

*Representations of Semisimple Lie Algebras  
in the BGG Category  $\mathcal{O}$*   
(Revisions)

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- xvi** In last paragraph of Preface, read “Stroppel” in place of “Stroppele”.
- 8** In line –11, delete the minus sign.
- 10** In line –9, replace  $L(3)$  by  $L(2)$ .
- 14** In line 7, replace 0.7 by 0.8.
- 25** In line 3, replace  $\mu$  by  $w\mu$ .
- 27–28** The proofs of (b) and (c) in 1.10 are out of focus at several points and should be revised as follows:
- 27** In line –6, replace  $W \cdot \lambda$  by  $W(\lambda + \rho)$  and  $W \cdot \mu$  by  $W(\mu + \rho)$ .
- 27** In line –4, replace “dot orbits” by “ $W$ -orbits”.
- 27** In lines –4, –3, replace “Using the assumption about  $\psi$ , take any pre-image ...” by “Using part (a), take the pre-image ...”.
- 27** In line –2, the expression should be  $\chi_\lambda(z) = (\lambda + \rho)(\psi(z))$  and similarly for  $\mu$ .
- 28** In lines 11–12, replace the three occurrences of  $\lambda$  by  $\lambda + \rho$ , and similarly for the first occurrence of  $\lambda$  in line 13.
- 29** In line –19, replace “for  $M \in \mathcal{O}$ ” by “for isomorphism class representatives  $M \in \mathcal{O}$ ”.
- 35** In line –17, replace “ $\lambda \geq 0$ ” by “ $\lambda \in \mathbb{Z}^+$ ”.
- 35** In line –6, the last symbol in the Exercise should be  $[M(\lambda)]$ .
- 55** In line –6, replace  $w' \cdot (\lambda - s_\alpha \cdot \lambda)$  by  $(w' \cdot \lambda - w \cdot \lambda)$ . Then in line –4, replace  $w' \cdot (\lambda - s_\alpha \cdot \lambda) = w(s_\alpha \cdot \lambda - \lambda)$  by  $w' \cdot \lambda - w \cdot \lambda = ws_\alpha \cdot \lambda - w \cdot \lambda$ .
- 60** In line –18, replace “the right exact functor ... is also left exact.” by “the contravariant functor ... is exact.”

- 60** In lines  $-11, -10$ , read “is dominant if  $-\lambda$  is antidominant.”
- 64** In the proof of Corollary 3.10, delete the sentence “Use induction on the length ...” at the end of the first paragraph. Then replace the last line of the proof by “After discarding these summands, induction on  $\sum c_\mu$  takes over for the remaining summand of  $P$ .”
- 65** Reword Exercise 3.11: “If  $\lambda \in \Lambda$ , prove that  $M(\lambda)$  is projective only when  $\lambda$  is dominant. [Using results of Chapter 4, this can be proved for all  $\lambda \in \mathfrak{h}^*$ .]”
- 80** In line  $-16$ , replace “at least” by “an”.
- 91** In the paragraph of the Notes starting with “Following ...”, replace the second sentence by: “Lutsyuk [204] independently develops a recursive formula for the elements of  $U(\mathfrak{n}^-)$  inducing such embeddings.”
- 95** In line  $-18$ , replace the occurrences of the symbol  $<$  by  $\uparrow$  and  $\leq$  respectively.
- 97** In line  $-6$ , replace “cases  $w = s_\beta s_\alpha$  and  $w = w_o$ ” by “case  $w = s_\beta s_\alpha$ , while the case  $w = w_o$  follows by comparing the alternating sum formula for  $\text{ch } L(w_o \cdot \lambda)$  in 2.4”
- 98** In lines  $9-10$ , replace the long sentence “By induction, ...” by the shorter “By induction,  $\mu \uparrow s_\alpha \cdot \lambda$ .”
- 101** In lines  $-9, -8$ , reword the sentence: “Since  $M(\lambda)$  has finite length, it follows that ...”
- 101** In line  $-5$ ,. replace “3.14” by “Exercise 3.14”.
- 118** In the last two lines, replace “case  $k = m$ ” by “case  $k = m - 1$ ” and “let  $k < m$ ” by “let  $k < m - 1$ ”.
- 121** In line 15, the second  $M(w \cdot 0)$  should be  $M(w' \cdot 0)$ .
- 121** The ideas sketched in the last paragraph for the proof of the theorem stated at the top of page 122 are probably not adequate for the purpose. It is safer to rely on the arguments given by Mazorchuk [214], though it would be interesting to minimize the prerequisites for the theorem.
- 122** In line  $-3$ , delete the extra right parenthesis.

**124** In line 8 of (1) in the proof, the first Ext term in the exact sequence should be  $\text{Ext}_{\mathcal{O}}$ .

**130-131** Revise the last paragraph of 7.1: At the end of the existing text in line -9, insert the first sentence at the top of page 131 “The rationale ...” In line -8, replace “a couple of” by “some”. Then reword the proposition as follows: “Let  $\lambda, \mu \in \mathfrak{h}^*$ .

