GSM 95 QUANTUM MECHANICS FOR MATHEMATICIANS

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Here I correct the formulation[1] and the proof of Proposition 1.1 and Corollary 1.2 in §1 of Chapter 2 and list some other smaller corrections[2]. I intend to update this file from time to time and so welcome further comments.

1. Errata

Page 26, Line 5 from the bottom. Replace $\boldsymbol{q}' = F_*(\boldsymbol{q})\boldsymbol{v}$ by $\dot{\boldsymbol{q}}' = F_*(\boldsymbol{q})\dot{\boldsymbol{q}}$.

Page 31, Problem 1.23. Replace "$i : TM \simeq T_V(TM)$" by "$i : \pi^*(TM) \simeq T_V(TM)$, where $\pi^*(TM) \to TM$ is the pullback of the tangent bundle TM of M under the map π."

Page 51, Problem 2.12, line 5. Replace $\mathcal{O}_u$ by $T_u\mathscr{M}$.
Page 70. Replace the formulation of Proposition 1.1 by the following.

Proposition 1.1. *Suppose that an observable $A \in \mathscr{A}$ and the state $M \in \mathscr{S}$ are such that $\langle A|M\rangle$ exists and $\operatorname{Im} B \subseteq D(A)$, where B is a positive square root of M. Then $BAB \in \mathscr{S}_1$ and*

$$\langle A|M\rangle = \operatorname{Tr} BAB.$$

If $B \in \mathscr{S}_1$, then $AM \in \mathscr{S}_1$ and

$$\langle A|M\rangle = \operatorname{Tr} AM.$$

In particular, if $M = P_\psi$ and $\psi \in D(A)$, then

$$\langle A|M\rangle = (A\psi, \psi) \quad \textit{and} \quad \langle A^2|M\rangle = \|A\psi\|^2.$$

Page 71. Replace the proof of Proposition 1.1 by the following.

Proof. First note that $B\mathsf{P}_A(E)B \in \mathscr{S}_1$ and

$$\operatorname{Tr} B\mathsf{P}_A(E)B = \operatorname{Tr} \mathsf{P}_A(E)B^2.$$

Date: February 7, 2017.
[1] I am grateful to Dr. Arnold Neidhardt for useful remarks and suggestions.
[2] I am grateful to J. Jaracz, C. Meneses-Torres and J. Walsh for their input.

Indeed, since $B\mathsf{P}_A(E)B$ is positive, it is sufficient to verify that

$$\sum_{n=1}^{\infty}(B\mathsf{P}_A(E)Be_n, e_n) < \infty$$

for some orthonormal basis in $\mathscr{H}$. Choosing it to be the basis which contains the orthonormal set $\{\psi_n\}_{n=1}^N$, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}(B\mathsf{P}_A(E)Be_n, e_n) &= \sum_{n=1}^{N}(B\mathsf{P}_A(E)B\psi_n, \psi_n) = \sum_{n=1}^{N}\alpha_n(\mathsf{P}_A(E)\psi_n, \psi_n)\\ &= \operatorname{Tr}\mathsf{P}_A(E)M < \infty,\end{aligned}$$

where we used that $M = B^2$.

Now let $\{e_n\}_{n=1}^{\infty}$ be an arbitrary orthonormal basis for $\mathscr{H}$. Since

$$\mu_A(E) = \operatorname{Tr}\mathsf{P}_A(E)B^2 = \operatorname{Tr}B\mathsf{P}_A(E)B = \sum_{n=1}^{\infty}(B\mathsf{P}_A(E)Be_n, e_n)$$

and we get for every $E \in \mathscr{B}(\mathbb{R})$,

$$\mu_A(E) = \sum_{n=1}^{\infty}\mu_n(E),$$

where μ_n are finite measures on $\mathbb{R}$ defined by $\mu_n(E) = (\mathsf{P}_A(E)Be_n, Be_n) = \|\mathsf{P}_A(E)Be_n\|^2$. Since $D(AB) = \mathscr{H}$ and operator AB is closed, by the closed graph theorem $AB \in \mathscr{L}(\mathscr{H})$. It follows from the spectral theorem that

$$\begin{aligned}\sum_{n=1}^{\infty}(BABe_n, e_n) &= \sum_{n=1}^{\infty}(ABe_n, Be_n) = \sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\lambda d\mu_n(\lambda)\\ &= \int_{-\infty}^{\infty}\lambda d\mu_A(\lambda) = \langle A|M\rangle < \infty.\end{aligned}$$

