CHAPTER 17

Sobolev Spaces

1. Absolutely Continuous Functions

In this setion we review absolute continuous functions.

Definition 17.1. *Let* $I \subseteq \mathbb{R}$ *be an interval and* (Y, d) *be a metric space. A function* $u : I \to Y$ *is said to be* absolutely continuous *on* I *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\sum_{i=1}^{n} d(u(b_i), u(a_i)) \leq \varepsilon \tag{17.1}$$

for every finite number of nonoverlapping intervals (a_i, b_i), $i = 1, \ldots, n$, *with* $[a_i, b_i] \subseteq I$ *and*

$$\sum_{i=1}^{n} (b_i - a_i) \leq \delta.$$

The space of all absolutely continuous functions $u : I \to Y$ *is denoted by* $AC(I; Y)$.

When $Y = \mathbb{R}$ we write $AC(I)$ for $AC(I; \mathbb{R})$..

Let $I \subseteq \mathbb{R}$ be an interval and (Y, d) a metric space. A function $u : I \to Y$ is *locally absolutely continuous* if it is absolutely continuous in $[a, b]$ for every interval $[a, b] \subseteq I$. The space of all locally absolutely continuous functions $u : I \to Y$ is denoted by $AC_{\text{loc}}(I; Y)$. As before, when $Y = \mathbb{R}$ we write $AC_{\text{loc}}(I)$ for $AC_{\text{loc}}(I; \mathbb{R})$. Note that $AC_{\text{loc}}([a, b]; Y) = AC([a, b]; Y)$.

Exercise 17.2. *Let* $u, v \in AC([a, b])$. *Prove the following.*

(i) $u \pm v \in AC([a, b])$.
(ii) $uv \in AC([a, b])$.
(iii) *If* $v(x) > 0$ *for all* $x \in [a, b]$, *then* $u/v \in AC([a, b])$.
(iv) $\max\{u, v\}$, $\min\{u, v\} \in AC([a, b])$.

Proposition 17.3. *Let* $I \subseteq \mathbb{R}$ *be an interval and* $u \in AC_{\text{loc}}(I)$. *Then* u *is differentiable* $\mathcal{L}^1$*-a.e. in* I *and* u' *is locally Lebesgue integrable.*

Theorem 17.4. *Let* $I \subseteq \mathbb{R}$ *be an open interval and* $u \in AC_{\text{loc}}(I)$ *be such that there exists* $u'(x) = 0$ *for* $\mathcal{L}^1$*-a.e.* $x \in I$. *Then* u *is constant.*

The next theorem shows the primitive of an integrable function is absolutely continuous.

Theorem 17.5. *Let* $I \subseteq \mathbb{R}$ *be an interval and* $v : I \to \mathbb{R}$ *a Lebesgue integrable function. Fix* $x_0 \in I$ *and let*

$$u(x) := \int_{x_0}^{x} v(t)\, dt, \quad x \in I.$$

Then the function u is absolutely continuous in I and $u'(x) = v(x)$ for $\mathcal{L}^1$-a.e. $x \in I$.

Using the previous theorem we have.

THEOREM 17.6 (Fundamental theorem of calculus). *Let $I \subseteq \mathbb{R}$ be an interval. A function $u : I \to \mathbb{R}^M$ belongs to $AC_{\text{loc}}(I)$ if and only if*

(i) *u is continuous in I,*
(ii) *u is differentiable $\mathcal{L}^1$-a.e. in I, and u' belongs to $L^1_{\text{loc}}(I)$,*
(iii) *the fundamental theorem of calculus is valid; that is, for all $x, x_0 \in I$,*

$$u(x) = u(x_0) + \int_{x_0}^{x} u'(t)\,dt.$$

As a corollary of Theorem **??** we recover the formula for integration by parts.

COROLLARY 17.7 (Integration by parts). *Let $I \subseteq \mathbb{R}$ be an interval and $u, v \in AC_{\text{loc}}(I)$. Then for all $x, x_0 \in I$,*

$$\int_{x_0}^{x} uv'\,dt = u(x)v(x) - u(x_0)v(x_0) - \int_{x_0}^{x} u'v\,dt.$$

We recall the following definition.

DEFINITION 17.8. *If $E \subseteq \mathbb{R}$ is a Lebesgue measurable set and $v : E \to \mathbb{R}$ is a Lebesgue measurable function, then v is* equi-integrable *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\int_F |v(x)|\,dx \le \varepsilon$$

for every Lebesgue measurable set $F \subseteq E$, with $\mathcal{L}^1(F) \le \delta$.

EXERCISE 17.9. *Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set, $1 \le p \le \infty$, and $v \in L^p(E)$. Prove that v is equi-integrable. Prove that if we only assume that $v \in L^1_{\text{loc}}(E)$, then the result may no longer be true.*

EXERCISE 17.10. *Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set with finite measure and $v : E \to \mathbb{R}^M$ equi-integrable. Prove that $v \in L^1(E)$.*

THEOREM 17.11 (Fundamental theorem of calculus, II). *Let $I \subseteq \mathbb{R}$ be an interval. A function $u : I \to \mathbb{R}^M$ belongs to $AC(I)$ if and only if*

(i) *u is continuous in I,*
(ii) *u is differentiable $\mathcal{L}^1$-a.e. in I, and u' belongs to $L^1_{\text{loc}}(I)$ and is equi-integrable,*
(iii) *the fundamental theorem of calculus is valid; that is, for all $x, x_0 \in I$,*

$$u(x) = u(x_0) + \int_{x_0}^{x} u'(t)\,dt.$$

COROLLARY 17.12. *Let $I \subseteq \mathbb{R}$ be an interval and $u : I \to \mathbb{R}$ be such that*

(i) *u is continuous on I,*
(ii) *u is differentiable $\mathcal{L}^1$-a.e. in I, and $u' \in L^p(I)$ for some $1 \le p \le \infty$,*
(iii) *the fundamental theorem of calculus is valid; that is, for all $x, x_0 \in I$,*

$$u(x) = u(x_0) + \int_{x_0}^{x} u'(t)\,dt.$$

Then u belongs to $AC(I)$.

2. Sobolev Functions of One Variable

DEFINITION 17.13. *Given an open interval $I \subseteq \mathbb{R}$, $n \in \mathbb{N}$, and $1 \le p \le \infty$, we say that a function $u \in L^1_{\mathrm{loc}}(I)$ admits a* weak *or* distributional derivative of order n in $L^p(I)$ *if there exists a function $v \in L^p(I)$ such that*

$$\int_I u\varphi^{(n)} dx = (-1)^n \int_I v\varphi \, dx$$

for all $\varphi \in C_c^\infty(I)$. The function v is denoted $u^{(n)}$.

A similar definition can be given when $L^p(I)$ is replaced by $L^p_{\mathrm{loc}}(I)$.

DEFINITION 17.14. *Given an open interval $I \subseteq \mathbb{R}$, $m \in \mathbb{N}$, and $1 \le p \le \infty$, the* Sobolev space $W^{m,p}(I)$ *is the space of all functions $u \in L^p(I)$ which admit weak derivatives of order n in $L^p(I)$ for every $n = 1, \dots, m$. The space $W^{m,p}(I)$ is endowed with the norm*

$$\|u\|_{W^{m,p}(I)} := \|u\|_{L^p(I)} + \sum_{n=1}^{m} \|u^{(n)}\|_{L^p(I)}.$$

The space $W^{m,p}_{\mathrm{loc}}(I)$ is defined as the space of all functions $u \in L^p_{\mathrm{loc}}(I)$ which admit weak derivatives of order n in $L^p_{\mathrm{loc}}(I)$ for every $n = 1, \dots, m$.

The connection between Sobolev functions and absolutely continuous functions is explained in the following theorem.

