Errata for **Geometric Relativity** by Dan A. Lee, as of November 19, 2021.

- In Exercise 1.15 on page 14, the coefficient of r^2 in the given formula should be $\frac{1}{6(n+2)}R(p)$ instead of $\frac{6}{(n+2)}R(p)$.
- On page 86, in the second displayed equation, the $=$ should be $\leq$.
- Exercise 3.49 on page 102 is false, and as a result, the entire proof of Lemma 3.48 is incorrect. Since this is a substantial error, we include a complete proof of Lemma 3.48 at the end of these Errata. We thank Ryan Unger for pointing out this error, as well as some others.
- In Definition 7.17, on page 225, the required decay rate for k should be $-q-1$ rather than $-q$. That is,

$$k_{ij} = O_1(|x|^{-q-1}).$$

- On page 231, there is a missing right parenthesis in the first line of the group of equations at the bottom of the page.
- In the second sentence of the second paragraph on page 240, there is a θ_+ which should be θ^+.
- In the second equation of Definition 7.35 on page 243, the ∇u should be $\nabla\varphi$.
- In Proposition 9.4, on the fourth line of page 287, the sign on $\langle V, \operatorname{div}\pi\rangle_g$ should be a plus sign rather than a minus sign. Furthermore, there is a factor of $\frac{1}{2}$ missing in the definition of $a \odot b$. It should be

$$(a \odot b)^{ij} = \tfrac{1}{2}(a^i b^j + a^j b^i).$$

- On page 288, the formula $\tilde{\pi} := u^{s/2}(\pi + \mathfrak{L}_g Y)$ should actually be

$$\tilde{\pi}_{ij} := u^{s/2}(\pi_{ij} + (\mathfrak{L}_g Y)_{ij}),$$

  or in other words, the formula $\tilde{\pi} := u^{s/2}(\pi + \mathfrak{L}_g Y)$ is only correct if one thinks of π as a $(0,2)$ tensor, contrary to the fact that we defined π as a $(2,0)$ tensor on page 223 and treat it as a $(2,0)$ tensor throughout Chapter 9. An equivalent way to write the correct formula is

$$\tilde{\pi}^{ij} := u^{-3s/2}(\pi^{ij} + (\mathfrak{L}_g Y)^{ij}),$$

  or in other words, since π is a $(2,0)$ tensor, we can write this as

$$\tilde{\pi} := u^{-3s/2}(\pi + \mathfrak{L}_g Y).$$

  Unfortunately, this error repeats throughout Chapter 9, and the corresponding correction should be made on pages 289, 293–295, and 298–299.

  Here we would also like to note that the *reason* for including a power of the conformal factor u in the formula for $\tilde{\pi}$ is for natural scaling purposes. However, all of the *analysis* in Chapter 9 works just as well regardless of what power of u is used. In particular, we *could have* defined $\tilde{\pi} := \pi + \mathfrak{L}_g Y$, and all of the arguments would work just as well, but the details of various formulas would be different.
- In Exercise 9.7 on page 288, the second part of the formula for $T(u, Y)$, corresponding to $\tilde{J}$, should be

$$\tilde{J}^i = u^{-3s/2}\left[(\operatorname{div}_g \mathfrak{L}_g Y + \operatorname{div}_g \pi)^i + \frac{s}{2}(n-1)(\pi + \mathfrak{L}_g Y)^{ij}u^{-1}u_{,j} - \frac{s}{2}\operatorname{tr}_g(\pi + \mathfrak{L}_g Y)g^{ij}u^{-1}u_{,j}\right],$$

and consequently, the second part of the formula for $DT|_{(1,0)}(v, Z)$, corresponding to the linearization of $\tilde{J}$, should be

$$(\operatorname{div}_g \mathfrak{L}_g Z)^i + \frac{s}{2}(n-1)\pi^{ij}v_{,j} - \frac{s}{2}(\operatorname{tr}_g \pi)\nabla^i v - \frac{3s}{2}J^i v.$$

The formula for $\tilde{J}$ above is used in the proof of Lemma 9.8 on page 289, so the second equation appearing in that proof should be

$$u^{3s/2}J^i = \overline{\Delta}Y^i + \frac{s}{2}(n-1)(\overline{\mathfrak{L}}Y)^{ij}u^{-1}u_{,j} - \frac{s}{2}\overline{\operatorname{tr}}(\overline{\mathfrak{L}}Y)u^{-1}\partial^i u.$$

- On page 290, the two formulas for $\operatorname{Hess}\xi$ should include $\pi * \nabla V$ terms, but after taking the trace, the $\Delta_g \xi$ will have the form described in equation (9.1), which is what is used on page 291.
- On page 294, in the second paragraph, where it says $D\Phi|_{g,\pi}(K_2)$, it should say $D\Phi|_{(g,\pi)}(K_2)$.
- In equation (9.4) on page 298, there should be a factor of 2 in $t(f + |J|_g)$, because of the factor of 2 in the definition of Φ.
- On page 336, in the first paragraph of the proof of Theorem A.40, it is implicitly assumed, without loss of generality, that the sequence u_i is bounded in $C^{2,\alpha}_s$ in order for the argument to be correct. We leave it as an exercise to prove this. This proof uses the fact that Δ_g has finite dimensional kernel, which we also leave as an exercise.