Thus $BAB \in \mathscr{S}_1$ and $\langle A|M\rangle = \operatorname{Tr}BAB$. If $B \in \mathscr{S}_1$, then $\operatorname{Tr}BAB = \operatorname{Tr}AB^2 = \operatorname{Tr}AM$. In particular, when $M = P_\psi$ and $\psi \in D(A)$,

$$\langle A|M\rangle = \int_{-\infty}^{\infty}\lambda d(\mathsf{P}_A(\lambda)\psi, \psi) = (A\psi, \psi).$$

Finally, from the spectral theorem and the change of variables formula we get

$$\|A\psi\|^2 = \int_{-\infty}^{\infty}\lambda^2 d(\mathsf{P}_A(\lambda)\psi, \psi) = \int_{0}^{\infty}\lambda d(\mathsf{P}_{A^2}(\lambda)\psi, \psi) = \langle A^2|M\rangle.$$

□

Page 71. Replace the formulation of Corollary 1.2 by the following.

Corollary 1.2. *If $\langle A^2|M\rangle < \infty$, then $BAB \in \mathscr{S}_1$ and $\langle A|M\rangle = \operatorname{Tr}BAB$.*

Page 71. Replace the proof of Corollary 1.2 by the following.

Proof. Since

$$\int_{-\infty}^{\infty} \lambda^2 d\mu_A(\lambda) = \int_0^{\infty} \lambda d\mu_{A^2}(\lambda) = \langle A^2|M\rangle < \infty,$$

for every orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for $\mathscr{H}$ we have by the spectral theorem

$$\sum_{n=1}^{\infty} \|ABe_n\|^2 = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \lambda^2 d\mu_n(\lambda) = \int_{-\infty}^{\infty} \lambda^2 d\mu_A(\lambda) < \infty.$$

Thus $AB \in \mathscr{L}(\mathscr{H})$ and the result follows from the proof of Proposition 1.1. □

Page 71, Line 4 from the bottom. Replace $A_n = f_n(A)$ by $A_n = Af_n(A)$.

Page 76, Line 17, after formula (1.9). Replace "strongly one-parameter group" by "strongly continuous one-parameter group".

Page 82 In the definition of the Heisenberg algebra the relations $[e^k, e^l] = 0$, $[f_k, f_l] = 0$ for all $k, l = 1, \dots, n$ are understood.

Page 83, Line 2 from the bottom. Replace 'Lie algebra' by 'Lie group'.

Page 89, Lines 14-16. Replace T by B.

Page 95, Last line. Replace $\sqrt{\frac{m}{t}}$ by $\frac{m}{t}$.

Page 103, Line 12. Replace 'on $\mathbb{R}^2 \simeq \mathbb{C}$' by '$\mathbb{R}^2 \simeq \mathbb{C}$'.

Page 103, In formulas (2.30) replace p by $\frac{p}{m}$ and in formulas below (2.32) replace p_0 by $\frac{p_0}{m}$.

Page 111, Problem 2.14. Replace $\mathrm{sl}(n, \mathbb{C})$ by $\mathrm{gl}(n, \mathbb{C})$.

Page 112, Line 9 from the bottom. Replace 'harmonic operator' by 'harmonic oscillator'.

Page 140, Line 5. Replace $\mathcal{A}_t = \mathbb{C}[[t]] \otimes_{\mathbb{C}} \mathcal{A}$ by $\mathcal{A}_t = \mathcal{A}[[t]]$.

Page 156, Second line of equations in part (ii) of Theorem 2.1. Replace

$$\lim_{x \to -\infty} e^{ikx} f_1'(x,k) = -ik \quad \text{by} \quad \lim_{x \to -\infty} e^{ikx} f_2'(x,k) = -ik.$$

Page 161, In the inequality in the sixth line replace $\dfrac{1}{4\varkappa}e^{-\varkappa A}$ by $\dfrac{1}{4\varkappa_n}e^{-\varkappa_n A}$.