THEOREM 17.15. *Let $I \subseteq \mathbb{R}$ be an open interval and $1 \le p \le \infty$. Then a function $u : I \to \mathbb{R}$ belongs to $W^{1,p}(I)$ if and only if it admits an absolutely continuous representative $\bar{u} : I \to \mathbb{R}$ such that $\bar{u}$ and its classical derivative $\bar{u}'$ belong to $L^p(I)$. Moreover, if $p > 1$, then $\bar{u}$ is Hölder continuous of exponent $1/p'$.*

THEOREM 17.16. *Let $I \subseteq \mathbb{R}$ be an open interval, $m \in \mathbb{N}$, and $1 \le p < \infty$. Then functions in $C^\infty(I) \cap W^{m,p}(I)$ are dense in $W^{m,p}(I)$.*

THEOREM 17.17 (Poincaré's inequality). *Let $I = (a, b)$ and $1 \le p < \infty$. Then*

$$\int_a^b |u(x) - u_I|^p dx \le (b-a)^p \int_a^b |u'(x)|^p dx \tag{17.2}$$

for all $u \in W^{1,p}(I)$, where

$$u_I := \frac{1}{b-a} \int_a^b u(x)\, dx.$$

We conclude this section with some interpolation inequalities.

THEOREM 17.18. *Let $I \subseteq \mathbb{R}$ be an open interval, $1 \le p, q, r \le \infty$ be such that $r \ge q$, and $u \in W^{1,1}_{\mathrm{loc}}(I)$. Then*

$$\|u\|_{L^r(I)} \le \ell^{1/r-1/q} \|u\|_{L^q(I)} + \ell^{1-1/p+1/r} \|u'\|_{L^p(I)} \tag{17.3}$$

for every $0 < \ell < \mathcal{L}^1(I)$.

Next we consider the case $m = 2$ and $k = 1$.

THEOREM 17.19. *Let $I \subseteq \mathbb{R}$ be an open interval, $1 \le p, q, r \le \infty$ be such that*

$$\frac{1}{2q} + \frac{1}{2p} \ge \frac{1}{r}, \tag{17.4}$$

and $u \in W^{2,1}_{\mathrm{loc}}(I)$*. Then*

$$\|u'\|_{L^r(I)} \preceq \ell^{1/r-1-1/q}\|u\|_{L^q(I)} + \ell^{1-1/p+1/r}\|u''\|_{L^p(I)} \tag{17.5}$$

for every $0 < \ell < \mathcal{L}^1(I)$*.*

We now consider the general case $m \geq 2$.

THEOREM 17.20. *Let* $I \subseteq \mathbb{R}$ *be an open interval,* $1 \leq p, q, r \leq \infty$*,* $m \in \mathbb{N}$*,* $k \in \mathbb{N}_0$*, with* $0 \leq k < m$*, be such that*

$$\left(1 - \frac{k}{m}\right)\frac{1}{q} + \frac{k}{m}\frac{1}{p} \geq \frac{1}{r}, \tag{17.6}$$

and $u \in L^q(I) \cap \dot{W}^{m,1}(I)$*. Then*

$$\|u^{(k)}\|_{L^r(I)} \preceq \ell^{1/r-k-1/q}\|u\|_{L^q(I)} + \ell^{m-k-1/p+1/r}\|u^{(m)}\|_{L^p(I)} \tag{17.7}$$

for every $0 < \ell < \mathcal{L}^1(I)$*. In particular, for* $p = q = r$*,*

$$\|u^{(k)}\|_{L^p(I)} \preceq \ell^{-k}\|u\|_{L^p(I)} + \ell^{m-k}\|u^{(m)}\|_{L^p(I)}. \tag{17.8}$$

THEOREM 17.21. *Let* $1 \leq p, q, r \leq \infty$*,* $m \in \mathbb{N}$*,* $k \in \mathbb{N}_0$*, with* $0 \leq k < m$*, be such that*

$$\left(1 - \frac{k}{m}\right)\frac{1}{q} + \frac{k}{m}\frac{1}{p} \geq \frac{1}{r},$$

and $u \in L^q(\mathbb{R}) \cap \dot{W}^{m,1}(\mathbb{R})$*. Then*

$$\|u^{(k)}\|_{L^r(\mathbb{R})} \preceq \|u\|^{\theta}_{L^q(\mathbb{R})}\|u^{(m)}\|^{1-\theta}_{L^p(\mathbb{R})}$$

where $\theta = (m - k - 1/p + 1/r)/(m - 1/p + 1/q)$*. In particular, for* $p = q = r$*,*

$$\|u^{(k)}\|_{L^p(\mathbb{R})} \preceq \|u\|^{\theta}_{L^p(\mathbb{R})}\|u^{(m)}\|^{1-\theta}_{L^p(\mathbb{R})}.$$

3. Sobolev Spaces

DEFINITION 17.22. *Given an open set* $\Omega \subseteq \mathbb{R}^N$*,* $i = 1, \dots, N$*, and* $1 \leq p \leq \infty$*, we say that a function* $u \in L^1_{\mathrm{loc}}(\Omega)$ *admits a* weak *or* distributional partial derivative in $L^p(\Omega)$ with respect to x_i *if there exists a function* $v_i \in L^p(\Omega)$ *such that*

$$\int_\Omega u\frac{\partial \phi}{\partial x_i}\,dx = -\int_\Omega v_i\phi\,dx$$

for all $\phi \in C^\infty_c(\Omega)$*. The function* v_i *is denoted* $\partial_i u$ *or* $\frac{\partial u}{\partial x_i}$*.*

DEFINITION 17.23. *Given an open set* $\Omega \subseteq \mathbb{R}^N$ *and* $1 \leq p \leq \infty$*, the* Sobolev space $W^{1,p}(\Omega)$ *is the space of all functions* $u \in L^p(\Omega)$ *which admit weak partial derivatives* $\frac{\partial u}{\partial x_i}$ *in* $L^p(\Omega)$ *for every* $i = 1, \dots, N$*. The space* $W^{1,p}(\Omega)$ *is endowed with the norm*

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \sum_{i=1}^N \|\partial_i u\|_{L^p(\Omega)}.$$

The space $W^{1,p}_{\mathrm{loc}}(\Omega)$ is defined as the space of all functions $u \in L^p_{\mathrm{loc}}(\Omega)$ which admit weak partial derivatives $\frac{\partial u}{\partial x_i}$ in $L^p_{\mathrm{loc}}(\Omega)$ for every $i = 1, \dots, N$.

DEFINITION 17.24. *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set and* $1 \leq p < \infty$*. The* homogeneous Sobolev space $\dot{W}^{1,p}(\Omega)$ *is the space of all functions* $u \in L^1_{\mathrm{loc}}(\Omega)$ *whose weak partial derivative* $\frac{\partial u}{\partial x_i}$ *belongs to* $L^p(\Omega)$ *for every* $i = 1, \dots, N$*.*

THEOREM 17.25. *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set and* $1 \le p \le \infty$. *Then*

(i) *the space* $W^{1,p}(\Omega)$ *is a Banach space,*
(ii) *the space* $H^1(\Omega) := W^{1,2}(\Omega)$ *is a Hilbert space with the inner product*

$$\langle u, v\rangle_{H^1(\Omega)} := \int_\Omega uv\,dx + \sum_{i=1}^N \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx.$$

THEOREM 17.26 (Meyers–Serrin). *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set and* $1 \le p < \infty$. *Then the space* $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ *is dense in* $W^{1,p}(\Omega)$.

THEOREM 17.27. *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set whose boundary is of class* C, *and* $1 \le p < \infty$. *Then the restriction to* Ω *of functions in* $C_c^\infty(\mathbb{R}^N)$ *is dense in* $W^{1,p}(\Omega)$.

THEOREM 17.28. *Let* $u \in W^{1,p}(\mathbb{R}^N)$, *where* $1 \le p < \infty$. *Then there exists a sequence* $\{u_n\}_n$ *of functions in* $C_c^\infty(\mathbb{R}^N)$ *such that* $u_n \to u$ *in* $W^{1,p}(\mathbb{R}^N)$.

THEOREM 17.29. *Let* $u \in \dot{W}^{1,p}(\mathbb{R}^N)$, *where* $1 \le p < \infty$. *Then there exists a sequence* $\{u_n\}_n$ *of functions in* $C_c^\infty(\mathbb{R}^N)$ *such that* $\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i}$ *in* $L^p(\mathbb{R}^N)$ *for every* $i = 1, \dots, N$ *if and only if* $N \ge 2$ *or* $p > 1$.

THEOREM 17.30 (Absolute continuity on lines). *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set and* $1 \le p < \infty$. *A function* $u \in L^p(\Omega)$ *belongs to the space* $W^{1,p}(\Omega)$ *if and only if it has a representative* $\overline{u}$ *that is absolutely continuous on* $\mathcal{L}^{N-1}$*-a.e. line segments of* Ω *that are parallel to the coordinate axes and whose first-order (classical) partial derivatives belong to* $L^p(\Omega)$. *Moreover the (classical) partial derivatives of* $\overline{u}$ *agree* $\mathcal{L}^N$*-a.e. with the weak derivatives of* u.

