1}(x),$$

which shows that

$$|w|^{\diamond}_{B^{s,p}(\partial\Omega)} \le (2M)^{1/p'}\left(\sum_{n=1}^{\infty}(|w_n|^{\diamond}_{B^{s,p}(\partial\Omega)})^p\right)^{1/p}.$$

p. 617 In line 12 $\phi_n := \varphi_{\varepsilon/4} * \chi_{\Omega_n^{3\varepsilon/4}}$ should be replaced by $\phi_n := \varphi_{\varepsilon/8} * \chi_{\Omega_n^{3\varepsilon/8}}$ . [Enzo Vitillaro]

p. 628 One line above formula (18.58), $\eta \in \mathcal{S}(\mathbb{R})$ and $\psi \in \mathcal{S}(\mathbb{R}^{N-1})$ should be replaced by $\eta \in \mathcal{S}(\mathbb{R}^{N-1})$ and $\psi \in \mathcal{S}(\mathbb{R})$.

## APPENDIX B:

p. 675 In Exercise B.105 the measure $\mu$ should be finite. [Ian Tice]

## APPENDIX C:

p. 693 In Exercise C.22 assume that $\Omega$ is bounded and its closure is contained in the union of the $U_\alpha$. [Andrea Braides]

p. 698 In the statement of Hardy's inequality, where $0 \le a \le b \le \infty$ should be removed.

**More to come for sure... :(**