Page 165, Line 4. Below formula (2.24) add "Recall that $C_0^2(\mathbb{R})$ is the space of twice continuously differentiable functions on $\mathbb{R}$ with compact support."

Page 168, Line 15. Replace $u(x,k)$ by $u_1(x,k)$.

Page 169, Line 14 from the bottom. Replace "...the normalization condition (2.14)..." by "...the analog of normalization condition (2.14)..."

Page 169, Line 13 from the bottom. In the right hand side of the formula, replace δ_{jl} by $2\pi\delta_{jl}$.

Page 179, Line 4. Replace "...are the corresponding normalized eigenfunctions..." by "...are the analogs of normalized eigenfunctions..."

Page 179, Line 7. Replace "...are the corresponding normalized eigenfunctions..." by "...are the analogs of normalized eigenfunctions..."

Page 179, Line 10. In the right hand side of the formula, replace

$$\int_0^\infty \quad \text{by} \quad \frac{1}{2\pi}\int_0^\infty$$

Page 179, Line 11. In the right hand side of the formula, replace

$$\int_c^\infty \quad \text{by} \quad \frac{1}{2\pi}\int_c^\infty$$

Page 180, Line 6. In the right hand side of the formula, replace

$$\int_0^\infty \quad \text{by} \quad \frac{1}{2\pi}\int_0^\infty$$

Page 181, Line 12. Replace $M_3 = Q_2P_3 - Q_3P_2$ by $M_3 = Q_1P_2 - Q_2P_1$.

Page 183, Line 12 from the bottom. Remove 'following'.

Page 189, Second line from the bottom. Delete "with the boundary condition $f_l(0) = 0$".

Page 189, Last line. Add "The operator H_0 should be supplemented by the boundary condition $f_0(0) = 0$, whereas in the case $l > 0$ no boundary conditions are required."

Page 196, In the left hand side of the first equation, replace ω_{mn} by ω_{nm} and also replace $m < n$ by $m > n$.

Page 207, Line 7 from the bottom. Replace "pure" by "purely".

Page 207, Equation (6.10). Replace $\varphi(q)$ by $\log\varphi(q)$.

Page 207, Equation (6.12). In the left-hand side replace $\left(\frac{\partial}{\partial t}+\frac{\partial S_0}{\partial q}\right)$ by $\left(\frac{\partial}{\partial t}+\frac{1}{m}\frac{\partial S_0}{\partial q}\frac{\partial}{\partial q}\right)$.

Page 210, Line 3. Replace $\sigma'^2 \ll \hbar\sigma''$ by $\hbar|\sigma''| \ll (\sigma')^2$.

Page 225, Line 6 from the bottom. Replace $S_3(S_3+I)$ by $S_3(S_3+\hbar I)$.

Page 228, Line 5 from the bottom. Replace *exchange energy* by *exchange term.*

Page 229, Equation (3.5). Replace $s(s+1)\Phi$ by $\hbar^2 s(s+1)\Phi$.

Page 233, Equation (3.8). Replace $s(s+1)\chi_{mj}$ by $\hbar^2 s(s+1)\chi_{mj}$ and $m\chi_{mj}$ by $\hbar m\chi_{mj}$.

Page 254. Clarification of the periodic boundary conditions in equation (2.16). Remove seven lines from the bottom of page 254 and first three lines on the top of page 255, starting from "Indeed" up to "periodic boundary conditions", and replace them by the following text.

Indeed, it immediately follows from the Euclidean version of (2.12) that

$$\operatorname{Tr}\tilde{U}(\Delta t)^n = \frac{1}{\pi\hbar}\int_{\mathbb{C}} U_n(\bar{a},a;-iT)e^{-\frac{1}{\hbar}\bar{a}a}d^2a$$

$$= \int\cdots\int_{\mathbb{C}^n} e^{-\frac{1}{\hbar}\sum_{k=1}^{n}(-\bar{a}_k(a_{k-1}-a_k)+H(\bar{a}_k,a_{k-1})\Delta t)}e^{-\frac{1}{\hbar}\bar{a}_n a_n}\prod_{k=1}^{n}\frac{d^2a_k}{\pi\hbar},$$

where now variables of integration $a=a_0$ and $\bar{a}=\bar{a}_n$ are complex-conjugated. This implies $a_0=a=a_n$, $\bar{a}_0=\bar{a}=\bar{a}_n$, and in the limit $n\to\infty$ one gets formula (2.16).