THEOREM 17.31 (Chain rule). *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set,* $1 \le p \le \infty$, *and* $f : \mathbb{R} \to \mathbb{R}$ *be Lipschitz continuous and* $u \in W^{1,p}(\Omega)$. *Assume that* $f(0) = 0$ *if* Ω *has infinite measure. Then* $f \circ u \in W^{1,p}(\Omega)$ *and that for all* $i = 1, \dots, N$ *and for* $\mathcal{L}^N$*-a.e.* $x \in \Omega$,

$$\partial_i(f \circ u)(x) = f'(u(x))\partial_i u(x),$$

where $f'(u(x))\partial_i u(x)$ *is interpreted to be zero whenever* $\partial_i u(x) = 0$.

THEOREM 17.32 (Change of variables). *Let* $\Omega, U \subseteq \mathbb{R}^N$ *be open sets,* $\Psi : U \to \Omega$ *be invertible, with* Ψ *and* Ψ^{-1} *Lipschitz continuous functions, and* $u \in \dot{W}^{1,p}(\Omega)$, $1 \le p \le \infty$. *Then* $u \circ \Psi \in \dot{W}^{1,p}(U)$ *and for all* $i = 1, \dots, N$ *and for* $\mathcal{L}^N$*-a.e.* $y \in U$,

$$\frac{\partial (u \circ \Psi)}{\partial y_i}(y) = \sum_{j=1}^N \frac{\partial u}{\partial x_j}(\Psi(y))\frac{\partial \Psi_j}{\partial y_i}(y).$$

THEOREM 17.33. *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set and* $u \in \dot{W}^{1,p}(\Omega)$, $1 \le p < \infty$. *Then for every* $h \in \mathbb{R}^N \setminus \{0\}$,

$$\int_{\Omega_h} \frac{|u(x+h) - u(x)|^p}{\|h\|^p}\,dx \le \int_\Omega \|\nabla u(x)\|^p dx \tag{17.9}$$

while,

$$\kappa_{N,p}\int_\Omega \|\nabla u(x)\|^p dx \le \limsup_{h\to 0}\int_{\Omega_h} \frac{|u(x+h) - u(x)|^p}{\|h\|^p}\,dx, \tag{17.10}$$

where

$$\kappa_{N,p} := \frac{1}{\beta_N} \int_{S^{N-1}} |e_1 \cdot \xi|^p d\mathcal{H}^{N-1}(\xi). \tag{17.11}$$

Conversely, if $1 < p < \infty$ *and* $u \in L^1_{\text{loc}}(\Omega)$ *is such that*

$$\limsup_{h \to 0} \int_{\Omega_h} \frac{|u(x+h) - u(x)|^p}{\|h\|^p} dx < \infty, \tag{17.12}$$

then $u \in \dot{W}^{1,p}(\Omega)$.

Exercise 17.34. *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set and for every* $i = 1, \dots, N$ *and* $h > 0$, $\Omega_{h,i} := \{x \in \Omega : x + te_i \in \Omega \quad \text{for all } 0 < t \le h\}$.

(i) *Let* $u \in \dot{W}^{1,p}(\Omega)$, $1 \le p < \infty$. *Prove that for every* $i = 1, \dots, N$ *and* $h > 0$,
$$\int_{\Omega_{h,i}} \frac{|u(x+he_i) - u(x)|^p}{h^p} dx \le \int_\Omega |\partial_i u(x)|^p dx$$
and
$$\lim_{h \to 0^+} \int_{\Omega_{h,i}} \frac{|u(x+he_i) - u(x)|^p}{h^p} dx = \int_\Omega |\partial_i u(x)|^p dx.$$

(ii) *Prove that if* $1 < p < \infty$ *and* $u \in L^1_{\text{loc}}(\Omega)$ *is such that*
$$\liminf_{h \to 0^+} \int_{\Omega_{h,i}} \frac{|u(x+he_i) - u(x)|^p}{h^p} dx < \infty$$
for every $i = 1, \dots, N$, *then* $u \in \dot{W}^{1,p}(\Omega)$.

The following theorem follows from Theorems **??** and **??**

Theorem 17.35. *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set and* $1 < p < \infty$. *Then* $W^{1,p}(\Omega)$ *is reflexive. In particular, if* $\{u_n\}_n$ *is a bounded sequence in* $W^{1,p}(E)$, *then there exist a subsequence* $\{u_{n_k}\}_k$ *and* $u \in W^{1,p}(\Omega)$ *such that* $u_{n_k} \rightharpoonup u$ *in* $W^{1,p}(\Omega)$, *that is,* $u_{n_k} \rightharpoonup u$ *in* $L^p(\Omega)$ *and* $\partial_i u_{n_k} \rightharpoonup \partial_i u$ *in* $L^p(\Omega)$ *for every* $i = 1, \dots, N$.

4. Embeddings

The number

$$p^* := \frac{Np}{N-p}. \tag{17.13}$$

is called the *Sobolev critical exponent.*

Theorem 17.36 (Sobolev–Gagliardo–Nirenberg's embedding in $W^{1,p}$). *Let* $1 \le p < N$. *Then for every function* $u \in \dot{W}^{1,p}(\mathbb{R}^N)$ *vanishing at infinity,*

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \preceq \|\nabla u\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)}.$$

In particular, $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ *for all* $p \le q \le p^*$.

Theorem 17.37 (Sobolev–Gagliardo–Nirenberg's embedding in $\dot{W}^{1,p}$). *Let* $1 \le p < \infty$ *be such* $p < N$. *Then for every function* $u \in \dot{W}^{1,p}(\mathbb{R}^N)$ *there exists a constant* c_u *such that*

$$\|u - c_u\|_{L^{p^*}(\mathbb{R}^N)} \preceq \|\nabla u\|_{L^p(\mathbb{R}^N)}.$$

THEOREM 17.38. *There for every function* $u \in \dot{W}^{1,N}(\mathbb{R}^N)$,

$$|u|_{\mathrm{BMO}(\mathbb{R}^N)} \preceq \|\nabla u\|_{L^N(\mathbb{R}^N;\mathbb{R}^N)}.$$

In particular, $W^{1,N}(\mathbb{R}^N) \hookrightarrow \mathrm{BMO}(\mathbb{R}^N)$.

THEOREM 17.39. *Let* $N \in \mathbb{N}$ *be such* $N \geq 2$. *Then there exists a constant* $c = c(N) > 0$ *such that for every function* $u \in W^{1,N}(\mathbb{R}^N)$,

$$\|u\|_{L^q(\mathbb{R}^N)} \leq cq^{1-1/p+1/q}\|u\|_{W^{1,N}(\mathbb{R}^N)}.$$

In particular,

$$W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$$

for all $N < q < \infty$.

THEOREM 17.40 (Morrey's embedding in $W^{1,p}$). *Let* $N < p < \infty$. *Then* $W^{1,p}(\mathbb{R}^N) \hookrightarrow C^{0,1-N/p}(\mathbb{R}^N)$. *Moreover, if* $u \in W^{1,p}(\mathbb{R}^N)$ *and* $\bar{u}$ *is its representative in* $C^{0,1-N/p}(\mathbb{R}^N)$, *then*

$$\lim_{\|x\|\to\infty} \bar{u}(x) = 0.$$

REMARK 17.41. *If* $N = 1$, *we have* $W^{1,1}(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$.

COROLLARY 17.42. *Let* $N < p < \infty$. *If* $u \in \dot{W}^{1,p}(\mathbb{R}^N)$, *then a representative* $\bar{u}$ *of* u *is Hölder continuous with exponent* $1 - N/p$ *and*

$$|\bar{u}(x) - \bar{u}(y)| \preceq \|x - y\|^{1-N/p}\|\nabla u\|_{L^p(\mathbb{R}^N)}$$

for all $x, y \in \mathbb{R}^N$.

DEFINITION 17.43. *Given* $1 \leq p \leq \infty$, *an open set* $\Omega \subseteq \mathbb{R}^N$ *is called an* extension domain for the Sobolev space $W^{1,p}(\Omega)$ *if there exists a continuous linear operator*

$$\mathcal{E} : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$$

with the property that for all $u \in W^{1,p}(\Omega)$, $\mathcal{E}(u)(x) = u(x)$ *for* $\mathcal{L}^N$*-a.e.* $x \in \Omega$.

