ULTRAFILTERS THROUGHOUT MATHEMATICS
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ERRATA AND UPDATES

- In the proof of Theorem 7.3.1, it should read that $[t_R]^n_{\mathcal{U}}, [t_S]^n_{\mathcal{U}} = 0$ and $[t_R]^{n-1}_{\mathcal{U}}, [t_R]^{n-1}_{\mathcal{U}} \neq 0$.
- In the statement of Corollary 7.3.17, the polynomials should satisfy the same degree constraint as in Fact 7.3.16.
- The statements of Theorem 7.3.20, Conjecture 7.3.21, Theorem 7.3.22, and Corollary 7.3.23 should all be about *nontrivial* zeros; the polynomials in question, all being homogeneous, always have the trivial (zero) solution.
- In Theorem 7.3.22 and the statement preceding it, $\mathbb{F}_p[T]$ should be replaced by $\mathbb{F}_p[[T]]$.
- The conclusion of Proposition 8.1.13 should read: $\mathcal{M}$ is κ^+-saturated if and only if: for every $A \subseteq M$ with $|A| \leqslant \kappa$, $\mathcal{M}_A$ is κ^+-universal. Without this reformulation, the proof of Theorem 8.4.17 (in particular, the implication (3) implies (4)) is unclear.
- In the proof of Theorem 8.3.9, one should replace any mention of the set I by the cardinal κ.
- In Section 8.4, in the paragraph introducing the notion of an anti-monotonic function, it is stated that the concern function C satisfies $C(v) \subseteq C(u)$ when $u \subseteq v$. This does not technically follow from the conditions imposed on C earlier in the section but rather follows from the heuristic that $C(u)$ captures those indices $i \in I$ which should be concerned about satisfying the formulae in u. Consequently, if i is concerned with satisfying the formulae in the larger set v, then it is also concerned with satisfying the formulae in the smaller set u.
- After Definition 8.4.4, it is incorrectly stated that every refinement of a distribution is once again a distribution. The correct statement is that every anti-monotonic refinement of a distribution is once again a distribution.

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- In the proof of Theorem 8.4.16, the partial types $\Gamma_i(y_j)$ should demand that the witnesses x to the existential statements be *nonempty*. More precisely, the formulae in the first portion of Γ_i should read

$$\exists x \left(\bigwedge_{l \leqslant k} x \subseteq y_{j_l} \wedge \exists z (z \subseteq x \wedge z \neq x) \right)$$

and similary for the formulae in the second portion of Γ_i. Likewise, in the definition of $\varphi_j(x)$, one should add a conjunct stating that x is nonempty.
- In the proof of Corollary 8.4.21, the assertion that $\mathcal{U} \times \mathcal{V}$ is not $\aleph_2$-good is true but not entirely justified based on earlier material. Indeed, the intended proof was to use Exercise 8.4.20; however, this exercise does not apply since no ultrafilter on ω can be $\aleph_1$-regular. Instead, one needs to use the following elementary fact: if $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters on sets I and J respectively for which $\mathcal{U} \times \mathcal{V}$ is κ^+-good, then $\mathcal{V}$ is also κ^+-good. To see this, suppose that $\mathcal{U} \times \mathcal{V}$ is κ^+-good and fix an anti-monotonic function $f : \mathcal{P}_f(\kappa) \to \mathcal{V}$; we seek a multiplicative refinement for f. Define $f' : \mathcal{P}_f(\kappa) \to \mathcal{P}(I \times J)$ by $f'(u) = I \times f(u)$. Note that $f'(u) \in \mathcal{U} \times \mathcal{V}$ since $f(u) \in \mathcal{V}$ and f' is anti-monotonic since f is anti-monotonic. Since $\mathcal{U} \times \mathcal{V}$ is κ^+-good, there is a multiplicative $g' : \mathcal{P}_f(\kappa) \to \mathcal{U} \times \mathcal{V}$ refining f'. We define $g : \mathcal{P}_f(\kappa) \to \mathcal{P}(J)$ by setting $g(u) := \{j \in J \ : \ g'(u)_j \in \mathcal{U}\}$. Since g' takes values in $\mathcal{U} \times \mathcal{V}$, we see that g takes values in $\mathcal{V}$. Also, if $j \in g(u)$, then $g'(u)_j \in \mathcal{U}$, whence $f'(u)_j \in \mathcal{U}$; in particular, $f'(u)_j \neq \emptyset$, whence $j \in f(u)$. It remains to see that g is multiplicative. To see this, note that

$$j \in g(u \cup v) \Leftrightarrow g'(u \cup v)_j \in \mathcal{U} \Leftrightarrow g'(u)_j \cap g'(v)_j \in \mathcal{U} \Leftrightarrow j \in g(u) \cap g(v).$$

- In the statement of Corollary 8.4.23, the assumption that $\mathcal{U}$ is countably incomplete is missing.
- In the proof of Proposition 8.6.10, one should replace any mention of the set I by the cardinal κ.
- In the statement of Theorem 8.6.12, the conclusion should be "$\mathcal{M}^{\mathcal{U}}$ is λ-saturated", not "$\mathcal{M}$ is λ-saturated."
- In the proof of Theorem 8.6.12, in the statement "$\mathcal{M} \models \varphi_{\Delta,n}(a_\alpha)$ if and only if w_α is a Δ-n-indiscernible set" $\mathcal{M}$ should be replaced with $\mathcal{M}'$.
- In Question 9.8.4, it was asked if one can prove in ZFC that Hausdorff ultrafilters exist. The question is now known to have a negative answer as shown by J. Cancino-Manríquez in *Every maximal ideal may be Katetov above of all F_σ ideals*, Transactions of the American Mathematical Society, Volume 375 (2022), 1861-1881.
- In Exercise 15.3.8, the sequence $(a_n)_{n \in \mathbb{N}}$ should be assumed to come from $M^{\mathcal{U}}$.

- In the proof of Theorem 15.3.1, the reference to Upward Löwenheim-Skolem should be to Downward Löwenheim-Skolem.
- In the proof of Theorem 15.3.1, the B_1-independent set D in the second paragraph should be assumed to consist of realizations of the type p.
- In the notes for Chapter 15, Theorem 15.2.14 is correctly attributed to Alan Dow but the wrong paper of Dow's is referenced. Instead, it should refer to A. S. Dow, *On ultrapowers of Boolean algebras*, in Topology Proc. 9 (1984), no. 2, 269–291; MR0828984. (Thanks to K.P. Hart for pointing this out in his MathSciNet review of the book.)
- In Appendix A, in the line directly after Definition A.1.1, "authers" should be "authors."