Lemma 3.48 (Density lemma for nonnegative scalar curvature)**.** *Let (M^n, g) be a complete asymptotically flat manifold with nonnegative scalar curvature. Let $p > n/2$ and $q < n-2$ such that q is less than the asymptotic decay rate of g in Definition 3.5. Then for any $\epsilon > 0$, there exists a complete asymptotically flat metric $\tilde{g}$ with nonnegative scalar curvature on M that is harmonically flat outside a compact set, such that $\|\tilde{g} - g\|_{W^{2,p}_{-q}} < \epsilon$ and $\|R_{\tilde{g}} - R_g\|_{L^1} < \epsilon$.*

Note that if we take $p > n$ and $q > \frac{n-2}{2}$, then Lemma 3.35 tells us that in the conclusion of Lemma , we can also demand that

$$|m_{\mathrm{ADM}}(\tilde{g}) - m_{\mathrm{ADM}}(g)| < \epsilon.$$

Proof. The proof starts off the same way as in Lemma 3.34, with the same definition of g_λ, which converges to g in $W^{2,p}_{-q}$. The difference is that instead of trying to make a conformal factor that deforms g_λ to a scalar-flat metric (which would result in a large deformation since g is not already scalar-flat), we make a conformal change to return the scalar curvature to something close to the original R_g but with compact support.

We *could* attempt to prescribe the scalar curvature *exactly*, but that would require solving a nonlinear equation: One way to do this is to use use the inverse function theorem, but it is not clear whether the relevant linearization is an isomorphism. This problem can be circumvented using surjectivity of the linearized scalar curvature operator (Corollary 6.11), at the cost of making a deformation that is no longer conformal. This is indeed the approach that we take in Chapter 9, where we prove density theorems for initial data sets, but for this proof of Lemma 3.48, we can use a simpler trick that requires only the linear theory.

Given a conformal factor u_λ, if we define $\tilde{g}_\lambda = u_\lambda^{\frac{4}{n-2}} g_\lambda$, then equation (1.8) tells us that $R_{\tilde{g}_\lambda} = u_\lambda^{-\frac{n+2}{n-2}} L_{g_\lambda} u_\lambda$. Rather than prescribing $R_{\tilde{g}_\lambda}$, we instead prescribe

$L_{g_\lambda} u_\lambda$. This turns out to be good enough because in the limit as $\lambda \to \infty$, we expect u_λ to converge to 1 uniformly. Explicitly, we attempt to solve

$$-\tfrac{4(n-1)}{n-2}\Delta_{g_\lambda} u_\lambda + R_{g_\lambda} u_\lambda = \chi_\lambda R_g \tag{1}$$

with $u_\lambda(\infty) = 1$, where χ_λ is the cutoff function described in the proof of Lemma 3.34. As we have done before, set $v_\lambda = u_\lambda - 1$ so that this is equivalent to

$$-\tfrac{4(n-1)}{n-2}\Delta_{g_\lambda} v_\lambda + R_{g_\lambda} v_\lambda = \chi_\lambda R_g - R_{g_\lambda}$$

with $v_\lambda(\infty) = 0$. Next, we assume $q > 0$ without loss of generality and then show that

$$-\tfrac{4(n-1)}{n-2}\Delta_{g_\lambda} + R_{g_\lambda} : W^{2,p}_{-q} \longrightarrow L^p_{-q-2} \tag{2}$$

is an isomorphism. By the construction and asymptotic flatness of g, it is not hard to see that $R_{g_\lambda} \to R_g$ in L^∞_{-2}. This is good enough to see that the operator (2) is close enough to $-\frac{4(n-1)}{n-2}\Delta_g + R_g$ (which is an isomorphism as explained in the proof of Lemma 3.31) to be an isomorphism for large enough λ, with a uniform injectivity estimate independent of λ.

Hence, the desired solution u_λ exists, and it is smooth by elliptic regularity. Next observe that both $\chi_\lambda R_g$ and R_{g_λ} converge to R_g in L^p_{-q-2} as $\lambda \to \infty$, and thus their difference converges to zero in L^p_{-q-2}. By our definition of v_λ and the uniform injectivity estimate on the operators (2), we see that $v_\lambda \to 0$ in $W^{2,p}_{-q}$, and hence $u_\lambda \to 1$ in $W^{2,p}_{-q} \subset C^0_{-q}$, and thus $u_\lambda > 0$ for large λ. So $\tilde{g}_\lambda$ is a well-defined asymptotically flat metric, and we can see that $\tilde{g}_\lambda \to g$ in $W^{2,p}_{-q}$. Moreover, since u_λ is harmonic outside a compact set, we see that $\tilde{g}_\lambda$ is harmonically flat outside a compact set. Finally, from our construction, we have $R_{\tilde{g}_\lambda} = u_\lambda^{-\frac{n+2}{n-2}} \chi_\lambda R_g$. This is clearly nonnegative, and the only thing thing left to verify is that $R_{\tilde{g}_\lambda} \to R_g$ in L^1. Since $u_\lambda \to 1$ uniformly, this reduces to observing that $\chi_\lambda R_g \to R_g$ in L^1, which is a simple consequence of R_g being integrable. □

The earliest proof of Lemma 3.48 in the literature seems to be in [**LP87**], where it appears as part of the proof of Lemma 10.6. The proof given here is similar to the one there except that instead of solving (1), they use $\chi_\lambda R_g u_\lambda$ as the right-hand side.