THEOREM 17.44 (Rellich–Kondrachov's compactness). *Let* $1 \leq p < \infty$ *and* $\Omega \subset \mathbb{R}^N$ *be an extension domain for* $W^{1,p}(\Omega)$ *with finite measure. Let* $\{u_n\}_n$ *be a bounded sequence in* $W^{1,p}(\Omega)$. *Then there exist a subsequence* $\{u_{n_k}\}_k$ *of* $\{u_n\}_n$ *and a function* $u \in L^p(\Omega)$ *such that* $u_{n_k} \to u$ *in* $L^p(\Omega)$. *Moreover, if* $p > 1$, *then* $u \in W^{1,p}(\Omega)$.

To give a unified treatment, only in this section, we use a different notation for L^r norms. To be precise, given $r \in [-\infty, \infty]$, $r \neq 0$, and a function $u : \mathbb{R}^N \to \mathbb{R}$, we define

$$|u|_r := \begin{cases} \|u\|_{L^r(\mathbb{R}^N)} & \text{if } r > 0, \\ \|\nabla^n u\|_{L^\infty(\mathbb{R}^N)} & \text{if } r < 0 \text{ and } a = 0, \\ |\nabla^n u|_{C^{0,a}(\mathbb{R}^N)} & \text{if } r < 0 \text{ and } 0 < a < 1, \end{cases} \tag{17.14}$$

where if $r < 0$ we set $n := \lfloor -N/r \rfloor$ and $a := -n - N/r \in [0, 1)$, provided the right-hand sides are well-defined.

We begin with the case $m = 1$.

THEOREM 17.45 (Gagliardo–Nirenberg interpolation, $m = 1$). *Let $1 \le p, q \le \infty$, $0 \le \theta \le 1$, and r be such that*

$$(1-\theta)\Big(\frac{1}{p} - \frac{1}{N}\Big) + \frac{\theta}{q} = \frac{1}{r} \in (-\infty, 1]. \tag{17.15}$$

Then

$$|u|_r \preceq \|u\|^{\theta}_{L^q(\mathbb{R}^N)} \|\nabla u\|^{1-\theta}_{L^p(\mathbb{R}^N)} \tag{17.16}$$

for every $u \in L^q(\mathbb{R}^N) \cap \dot{W}^{1,p}(\mathbb{R}^N)$, with the following exceptions

(i) *if $p < N$ and $r < q = \infty$, we assume that u vanishes at infinity,*
(ii) *if $1 < p = N$ and $q = r = \infty$, then (17.16) fails for $0 \le \theta < 1$,*
(iii) *if $p = N > 1$ and $q \ne r$ we take $0 < \theta \le 1$.*

5. Extension

DEFINITION 17.46. *The boundary $\partial\Omega$ of an open set $\Omega \subseteq \mathbb{R}^N$ is* uniformly Lipschitz continuous *if there exist ε, $L > 0$, $M \in \mathbb{N}$, and a locally finite countable open cover $\{\Omega_n\}_n$ of $\partial\Omega$ such that*

(i) *if $x \in \partial\Omega$, then $B(x, \varepsilon) \subseteq \Omega_n$ for some $n \in \mathbb{N}$,*
(ii) *no point of $\mathbb{R}^N$ is contained in more than M of the Ω_n's,*
(iii) *for each n there exist local coordinates $y = (y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and a Lipschitz continuous function $f : \mathbb{R}^{N-1} \to \mathbb{R}$ (both depending on n), with $\operatorname{Lip} f \le L$, such that $\Omega_n \cap \Omega = \Omega_n \cap V_n$, where V_n is given in local coordinates by*

$$\{(y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : \, y_N > f(y')\}.$$

REMARK 17.47. *Similarly, given $m \in \mathbb{N}_0$ and $0 < \alpha \le 1$ we can define open sets $\Omega \subseteq \mathbb{R}^N$ whose boundary is* uniformly of class C^m *(respectively,* of class $C^{m,\alpha}$*, see Definition* **??***) with parameters ε, $L > 0$, M provided (i), (ii), and (iii) hold but with f of class C^m (respectively of class $C^{m,\alpha}$) and with $\|f\|_{C^m(\mathbb{R}^{N-1})} \le L$ (respectively, $\|f\|_{C^{m,\alpha}(\mathbb{R}^{N-1})} \le L$).*

THEOREM 17.48 (Stein). *Let $\Omega \subseteq \mathbb{R}^N$ be an open set with uniformly Lipschitz continuous boundary. Then for all $1 \le p \le \infty$ there exists a continuous linear operator $\mathcal{E} : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$ such that for all $u \in W^{1,p}(\Omega)$, $\mathcal{E}(u)(x) = u(x)$ for $\mathcal{L}^N$-a.e. $x \in \Omega$, and*

$$\|\mathcal{E}(u)\|_{L^p(\mathbb{R}^N)} \preceq_M \|u\|_{L^p(\Omega)}, \tag{17.17}$$

$$\|\nabla^k \mathcal{E}(u)\|_{L^p(\mathbb{R}^N)} \preceq_{\varepsilon, L, M} \|u\|_{W^{1,p}(\Omega)} \tag{17.18}$$

for every multi-index $\alpha \in \mathbb{N}_0^N$ with $1 \le |\alpha| \le m$.

6. Poincaré's Inequalities

Given an open set $\Omega \subseteq \mathbb{R}^N$, a Lebesgue measurable set $E \subseteq \Omega$ with finite positive measure, and an integrable function $u : \Omega \to \mathbb{R}$, we define

$$u_E := \frac{1}{\mathcal{L}^N(E)} \int_E u(x)\, dx. \tag{17.19}$$

THEOREM 17.49 (Poincaré's inequality in $W^{1,p}(\Omega)$). *Let $1 \le p < \infty$ and $\Omega \subset \mathbb{R}^N$ be a connected extension domain for $W^{1,p}(\Omega)$ with finite measure. Let $E \subseteq \Omega$ be a Lebesgue measurable set with positive measure. Then for all $u \in W^{1,p}(\Omega)$,*

$$\|u - u_E\|_{L^p(\Omega)} \preceq_{\Omega,E} \|\nabla u\|_{L^p(\Omega)}.$$

PROPOSITION 17.50 (Poincaré's inequality for rectangles). *Let $1 \le p < \infty$ and $R = (0, a_1) \times \cdots \times (0, a_N) \subset \mathbb{R}^N$. Then for all $u \in W^{1,p}(R)$,*

$$\|u - u_R\|_{L^p(R)} \preceq \max\{a_1, \dots, a_N\} \|\nabla u\|_{L^p(R)}.$$

COROLLARY 17.51. *Let $1 \le p < \infty$ and $R = (0, a_1) \times \cdots \times (0, a_N) \subset \mathbb{R}^N$ and let $R_1 \Subset R$ be a rectangle. Then for all $u \in \dot{W}^{1,p}(R)$,*

$$\|u - u_{R_1}\|_{L^p(R)} \preceq \max\{a_1, \dots, a_N \|\nabla u\|_{L^p(R)}.$$

THEOREM 17.52 (Poincaré's inequality for convex sets). *Let $1 \le p < \infty$ and $\Omega \subset \mathbb{R}^N$ be an open bounded convex set. Then for all $u \in W^{1,p}(\Omega)$,*

$$\|u - u_\Omega\|_{L^p(\Omega)} \preceq \operatorname{diam} \Omega \|\nabla u\|_{L^p(\Omega)}.$$

Next we consider star-shaped sets. We recall that a set $E \subseteq \mathbb{R}^N$ is *star-shaped with respect to a point* $x_0 \in E$ if $\theta x + (1 - \theta) x_0 \in E$ for all $\theta \in (0, 1)$ and for all $x \in E$.