(a) The exact functor  $T_{\lambda}^{\mu}$  commutes with the duality functor.

(b)  $T_{\lambda}^{\mu}$  takes projective modules to projective modules.

(c)  $T_{\lambda}^{\mu}M(\lambda)$  has a standard filtration involving  $M(\mu)$  as a subquotient.”

For the proof, follow the current wording for the first two parts and the current wording on page 131 for (c) with obvious modifications (omitting parentheses around the last sentence):

(a) “To see that ... in that section.”

(b) “This follows from ... to a direct summand.”

(c) “When  $M = M(\lambda)$ , a standard filtration ... This module also has a standard filtration, thanks to Proposition 3.7(b).”

**134** In line -15, begin (c) with: “Suppose  $\Phi_{[\lambda]} = \Phi_{[\mu]} = \Phi_{[\lambda+\mu]}$ .” Then in line -6, begin (c) with: “By assumption,  $\Phi_{[\lambda+\mu]} = \Phi_{[\lambda]}$ .”

**136** In line 14, replace  $C'$  by  $\overline{C'}$ .

**136** In line -14, delete “and (7)”.

**136** In line -1, read:  $\xi = \lambda^{\natural} + \nu = \mu^{\natural}$ .

**137** Delete the bottom text on the page, starting at line -10: “But it ...”

**141** Replace line 5 by “ $\alpha \in w\Phi_{F'}^0$ , with  $F'$  the facet of  $\mu$ .” Then at the end of line 6, replace  $F$  by  $F'$ , and in line 8 replace  $F$  by  $F'$ .

**143** In line 3, replace  $W_{\mu}$  by  $W_{\mu}^{\circ}$ .

**144** In line 7, replace “Corollary 7.6” by “Proposition 7.1(c)”.

**144** In line -3, replace  $\mathcal{O}$  by  $\mathcal{O}_{\chi_{\mu}}$ .

**146** In line 1, replace  $P(w_{\lambda} \cdot \lambda)$  by  $P(w_{\lambda}w_{\mu^{\circ}} \cdot \lambda)$ .

**147** In line 7, replace “Corollary 7.6” by “Proposition 7.1(c)”.

- 147** In line  $-2$  expand “ $> 0$ ” to “ $> 0$  if  $T_\lambda^\mu L(w' \cdot \lambda) \neq 0$ ”.
- 150** In line  $-1$ , replace “bonce” by “once”.
- 151** In line 2, replace “just one” by “no”. Then in lines  $-16, -15$ , remove the sentence “Using the projective property ...”
- 151-152** In lines 151 :  $-2$  and 152:1, replace  $L(\lambda_\circ)$  by  $M(\lambda_\circ)$ .
- 163** In line 14, replace “which means” by “which implies”. Then in the lemma replace “ $\ell(w) - \ell(x) = 1$ ” by “ $w = rx$  for some reflection  $r$ ”.
- 176** In the second paragraph of (2), lines 11–15 are seriously out of focus in the discussion of  $C_3$ . The reader should work out the details as an exercise, keeping in mind from 8.4(2) how to relate the composition factor multiplicities of Verma modules with the values at 1 of certain KL polynomials. Here  $x < w$  in  $W$  is equivalent to  $w_\circ w < w_\circ x$ , while  $\ell(w_\circ w) = \ell(w_\circ) - \ell(w) = 9 - \ell(w)$ , etc.
- 177** In line 3, replace  $\ell(w_\circ) + 1$  by  $\ell(w_\circ)$ .
- 178** In line 9, replace  $\mathfrak{h}$  by  $\mathfrak{h}^*$ .
- 191** In line 6, read ”Conjecture”.
- 200** In the third line of 9.15, read “more refined partition”.
- 233** My reading of Soergel’s notation was too hasty, so several lines must be modified:  
 In (1), replace the term on the right side by  $[M(w \cdot \lambda) : L(x \cdot \lambda)]$ , and in line 15, replace “Theorem 3.9” by “Theorem 3.7”.  
 In the statement of Soergel’s Theorem, replace  $P_{x,w}(1)$  by  $P_{w_\circ w, w_\circ x}(1)$ . Then in the Corollary, delete all but the first sentence.
- 243** Revise the third paragraph of 12.7, starting with line 4:  
 “and  $N_w = \mathbb{C}[y_\alpha]$ . The algebra  $N_w$  is  $\mathbb{Z}$ -graded: the standard grading of  $U$  by  $\Lambda_r$  induces a  $\mathbb{Z}$ -grading on  $U$  if all simple root vectors are placed in  $U_1$ . The graded dual  $N_w^*$ , with  $n$ th graded piece the dual space  $(N_w)_{-n}^*$ , then becomes a  $\mathbb{Z}$ -graded  $N_w$ -bimodule. Define  $S_w := U \otimes_{N_w} N_w^*$ . Somewhat miraculously, ...”
- 265** In line  $-4$ , replace “on  $W$ ” by “on  $V$ ”.
- 269** In line  $-15$ , replace  $W_I$  by  $W_1$ .

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