Page 258, Line 5 from the top and Line 11 from the bottom. Replace $(A_{n-1}\boldsymbol{p},\boldsymbol{p})$ by $(A_{n-1}^{-1}\boldsymbol{p},\boldsymbol{p})$.

Page 258, Line 22. Replace

$$B_{1n-1}=B_{n-11}=\frac{1}{a_n}=\frac{\sin n\theta}{\sin\theta}$$

by

$$B_{1n-1} = B_{n-11} = \frac{1}{a_{n-1}} = \frac{\sin n\theta}{\sin\theta}$$

Page 276, Line 13. Replace $\det(T(\lambda) - I_2)$ by $\det_2(T(\lambda) - I_2)$.

Page 292, Lines 14-15. Replace $(1 + \varepsilon^2\alpha_i^2)$ by $(1 + 2\varepsilon^2\alpha_i^2)$.

Page 297, Line 18. The statement of Theorem 2.1 should read: "... is the conditional Wiener measure with the diffusion coefficient $D = \dfrac{\hbar}{2m}$."

Page 300, Line 10. Replace 'bounded below' by 'bounded'.

Page 300, Line 12. Replace $L_\hbar(\boldsymbol{q}', t'; \boldsymbol{q}, t)$ by "$\tilde{L}_\hbar(\boldsymbol{q}', t'; \boldsymbol{q}, t)$ for the operator $\tilde{H} = H_0 - V$".

Page 300, Equation (2.9). Replace $L_{i\hbar+\varepsilon}(\boldsymbol{q}', t'; \boldsymbol{q}, t)$ by $\tilde{L}_{i\hbar+\varepsilon}(\boldsymbol{q}', t'; \boldsymbol{q}, t)$.

Page 332, Line 18. Replace $\tilde{\Phi}_{\boldsymbol{\alpha}}(\boldsymbol{\theta})$ by $\tilde{\Phi}_{\boldsymbol{\alpha}}(\bar{\boldsymbol{\theta}})$.

Page 335, First line of the proof of Lemma 4.4. Replace $\hat{H}$ by H.

Page 335, Last equation in the proof of Lemma 4.4. In the second equality, replace $U(\Delta t)$ by $\tilde{U}(\Delta t)$.

Page 337, Line 4. Replace $\boldsymbol{\alpha}(T)$ by $\boldsymbol{\alpha}_N$.

Page 337, Lines 7-8. Clarification of the periodic boundary conditions in the formula for $\operatorname{Tr}_s e^{-iTH}$. Remove the sentence starting with "Namely, since $\bar{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}$..." and continue the previous sentence as follows: "... as in Section 2.4 of Chapter 5, where now one uses Theorem 4.1 and Problem 4.2."

Page 338, Last equation in the proof of Theorem 4.2. Replace $\lim\det A_N$ by $\lim\limits_{N\to\infty}\det A_N$.

Page 344, Third line. Replace $\subset$ by $\subseteq$.

Page 346, Line 16. Replace 'necessary' by 'usually'.

Page 346, Line 18. Replace $d\alpha_{n-1} = 0$ by $i_V\alpha_1 = 0,\ d\alpha_{n-1} = 0$.

Page 346, Line 12 from the bottom. Replace 'an a compact manifold' by 'on a compact manifold'.

Page 348, Line 8 from the bottom. Replace 'the method of steepest descent' by 'the Laplace method'.

Page 349, Line 12 from the bottom. Replace $\mathcal{L}M$ by $\mathcal{L}(M)$.

Page 362, Line 6 from the bottom. Replace 'local coordinantes' by 'local coordinates'.

Page 368, Line 13 from the bottom. Replace $\mathbb{R}^n \times \mathscr{H}_F$ by $\mathbb{C}^n \times \mathscr{H}_F$.

Page 368, Line 10 from the bottom. Replace S by $\mathcal{S}$.

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