THEOREM 17.53 (Poincaré's inequality for star-shaped sets). *Let $1 \le p < \infty$ and $\Omega \subset \mathbb{R}^N$ be an open set star-shaped with respect to $x_0 \in \Omega$ and such that*

$$Q(x_0, 4r) \subseteq \Omega \subseteq B(x_0, R)$$

for some $r, R > 0$. Then for all $u \in W^{1,p}(\Omega)$,

$$\|u - u_\Omega\|_{L^p(\Omega)} \preceq R(R/r)^{(N-1)/p} \|\nabla u\|_{L^p(\Omega)}.$$

COROLLARY 17.54. *Let $1 \le p < \infty$ and $\Omega \subset \mathbb{R}^N$ be an open set star-shaped with respect to $x_0 \in \Omega$ and such that*

$$Q(x_0, 4r) \Subset \Omega \subseteq B(x_0, R)$$

for some $r, R > 0$. Then for all $u \in \dot{W}^{1,p}(\Omega)$,

$$\|u - u_\Omega\|_{L^p(\Omega)} \preceq R(R/r)^{(N-1)/p} \|\nabla u\|_{L^p(\Omega)}.$$

7. Trace Theory

THEOREM 17.55. *Let $\Omega \subseteq \mathbb{R}^N$, $N \ge 2$, be an open set whose boundary $\partial\Omega$ is Lipschitz continuous, let $1 \le p < \infty$. There exists a unique linear operator*

$$\operatorname{Tr} : W^{1,p}(\Omega) \to L^p_{\mathrm{loc}}(\partial\Omega)$$

such that

(i) $\operatorname{Tr}(u) = u$ *on* $\partial\Omega$ *for all* $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$,

(ii) *the integration by parts formula*

$$\int_\Omega u \partial_i \psi \, dx = -\int_\Omega \psi \partial_i u \, dx + \int_{\partial\Omega} \psi \operatorname{Tr}(u) \nu_i \, d\mathcal{H}^{N-1}$$

holds for all $u \in W^{1,p}(\Omega)$, all $\psi \in C^1_c(\mathbb{R}^N)$, and all $i = 1, \dots, N$,

(iii) *for every* $R > 0$ *there exist two constants* c_R, $\varepsilon_R > 0$ *depending on* R, Ω *and* p *such that*

$$\int_{B(0,R)\cap\partial\Omega} |\operatorname{Tr}(u)|^p d\mathcal{H}^{N-1} \le c_R \varepsilon^{-1} \int_{B(0,R)\cap(\Omega\setminus\Omega_\varepsilon)} |u|^p dx + c_R \varepsilon^{p-1} \int_{B(0,R)\cap(\Omega\setminus\Omega_\varepsilon)} \|\nabla u\|^p dx$$

for every $0 < \varepsilon \le \varepsilon_R$, *where* $\Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$.

The function $\operatorname{Tr}(u)$ is called the *trace* of u on $\partial\Omega$.

When $p = 1$ and Ω sufficiently regular the trace operator

$$\operatorname{Tr} : W^{1,1}(\Omega) \to L^1(\partial\Omega)$$

is onto. This fact is explained in the following theorems. As usual, we begin with the case of the half-space $\mathbb{R}^N_+$.

THEOREM 17.56 (Gagliardo). *Let* $N \ge 2$. *Then for all functions* $u \in \dot{W}^{1,1}(\mathbb{R}^N_+)$ *vanishing at infinity,*

$$\|\operatorname{Tr}(u)(\cdot, 0)\|_{L^1(\mathbb{R}^{N-1})} \le \|\partial_N u\|_{L^1(\mathbb{R}^N_+)}. \tag{17.20}$$

THEOREM 17.57 (Gagliardo). *Let* $g \in L^1(\mathbb{R}^{N-1})$, $N \ge 2$. *Then for every* $0 < \delta < 1$ *there exists a function* $u \in W^{1,1}(\mathbb{R}^N_+)$ *such that* $\operatorname{Tr}(u) = g$ *and*

$$\|u\|_{L^1(\mathbb{R}^N_+)} \le \delta \|g\|_{L^1(\mathbb{R}^{N-1})}, \quad \|\nabla u\|_{L^1(\mathbb{R}^N_+;\mathbb{R}^N)} \le (1 + \delta)\|g\|_{L^1(\mathbb{R}^{N-1})}.$$

For more general domains, we have the following result.

THEOREM 17.58. *Let* $\Omega \subseteq \mathbb{R}^N$, $N \ge 2$, *be an open set whose boundary* $\partial\Omega$ *is uniformly Lipschitz continuous. Then*

$$\|\operatorname{Tr}(u)\|_{L^1(\partial\Omega)} \preceq_{\varepsilon,L,M} \|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^1(\Omega;\mathbb{R}^N)}$$

for all $u \in W^{1,1}(\Omega)$, *where* ε, $L > 0$, *and* $M \in \mathbb{N}$ *are given in Definition 17.46.*

Moreover, for every $g \in L^1(\partial\Omega)$ *there exists a function* $u \in W^{1,1}(\Omega)$ *such that* $\operatorname{Tr}(u) = g$ *and*

$$\|u\|_{L^1(\Omega)} \le \|g\|_{L^1(\partial\Omega)}, \quad \|\nabla u\|_{L^1(\Omega;\mathbb{R}^N)} \le 4(1 + L)\|g\|_{L^1(\partial\Omega)}.$$

When $1 < p < \infty$, the trace operator

$$\operatorname{Tr} : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

is not onto. Indeed, when Ω is sufficiently regular, its image $\operatorname{Tr}(W^{1,p}(\Omega))$ can be identified with the fractional Sobolev space $W^{1-1/p,p}(\partial\Omega)$.

THEOREM 17.59 (Gagliardo). *Let* $1 < p < \infty$ *and* $N \ge 2$. *Then for all* $u \in \dot{W}^{1,p}(\mathbb{R}^N_+)$,

$$|\operatorname{Tr}(u)(\cdot, 0)|_{W^{1-1/p,p}(\mathbb{R}^{N-1})} \preceq \|\nabla u\|_{L^p(\mathbb{R}^N_+)}. \tag{17.21}$$

THEOREM 17.60 (Gagliardo). *Let* $1 < p < \infty$, $N \ge 2$, *and* $g \in \dot{W}^{1-1/p,p}(\mathbb{R}^{N-1})$. *Then there exists a function* $v \in \dot{W}^{1,p}(\mathbb{R}^N_+)$ *such that* $\operatorname{Tr}(v)(\cdot, 0) = g$ *and*

$$|v|_{W^{1,p}(\mathbb{R}^N_+)} \preceq |g|_{W^{1-1/p,p}(\mathbb{R}^{N-1})}.$$

COROLLARY 17.61. *Let $1 < p < \infty$, $N \geq 2$, and $g \in W^{1-1/p,p}(\mathbb{R}^{N-1})$. Then for every $0 < \varepsilon \leq 1$ there exists a function $u \in W^{1,p}(\mathbb{R}^N_+)$ such that* $\operatorname{Tr}(u)(\cdot, 0) = g$, $\operatorname{supp} u \subseteq \mathbb{R}^{N-1} \times [-\varepsilon, \varepsilon]$ *and*

$$\|u\|_{L^p(\mathbb{R}^N_+)} \leq \varepsilon^{1/p}\|g\|_{L^p(\mathbb{R}^{N-1})},$$
$$|u|_{W^{1,p}(\mathbb{R}^N_+)} \preceq \varepsilon^{-1/p'}\|g\|_{L^p(\mathbb{R}^{N-1})} + |g|_{W^{1-1/p,p}(\mathbb{R}^{N-1})}.$$

THEOREM 17.62. *Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be an open set whose boundary $\partial\Omega$ is uniformly Lipschitz continuous and let $1 < p < \infty$. Then*

$$\|\operatorname{Tr}(u)\|_{L^p(\partial\Omega)} \preceq_{\varepsilon,L,M} \|u\|_{W^{1,p}(\Omega)},$$
$$|\operatorname{Tr}(u)|^{\diamond}_{W^{1-1/p,p}(\partial\Omega)} \preceq_{\varepsilon,L,M} \|u\|_{W^{1,p}(\Omega)}$$

for all $u \in W^{1,1}(\Omega)$, where ε, $L > 0$, and $M \in \mathbb{N}$ are given in Definition 17.46.

Moreover, for every $g \in W^{1-1/p,p}(\partial\Omega)$ there exist a constant $c = c(N,p)$ and a function $u \in W^{1,p}(\Omega)$ such that $\operatorname{Tr}(u) = g$,

$$\|u\|_{L^p(\Omega)} \preceq_{\varepsilon,M} \|g\|_{L^p(\partial\Omega)}$$

and

$$\|\nabla u\|_{L^p(\Omega)} \preceq_{\varepsilon,L,M} \|g\|_{W^{1-1/p,p}(\partial\Omega)}.$$

Next we show that if the domain Ω is sufficiently regular, we may characterize $W^{1,p}_0(\Omega)$ as the subspace of functions in $W^{1,p}(\Omega)$ with trace zero.

