A First Course in Fractional Sobolev Spaces
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Graduate Studies in Mathematics, AMS, 2023
December 25, 2024

For the original text I use the color Red, for corrections the color Blue, and for improvements and additions the color Cyan. Names in brackets refer to the persons who called the error to my attention (to the best of my recollection) or suggested improvements and additions.[1]

CHAPTER 1:

p. 7 Applying Theorem 1.76 should be replaced by Applying Theorem 1.3 [Antonio Villagómez].

CHAPTER 4:

p. 121 & 142 **Corollaries 4.19 and 4.46.** The hypothesis that k is integrable in those corollaries is not needed. Let $k : [0, \infty) \to [0, \infty)$ be decreasing. Define $k_n := \chi_{[0,n]} \min\{k, n\}$. Then k_n is decreasing and integrable. Hence, by Corollary 4.19,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} g(u^\sigma(x) - v^\sigma(y)) k_n(|x-y|)\, dxdy \le \int_{\mathbb{R}} \int_{\mathbb{R}} g(u(x) - v(y)) k_n(|x-y|)\, dxdy.$$

Since $k_n \le k_{n+1}$, by the Lebesgue monotone convergence theorem,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} g(u^\sigma(x) - v^\sigma(y)) k(|x-y|)\, dxdy \le \int_{\mathbb{R}} \int_{\mathbb{R}} g(u(x) - v(y)) k(|x-y|)\, dxdy. \tag{1}$$

[Tristan Bullion-Gauthier]

p. 124 **An alternative proof of Theorem 4.20.** The following proof is due to Tristan Bullion-Gauthier "Espaces de Sobolev et de Besov: réarrangements, interpolation, extrapolation", master thesis. The inequality (4.24) follows from the (improved) Corollary 4.19 (see the bullet above), taking $k(t) = \frac{1}{t^{1+sp}}$.

To prove continuity, let $u_n \to u$ in $W^{s,p}(\mathbb{R})$. Given $\delta > 0$, write

$$\begin{aligned} |u_n^\sigma - u^\sigma|^p_{W^{s,p}(\mathbb{R})} &= \int_{\mathbb{R}} \int_{(y-\delta, y+\delta)} \frac{|(u_n^\sigma - u^\sigma)(x) - (u_n^\sigma - u^\sigma)(y)|^p}{|x-y|^{1+sp}} dxdy \\ &\quad + \int_{\mathbb{R}} \int_{\mathbb{R}\setminus(y-\delta, y+\delta)} \frac{|(u_n^\sigma - u^\sigma)(x) - (u_n^\sigma - u^\sigma)(y)|^p}{|x-y|^{1+sp}} dxdy \\ &=: \mathcal{A} + \mathcal{B}. \end{aligned}$$

[1] The style of this file is inspired by http://www.hss.caltech.edu/ kcb/IDA-Errata.pdf

By Tonelli's theorem and Corollary 4.17, we have

$$\begin{aligned}\mathcal{B} &\preceq \int_{\mathbb{R}} |(u_n^\sigma - u^\sigma)(y)|^p \left(\int_{\mathbb{R}\setminus(y-\delta,y+\delta)} \frac{1}{|x-y|^{1+sp}} dx \right) dy \\ &\quad + \int_{\mathbb{R}} |(u_n^\sigma - u^\sigma)(x)|^p \left(\int_{\mathbb{R}\setminus(x-\delta,x+\delta)} \frac{1}{|x-y|^{1+sp}} dy \right) dx \\ &\preceq \frac{1}{\delta^{sp}} \int_{\mathbb{R}} |(u_n^\sigma - u^\sigma)(y)|^p dy \preceq \frac{1}{\delta^{sp}} \int_{\mathbb{R}} |(u_n - u)(y)|^p dy \to 0\end{aligned}$$

as $n \to \infty$.

