Revision v2.0, Chapter I
Foundations of Geometry in the Plane

The primary goal of this book is to display the richness, unity, and insight that the concept of ***symmetry*** brings to the study of geometry. To do so, we must start from a solid grounding in elementary geometry. This chapter provides that necessary background.

A rigorous treatment of foundational geometry is hardly elementary: it requires a careful development of appropriate concepts, axioms, and theorems. We will not give all the details for such a full axiomatic treatment. We will, however, structure the discussion in this chapter to outline such a treatment. The logical clarity of this approach helps in understanding and remembering the material.[1]

There are various ways to build a foundation for geometry. Our approach is that of ***metric geometry***, which means that *distance between points* and *angular measure* are central organizing concepts. These concepts, in turn, are dependent on the basic properties of $\mathbb{R}$, the *real number system*. We thus begin our discussion with some comments about the real numbers.

§I.1 The Real Numbers

We denote the set of **real numbers** with the symbol $\mathbb{R}$. Other important number systems are $\mathbb{Z}$, the set of **integers**; $\mathbb{N}$, the set of **natural numbers** (non-negative integers); and $\mathbb{Q}$, the set of **rational numbers** (quotients of integers). Later we will also discuss $\mathbb{C}$, the set of **complex numbers**.

Intuitively the real numbers correspond to the points on a straight line. Given a line ℓ, fix a unit of length (e.g., an inch), designate a point $\mathcal{O}$ on ℓ as the *origin*, and specify the "positive" and "negative" sides of ℓ. Then every point p of ℓ respresents one unique real number x_p, where x_p is the distance of p to $\mathcal{O}$ (positive or negative depending on the side of $\mathcal{O}$) — this is shown in Figure 1.1. The algebraic operations of addition and multiplication of real numbers then correspond to geometric movements on the line.

[1] We closely follow the development of geometry found in Edwin Moise, *Elementary Geometry from an Advanced Standpoint*, Third Edition, Addison Wesley Publishers. This book is the primary reference for the basic geometry to be used in *Continuous Symmetry*.

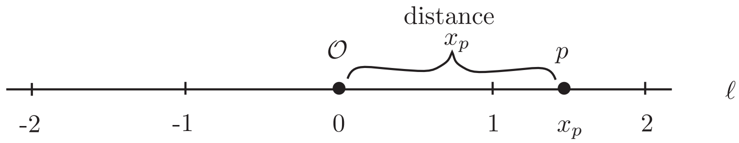


Figure 1.1. The real number line.

Using the number line, the *order* operation on real numbers, $x < y$, read "x is less than y," means that the point corresponding to x occurs "to the left" of the point corresponding to y. The inequality $x \leq y$, read "x is less than or equal to y," means either $x < y$ or $x = y$.

We will assume all the standard algebraic properties of the real numbers, i.e., all the standard properties of addition, multiplication, subtraction, division, and order. For example, if $x + r = y + r$, then $x = y$, or if $x < y$ and $r > 0$, then $rx < ry$. Do not underestimate the importance or the depth of this assumption! We base our development of geometry on the real number system, a system whose existence is non-trivial to establish and whose properties are sophisticated. A proper study of the real number system belongs to the mathematical subject known as *analysis*.

Using the order relations, we define various types of **intervals** for real numbers $a < b$:

$$\begin{aligned}
\text{bounded closed interval:} \quad & [a, b] = \{\text{all } x \text{ such that } a \leq x \leq b\}, \\
\text{bounded open interval:} \quad & (a, b) = \{\text{all } x \text{ such that } a < x < b\}.
\end{aligned}$$

We allow *open* endpoints with $a = -\infty$ or $b = \infty$. However, in those cases the intervals are *unbounded*.

The real numbers include all the natural numbers 1, 2, 3, $\ldots$. An important property of this inclusion is called the ***Archimedean Ordering Principle**:*

For any real number x there exists a natural number n greater than x.

One simple but highly important consequence of this property is that for any positive real number $\epsilon > 0$, no matter how small, there exists a natural number n such that $1/n < \epsilon$ (Exercise 1.1). Thus the fraction $1/n$ can be made "arbitrarily small" by choosing n sufficiently large.

Completeness of the Real Number System. An intuitive understanding of real numbers as developed in, say, advanced secondary school algebra, will suffice for most of our work. However, there is one far deeper fact that we will need at certain times: the real number line "has no holes." When rigorously formulated, this property is known as the ***completeness*** of the real number system. Some readers may wish to defer this sophisticated concept until needed later in the text.

Such readers should now skip to §I.2.