THEOREM 17.63 (Traces and $W^{1,p}_0$). *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open set whose boundary $\partial\Omega$ is Lipschitz continuous, let $1 \leq p < \infty$, and let $u \in W^{1,p}(\Omega)$. Then* $\operatorname{Tr}(u) = 0$ *if and only if $u \in W^{1,p}_0(\Omega)$.*

8. Higher Order Sobolev Spaces

DEFINITION 17.64. *Given an open set $\Omega \subseteq \mathbb{R}^N$, a multi-index $\alpha \in \mathbb{N}^N_0 \setminus \{0\}$, and $1 \leq p \leq \infty$, we say that a function $u \in L^1_{\text{loc}}(\Omega)$ admits a* weak *or* distributional αth derivative in $L^p(\Omega)$ *if there exists a function $v_\alpha \in L^p(\Omega)$ such that*

$$\int_\Omega u\partial^\alpha\phi\,dx = (-1)^{|\alpha|}\int_\Omega v_\alpha\phi\,dx \tag{17.22}$$

for all $\phi \in C^\infty_c(\Omega)$. The function v_α is denoted $\partial^\alpha u$ or $\frac{\partial^{|\alpha|}u}{\partial x^\alpha}$.

DEFINITION 17.65. *Given an open set $\Omega \subseteq \mathbb{R}^N$, $m \in \mathbb{N}$, and $1 \leq p \leq \infty$, the* Sobolev space $W^{m,p}(\Omega)$ *is the space of all functions $u \in L^p(\Omega)$ which admit weak derivatives $\partial^\alpha u$ in $L^p(\Omega)$ for every $\alpha \in \mathbb{N}^N_0$ with $1 \leq |\alpha| \leq m$. The space $W^{m,p}(\Omega)$ is endowed with the norm*

$$\|u\|_{W^{m,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \sum_{1\leq|\alpha|\leq m}\|\partial^\alpha u\|_{L^p(\Omega)}.$$

The space $W^{m,p}_{\text{loc}}(\Omega)$ is defined as the space of all functions $u \in L^p_{\text{loc}}(\Omega)$ which admit weak derivatives $\partial^\alpha u$ in $L^p_{\text{loc}}(\Omega)$ for every $\alpha \in \mathbb{N}^N_0$ with $1 \leq |\alpha| \leq m$.

DEFINITION 17.66. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, $m \in \mathbb{N}$, and $1 \leq p < \infty$. The* homogeneous Sobolev space $\dot{W}^{m,p}(\Omega)$ *is the space of all functions $u \in L^1_{\text{loc}}(\Omega)$ whose αth weak derivative $\partial^\alpha u$ belongs to $L^p(\Omega)$ for every $\alpha \in \mathbb{N}^N_0$ with $|\alpha| = m$.*

THEOREM 17.67. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, $m \in \mathbb{N}$, and $1 \leq p \leq \infty$. Then*

(i) *the space* $W^{m,p}(\Omega)$ *is a Banach space,*
(ii) *the space* $H^m(\Omega) := W^{m,2}(\Omega)$ *is a Hilbert space with the inner product*

$$\langle u, v\rangle_{H^1(\Omega)} := \int_\Omega uv\,dx + \sum_{1\le|\alpha|\le m}\int_\Omega \partial^\alpha u \partial^\alpha v\,dx.$$

THEOREM 17.68 (Meyers–Serrin). *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set,* $m \in \mathbb{N}$*, and* $1 \le p < \infty$*. Then the space* $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ *is dense in* $W^{m,p}(\Omega)$*.*

THEOREM 17.69. *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set whose boundary is of class* C*,* $m \in \mathbb{N}$*, and* $1 \le p < \infty$*. Then the restriction to* Ω *of functions in* $C_c^\infty(\mathbb{R}^N)$ *is dense in* $W^{m,p}(\Omega)$*.*

THEOREM 17.70. *Let* $u \in \dot{W}^{m,p}(\mathbb{R}^N)$*, where* $m \in \mathbb{N}$ *and* $1 \le p < \infty$*. Then there exists a sequence* $\{u_n\}_n$ *of functions in* $C_c^\infty(\mathbb{R}^N)$ *such that* $\partial^\alpha u_n \to \partial^\alpha u$ *in* $L^p(\mathbb{R}^N)$ *for every multi-index* α *with* $|\alpha| = m$ *if and only if* $N \ge 2$ *or* $p > 1$*.*

THEOREM 17.71. *Let* $\Omega, U \subseteq \mathbb{R}^N$ *be open sets, let* $m \in \mathbb{N}$*, let* $\Psi : U \to \Omega$ *be invertible, with* $\Psi \in C^{m-1,1}(\overline{U}; \mathbb{R}^N)$ *and* Ψ^{-1} *Lipschitz continuous, and let* $u \in W^{m,p}(\Omega)$*,* $1 \le p \le \infty$*. Then* $u \circ \Psi$ *belongs to* $W^{m,p}(U)$ *and and for every multi-index* $\alpha \in \mathbb{N}_0^M$*, with* $0 < |\alpha| \le m$*, and for* $\mathcal{L}^N$*-a.e.* $y \in U$*,*

$$\frac{\partial^{|\alpha|}}{\partial y^\alpha}(u\circ\Psi)(y) = \sum c_{\alpha,\beta,\gamma,l}\frac{\partial^{|\beta|}u}{\partial x^\beta}(\Psi(y))\prod_{i=1}^{|\beta|}\frac{\partial^{|\gamma_i|}\Psi_{l_i}}{\partial y^{\gamma_i}}(y),$$

where $c_{\alpha,\beta,\gamma,l} \in \mathbb{R}$*, the sum is done over all* $\beta \in \mathbb{N}_0^N$ *with* $1 \le |\beta| \le |\alpha|$*,* $\gamma = (\gamma_1, \dots, \gamma_{|\beta|})$*,* $\gamma_i \in \mathbb{N}_0^M$*, with* $|\gamma_i| > 0$ *and* $\sum_{i=1}^{|\beta|}\gamma_i = \alpha$*, and* $l = (l_1, \dots, l_{|\beta|})$*,* $l_i \in \{1, \dots, N\}$*,* $i = 1, \dots, |\beta|$*.*

To extend the Sobolev–Gagliardo–Nirenberg embedding theorem to functions in $\dot{W}^{m,p}(\mathbb{R}^N)$, where $m \ge 2$ and $1 \le p < \infty$ are such that $mp < N$, for every $k = 0$, $\dots$, m we define the *Sobolev critical exponent.*

$$p^*_{m,k} := \frac{Np}{N-(m-k)p}. \tag{17.23}$$

Note that $p^*_{m,m-1} = p^*$ and $p^*_{m,m} = p$.

COROLLARY 17.72 (Sobolev–Gagliardo–Nirenberg's embedding in $W^{m,p}$). *Let* $m \in \mathbb{N}$ *and* $1 \le p < \infty$ *be such* $mp < N$*. Then for every function* $u \in W^{m,p}(\mathbb{R}^N)$ *and for every* $k = 0, \dots, m-1$ *and*

$$\|\nabla^k u\|_{L^{p^*_{m,k}}(\mathbb{R}^N)} \preceq \|\nabla^m u\|_{L^p(\mathbb{R}^N)}.$$

In particular,

$$W^{m,p}(\mathbb{R}^N) \hookrightarrow L^{q_0}(\mathbb{R}^N) \cap W^{1,q_1}(\mathbb{R}^N) \cap \cdots \cap W^{m-1,q_{m-1}}(\mathbb{R}^N)$$

for all $p \le q_k \le p^*_{m,k}$*,* $k = 0, \dots, m-1$*.*

THEOREM 17.73 (Sobolev–Gagliardo–Nirenberg's embedding in $\dot{W}^{m,p}$). *Let* $m \in \mathbb{N}$ *and* $1 \le p < \infty$ *be such* $mp < N$*. Then for every function* $u \in \dot{W}^{m,p}(\mathbb{R}^N)$ *there exists a polynomial* P_u *of degree* $m-1$ *such that*

$$\|\nabla^k (u - P_u)\|_{L^{p^*_{m,k}}(\mathbb{R}^N)} \preceq \|\nabla^m u\|_{L^p(\mathbb{R}^N)}$$

for every $k = 0, \ldots, m-1$. *Moreover,*

$$P_u = u - \sum_{|\alpha|=m} (K_\alpha * \partial^\alpha u),$$

where $K_\alpha(x) := \frac{m}{\beta_N \alpha!} \frac{x^\alpha}{\|x\|^N}$, $x \in \mathbb{R}^N \setminus \{0\}$.