On the other hand, since $u \in W^{s,p}(\mathbb{R})$, given $\varepsilon > 0$ we can find $\delta > 0$ so small that

$$\int_{\mathbb{R}} \int_{(y-\delta,y+\delta)} \frac{|u(x)-u(y)|^p}{|x-y|^{1+sp}} dxdy \le \varepsilon.$$

Since $|u_n - u|_{W^{s,p}(\mathbb{R})} \to 0$ as $n \to \infty$, using the inequality $||a| - |b|| \le |a-b|$, we have that

$$\begin{aligned}&\left| \int_{\mathbb{R}} \int_{(y-\delta,y+\delta)} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{1+sp}} dxdy - \int_{\mathbb{R}} \int_{(y-\delta,y+\delta)} \frac{|u(x)-u(y)|^p}{|x-y|^{1+sp}} dxdy \right| \\ &\le \int_{\mathbb{R}} \int_{(y-\delta,y+\delta)} \left| \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{1+sp}} - \frac{|u(x)-u(y)|^p}{|x-y|^{1+sp}} \right| dxdy \\ &\le \int_{\mathbb{R}} \int_{(y-\delta,y+\delta)} \left| \frac{|(u_n-u)(x)-(u_n-u)(y)|^p}{|x-y|^{1+sp}} \right| dxdy \to 0\end{aligned}$$

as $n \to \infty$. Hence,

$$\int_{\mathbb{R}} \int_{(y-\delta,y+\delta)} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{1+sp}} dxdy \le 2\varepsilon$$

for all n sufficiently large. Using the (improved) Corollary 4.19 (see the bullet above) with $k(t) = \frac{\chi_{[0,\delta)}(t)}{t^{1+sp}}$, we have

$$\begin{aligned}\mathcal{A} &\preceq \int_{\mathbb{R}} \int_{(y-\delta,y+\delta)} \frac{|u_n^\sigma(x)-u_n^\sigma(y)|^p}{|x-y|^{1+sp}} dxdy + \int_{\mathbb{R}} \int_{(y-\delta,y+\delta)} \frac{|u^\sigma(x)-u^\sigma(y)|^p}{|x-y|^{1+sp}} dxdy \\ &\preceq \int_{\mathbb{R}} \int_{(y-\delta,y+\delta)} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{1+sp}} dxdy + \int_{\mathbb{R}} \int_{(y-\delta,y+\delta)} \frac{|u(x)-u(y)|^p}{|x-y|^{1+sp}} dxdy \\ &\preceq \varepsilon\end{aligned}$$

for all n sufficiently large. [Tristan Bullion-Gauthier]

CHAPTER 5:

p. 174 Right underneath equation (5.36), $u \in \dot{W}^{s_1,p_1}(\mathbb{R}^N) \cap \dot{W}^{s_2,p_2}(\mathbb{R}^N)$ should be replaced by $u \in \dot{W}^{s_1,p_1}(\mathbb{R}) \cap \dot{W}^{s_2,p_2}(\mathbb{R})$. [Joshua Siktar]

CHAPTER 6:

p. 189 In line 9, $(u_n, v_n) \rightharpoonup L^p(\Omega) \times L^p(\Omega \times \Omega)$ should be replaced by $(u_n, v_n) \rightharpoonup (u, v)$ in $L^p(\Omega) \times L^p(\Omega \times \Omega)$. [Wesley Caldwell]

p. 210 In the proof of Lemma 6.48 in the last equation block, y should be replaced by z . [Wesley Caldwell]

p. 211 In the proof of Theorem 6.47, $f(x, y) = |u(x + h) - u(x)|^p$ should be replaced by $f(x, y) = \frac{|u(x+h)-u(x)|^p}{|x-y|^{N+sp}}$. [Wesley Caldwell]

CHAPTER 10:

p. 390 & 402 **Corollaries 10.12 and 10.32.** In view of the bullet above on Corollaries 4.19 and 4.46**,** the hypothesis that k is integrable in those corollaries is not needed. [Tristan Bullion-Gauthier]

p. 391 **An alternative proof of Theorem 10.13.** The same proof in the bullet above on **An alternative proof of Theorem 4.20** can be used to give an alternative proof of Theorem 10.13. In particular, the inequality (10.3) follows from the (improved) Corollary 10.12 (see the bullet above), taking $k(t) = \frac{1}{t^{N+sp}}$. [Tristan Bullion-Gauthier]

CHAPTER 13:

p. 529 In the proof of Step 1 of Proof of Theorem 13.9, I used v_μ for two difference sequences. It is confusing. On line 23 and afterwards, $v_\mu := \mu u_\mu$ should be replaced by $w_\mu := \mu u_\mu$ [Indulekha Madathil Sasi]

p. 531 In the second displayed formula after (13.16), $+f\eta v$ should be replaced by $-f\eta v$. [Indulekha Madathil Sasi]

More to come for sure... :(