THEOREM 17.74. *Let* $m \in \mathbb{N}$, $m \geq 2$. *Then for every* $u \in \dot{W}^{m,N/m}(\mathbb{R}^N)$ *there exists a polynomial* P_u *of degree* $m-1$ *such that*

$$|u - P_u|_{\mathrm{BMO}(\mathbb{R}^N)} \preceq \|\nabla^m u\|_{L^{N/m}(\mathbb{R}^N)}$$

and

$$\|\nabla^k (u - P_u)\|_{L^{N/k}(\mathbb{R}^N)} \preceq \|\nabla^m u\|_{L^{N/m}(\mathbb{R}^N)}$$

for $k = 1, \ldots, m-1$. *In particular,*

$$W^{m,N/m}(\mathbb{R}^N) \hookrightarrow \mathrm{BMO}(\mathbb{R}^N) \cap W^{1,q_1}(\mathbb{R}^N) \cap \cdots \cap W^{m-1,q_{m-1}}(\mathbb{R}^N)$$

for all $N/m \leq q_k \leq N/k$, $k = 1, \ldots, m-1$.

THEOREM 17.75. *Let* $m, N \in \mathbb{N}$ *be such* $N > m$. *Then for every function* $u \in W^{m,N/m}(\mathbb{R}^N)$,

$$\|u\|_{L^q(\mathbb{R}^N)} \preceq_{m,N} q^{1-1/p+1/q} \|u\|_{W^{m,N/m}(\mathbb{R}^N)}$$

for every $N/m < q < \infty$ *and*

$$\|\nabla^k u\|_{L^{N/k}(\mathbb{R}^N)} \preceq \|\nabla^m u\|_{L^{N/m}(\mathbb{R}^N)}$$

for every $k = 1, \ldots, m-1$. *In particular,*

$$W^{m,N/m}(\mathbb{R}^N) \hookrightarrow L^{q_0}(\mathbb{R}^N) \cap W^{1,q_1}(\mathbb{R}^N) \cap \cdots \cap W^{m-1,q_{m-1}}(\mathbb{R}^N)$$

for all $N/m < q_0 < \infty$ *and all* $N/m \leq q_k \leq N/k$, $k = 1, \ldots, m-1$.

THEOREM 17.76 (Morrey's embedding in $W^{m,p}$). *Let* $m \in \mathbb{N}$, $m \geq 2$, *and* $1 \leq p < \infty$ *be such* $mp > N$. *Then* $W^{m,p}(\mathbb{R}^N) \hookrightarrow C^{\ell,\theta}(\mathbb{R}^N)$, *where if* $m - N/p$ *is not an integer,*

$$\ell := \lfloor m - N/p \rfloor, \quad \theta := m - \ell - N/p,$$

while if $m - N/p$ *is an integer,*

$$\ell := m - 1 - N/p, \quad \theta := \text{any number less than } 1.$$

DEFINITION 17.77. *Given* $m \in \mathbb{N}$ *and* $1 \leq p \leq \infty$, *an open set* $\Omega \subseteq \mathbb{R}^N$ *is called an* extension domain for the Sobolev space $W^{m,p}(\Omega)$ *if there exists a continuous linear operator*

$$\mathcal{E} : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^N)$$

with the property that for all $u \in W^{m,p}(\Omega)$, $\mathcal{E}(u)(x) = u(x)$ *for* $\mathcal{L}^N$*-a.e.* $x \in \Omega$.

THEOREM 17.78 (Gagliardo–Nirenberg interpolation, I). *Let* $1 \leq p, q \leq \infty$, $m \in \mathbb{N}$, $k \in \mathbb{N}$, *with* $1 \leq k < m$, r *be such that*

$$\frac{k}{m}\frac{1}{p} + \Big(1 - \frac{k}{m}\Big)\frac{1}{q} = \frac{1}{r}.$$

Then

$$\|\nabla^k u\|_{L^r(\mathbb{R}^N)} \preceq \|u\|_{L^q(\mathbb{R}^N)}^{1-k/m} \|\nabla^m u\|_{L^p(\mathbb{R}^N)}^{k/m}$$

for every $u \in L^q(\mathbb{R}^N) \cap \dot{W}^{m,p}(\mathbb{R}^N)$*. In particular, if* $1 \le p, q \le \infty$*, and* r *is given by*

$$\frac{1}{2p} + \frac{1}{2q} = \frac{1}{r},$$

then

$$\|\nabla u\|_{L^r(\mathbb{R}^N)} \preceq \|u\|_{L^q(\mathbb{R}^N)}^{1/2} \|\nabla^2 u\|_{L^p(\mathbb{R}^N)}^{1/2}$$

for every $u \in L^q(\mathbb{R}^N) \cap \dot{W}^{2,p}(\mathbb{R}^N)$*.*

Theorem 17.79 (Gagliardo–Nirenberg interpolation, general case). *Let* $1 \le p, q \le \infty$*,* $m \in \mathbb{N}$*,* $k \in \mathbb{N}_0$*, with* $0 \le k < m$*, and* θ*,* r *be such that*

$$0 \le \theta \le 1 - k/m$$

and

$$(1-\theta)\Big(\frac{1}{p} - \frac{m-k}{N}\Big) + \theta\Big(\frac{1}{q} + \frac{k}{N}\Big) = \frac{1}{r} \in (-\infty, 1]. \tag{17.24}$$

Then

$$|\nabla^k u|_r \preceq \|u\|_{L^q(\mathbb{R}^N)}^{\theta} \|\nabla^m u\|_{L^p(\mathbb{R}^N)}^{1-\theta} \tag{17.25}$$

for every $u \in L^q(\mathbb{R}^N) \cap \dot{W}^{m,p}(\mathbb{R}^N)$*, with the following exceptional cases:*

(i) *If* $q = r = \infty$ *and* $1 < p < \infty$*, then (17.25) fails for* $\theta \in (0, 1)$*.*
(ii) *If* $k = 0$*,* $mp < N$*,* $r < q = \infty$*, we assume that* u *vanishes at infinity.*
(iii) *If* $1 < p < \infty$ *and* $m - k - N/p$ *is a nonnegative integer, then (17.25) only holds for* $0 < \theta \le 1 - k/m$*.*

Exercise 17.80. *Let* $\Omega \subset \mathbb{R}^N$ *be an open bounded set, let* $E \subseteq \Omega$ *be a Lebesgue measurable set with finite positive measure, let* $m \in \mathbb{N}$*, and let* $1 \le p \le \infty$*. Prove that given* $u \in W^{m,p}(\Omega)$ *there exists a polynomial* $p_E(u)$ *of degree* $m - 1$ *such that for every multi-index* $\alpha \in \mathbb{N}_0^N$*, with* $0 \le |\alpha| \le m - 1$*,*

$$\int_E (\partial^\alpha u(x) - \partial^\alpha p_E(u)(x))\, dx = 0.$$

Theorem 17.81 (Poincaré's inequality in $W^{m,p}(\Omega)$). *Let* $m \in \mathbb{N}$*, let* $1 \le p < \infty$*, and let* $\Omega \subset \mathbb{R}^N$ *be a connected extension domain for* $W^{m,p}(\Omega)$ *with finite measure. Let* $E \subseteq \Omega$ *be a Lebesgue measurable set with positive measure. Then for all* $u \in W^{m,p}(\Omega)$*,*

$$\sum_{k=0}^{m-1} \|\nabla^k (u - p_E(u))\|_{L^p(\Omega)} \preceq_{E,\Omega} \|\nabla^m u\|_{L^p(\Omega)}.$$

Proposition 17.82 (Poincaré's inequality for rectangles). *Let* $m \in \mathbb{N}$*, let* $1 \le p < \infty$*, and let* $R = (0, a_1) \times \cdots \times (0, a_N) \subset \mathbb{R}^N$*. Then for all* $u \in W^{m,p}(R)$*, and every* $0 \le k \le m - 1$*,*

$$\|\nabla^k (u - p_R(u))\|_{L^p(R)} \preceq (\max\{a_1, \dots, a_N\})^{(m-k)} \|\nabla^m u\|_{L^p(R)}.$$

Exercise 17.83. *Let* R *be as in the previous theorem and let* $u \in \dot{W}^{m,p}(R)$*. Prove that* $u \in W^{m,p}(R)$*. Hint: Prove first that* $p_R(u)$ *can be replaced by* $p_{R_1}(u)$*, where* R_1 *is rectangle compactly contained in* R*.*

THEOREM 17.84 (Poincaré's inequality for convex sets). *Let $m \in \mathbb{N}$, let $1 \leq p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded convex set. Then for all $u \in W^{m,p}(\Omega)$, and every $0 \leq k \leq m-1$,*

$$\|\nabla^k (u - p_\Omega(u))\|_{L^p(\Omega)} \preceq (\operatorname{diam} \Omega)^{m-k} \|\nabla^m u\|_{L^p(\Omega)}.$$

THEOREM 17.85 (Poincaré's inequality for star-shaped sets). *Let $m \in \mathbb{N}$, let $1 \leq p < \infty$, and let $\Omega \subset \mathbb{R}^N$ be an open set star-shaped with respect to $x_0 \in \Omega$ and such that*

$$Q(x_0, 4r) \subseteq \Omega \subseteq B(x_0, R)$$

for some $r, R > 0$. Then for all $u \in W^{m,p}(\Omega)$, and every $0 \leq k \leq m-1$,

$$\|\nabla^k (u - p_\Omega(u))\|_{L^p(\Omega)} \preceq R^{m-k} (R/r)^{(N-1)(m-k)/p} \|\nabla^m u\|_{L^p(\Omega)}.$$

EXERCISE 17.86. *Let Ω be as in the previous theorem and let $u \in \dot{W}^{m,p}(\Omega)$. Prove that $u \in W^{m,p}(\Omega)$.*

THEOREM 17.87 (Stein). *Let $\Omega \subseteq \mathbb{R}^N$ be an open set with uniformly Lipschitz continuous boundary. Then for all $1 \leq p \leq \infty$ and $m \in \mathbb{N}$ there exists a continuous linear operator $\mathcal{E} : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^N)$ such that for all $u \in W^{m,p}(\Omega)$, $\mathcal{E}(u)(x) = u(x)$ for $\mathcal{L}^N$-a.e. $x \in \Omega$, and*

$$\|\mathcal{E}(u)\|_{L^p(\mathbb{R}^N)} \preceq_M \|u\|_{L^p(\Omega)},$$

$$\|\nabla^k \mathcal{E}(u)\|_{L^p(\mathbb{R}^N)} \preceq_{\varepsilon, L, M} \sum_{i=0}^{k} \|\nabla^i u\|_{L^p(\Omega)}$$

for every $1 \leq k \leq m$.

THEOREM 17.88. *Let $R := I_1 \times \cdots \times I_N \subseteq \mathbb{R}^N$, where each $I_i \subseteq \mathbb{R}$ is an open interval, let $1 \leq p, q, r \leq \infty$ be such that*

$$\frac{1}{r} = \frac{1}{2p} + \frac{1}{2q}, \tag{17.26}$$

and let $u \in L^q(R) \cap \dot{W}^{2,p}(R)$. Then

$$\|\partial_i u\|_{L^r(R)} \preceq \|u\|_{L^q(R)}^{1/2} \|\partial_i^2 u\|_{L^p(R)}^{1/2}$$

for all $i = 1, \ldots, N$, provided all the intervals I_i have infinite length, while

$$\|\partial_i u\|_{L^r(R)} \preceq \ell^{-1} (\mathcal{L}^N(R))^{1/r - 1/q} \|u\|_{L^q(R)} + \|u\|_{L^q(R)}^{1/2} \|\partial_i^2 u\|_{L^p(R)}^{1/2}$$

if all the intervals I_i have finite length and $p \leq q$, where $\ell := \min_i \mathcal{L}^1(I_i)$.

THEOREM 17.89 (Gagliardo–Nirenberg interpolation, $m = 2$). *Let $\Omega \subseteq \mathbb{R}^N$ be an open set with uniformly Lipschitz continuous boundary (with parameters ε, L, M), let $0 < \ell < \varepsilon/(4(1+L))$, and let $1 \leq p, q, r \leq \infty$ be such that $p \leq q$ and (17.26) holds. If $p < q$ assume further that Ω is bounded. Then for every $u \in L^q(\Omega) \cap \dot{W}^{2,p}(\Omega)$, if $p < q$,*

$$\|\nabla u\|_{L^r(\Omega)} \preceq \ell^{-1} (\mathcal{L}^N(\Omega))^{1/r - 1/q} \|u\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)}^{1/2} \|\nabla^2 u\|_{L^p(\Omega)}^{1/2}$$

if $p < q$, while if $p = q$,

$$\|\nabla u\|_{L^p(\Omega)} \preceq \ell^{-1} \|u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}^{1/2} \|\nabla^2 u\|_{L^p(\Omega)}^{1/2}.$$

REMARK 17.90. *We remark that when $\Omega \subset \mathbb{R}^N$ is an open, bounded, connected set with Lipschitz boundary and if $u \in L^q(\Omega) \cap \dot{W}^{2,p}(\Omega)$, where $1 \leq q \leq p$, then it follows from Poincaré's inequality that u and ∇u are in $L^p(\Omega)$ and so in all $L^r(\Omega)$ for $1 \leq r \leq p$. This is why we only considered the interesting case $p < q$ in the previous theorem.*

An important consequence of the previous theorem is the following result.

COROLLARY 17.91. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set with uniformly Lipschitz continuous boundary and let $1 \leq p \leq \infty$. Then $W^{2,p}(\Omega) = L^p(\Omega) \cap \dot{W}^{2,p}(\Omega)$ with equivalence of the norms $\sum_{k=0}^{2} \|\nabla^k u\|_{L^p(\Omega)}$ and $\|u\|_{L^p(\Omega)} + \|\nabla^2 u\|_{L^p(\Omega)}$.*

REMARK 17.92. *Note that for sets of infinite measure the previous corollary cannot be obtained using the Poincaré inequality.*

Next we consider the case $m \geq 2$.

THEOREM 17.93 (Gagliardo–Nirenberg interpolation, $m \geq 2$). *Let $\Omega \subseteq \mathbb{R}^N$ be an open set with uniformly Lipschitz continuous boundary (with parameters ε, L, M), let $0 < \ell < \varepsilon/(4(1+L))$, let $m, k \in \mathbb{N}$, with $m \geq 2$ and $1 \leq k < m$, and let $1 \leq p, q, r \leq \infty$ be such that $p \leq q$ and*

$$\frac{k}{m}\frac{1}{p} + \left(1 - \frac{k}{m}\right)\frac{1}{q} = \frac{1}{r}.$$

If $p < q$ assume further that Ω is bounded. Then for every $u \in L^q(\Omega) \cap \dot{W}^{m,p}(\Omega)$,

$$\|\nabla^k u\|_{L^r(\Omega)} \preceq \ell^{-k} (\mathcal{L}^N(\Omega))^{1/r - 1/q} \|u\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)}^{1-k/m} \|\nabla^m u\|_{L^p(\Omega)}^{k/m}$$

if $p < q$, while

$$\|\nabla^k u\|_{L^p(\Omega)} \preceq \ell^{-k} \|u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}^{1-k/m} \|\nabla^m u\|_{L^p(\Omega)}^{k/m}$$

if $p = q$.

As in the case $m = 2$, the previous inequality implies the following important consequence.

COROLLARY 17.94. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set with uniformly Lipschitz continuous boundary, let $m \in \mathbb{N}$ with $m \geq 2$ and let $1 \leq p \leq \infty$. Then $W^{m,p}(\Omega) = L^p(\Omega) \cap \dot{W}^{m,p}(\Omega)$ with equivalence of the norms $\sum_{k=0}^{m} \|\nabla^k u\|_{L^p(\Omega)}$ and $\|u\|_{L^p(\Omega)} + \|\nabla^m u\|_{L^p(\Omega)}$.*