There are various equivalent ways to describe completeness — we will give a description that is easy to picture using closed intervals. A sequence of bounded, closed intervals $[a_1, b_1]$, $[a_2, b_2]$, ... is said to be **nested** if each interval contains the next one as a subset. Written in set notation, this means

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots \supseteq [a_n, b_n] \supseteq \cdots .$$

The real number system $\mathbb{R}$ is said to be **complete** because every such nested sequence of bounded closed intervals has at least one real number x that belongs to *all* the intervals:

Theorem 1.2. ***The Completeness of*** $\mathbb{R}$**.**
For any nested sequence of bounded closed intervals in $\mathbb{R}$ there will always be at least one real number x that belongs to all the intervals (i.e., x is in the intersection of all the intervals).

This theorem is illustrated in Figure 1.3.

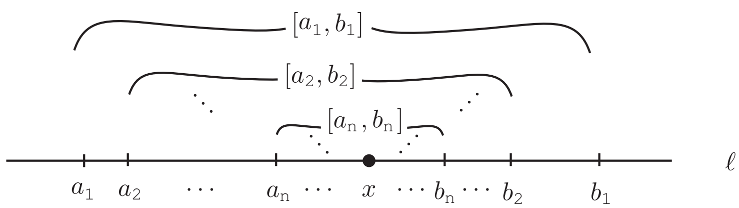


Figure 1.3. All the nested intervals $[a_n, b_n]$, $n = 1, 2, 3, \ldots$, contain x.

If the nested intervals are *not* closed, then there might or might *not* be a point common to all the intervals. This is examined in Exercise 1.3.

The intuitive meaning of completeness is that the real number line "has no holes." If you have not studied a rigorous formulation of the real number system (such as in an introductory analysis course), it's probably hard to appreciate the importance of completeness. In fact, it is of critical importance in much of mathematics — it guarantees that real numbers exist when and where we need them. An example of a number system which is *not* complete is $\mathbb{Q}$, the set of rational numbers (fractions). It would, for example, be very difficult to develop calculus with just rational numbers — there are too many "holes" because the rationals are not complete. These ideas are more fully explored in the exercises.

More on Completeness: Bolzano's Theorem. There are several properties of the real number system that are implied by completeness. One such property we will use later in the book is *every bounded sequence in $\mathbb{R}$ has a convergent subsequence*. This is known as *Bolzano's Theorem*. We now explain this result.

A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is termed **bounded** if all the elements of the sequence are contained in a bounded interval, i.e., there exists a bounded interval $[a, b]$ such that $a \leq x_n \leq b$ for all n. A sequence $\{x_n\}_{n=1}^{\infty}$ is said to **converge to x** if the terms in the sequence become arbitrarily close to x as the index n becomes large.

A sequence can be bounded but not converge: a simple example is the sequence

$$\{1, -1, 1, -1, 1, -1, \dots\}.$$

However, this sequence does have convergent subsequences; one example is $\{1, 1, 1, \dots\}$. In fact, the completeness of the real number system implies that *every* bounded sequence has a convergent subsequence. This is *Bolzano's Theorem.*

Theorem 1.4. ***Bolzano's Theorem.***

Every bounded sequence in $\mathbb{R}$ *has a convergent subsequence.*

The proof of Bolzano's Theorem, showing it to be a consequence of the completeness of the real number system as formulated in Theorem 1.2, is given in §14.

Exercises I.1

"Why," said the Dodo, "the best way to explain it is to do it." Lewis Carroll

Exercise 1.1.

(a) Show that for any positive real number $\epsilon > 0$, no matter how small, there exists a natural number n such that $1/n < \epsilon$.
Hint: Use the *Archimedean Ordering Principle* with $x = 1/\epsilon$.

(b) For any real number x' show there exists an integer n' less than x'.
Hint: Use the *Archimedean Ordering Principle* with $x = -x'$.

Exercise 1.2.

(a) Given any real number x, show there exists a *smallest* integer n_0 such that $x < n_0$. *Hint:* Use *Archimedean order* to show there exists an integer N such that $x < N$. Then use Exercise 1.1b to show that the set of integers $\{k \mid x < k \leq N\}$ is non-empty and *finite.* Any finite set of real numbers must have a smallest member.

(b) Suppose x_1 and x_2 are two real numbers such that $x_1 < x_2$. Prove there exists a rational number $r = n/m$ such that $x_1 < r < x_2$.
Outline: First show there exists a positive integer m large enough so that $1/m < x_2 - x_1$. Then show there exists a *smallest* integer n such that $x_1 < n/m$. To verify that $r = n/m$ is what you need, you must only verify $r < x_2$.

(c) Suppose x_1 and x_2 are two real numbers such that whenever r_1 and r_2 are rational numbers satisfying $r_1 < x_1 < r_2$, then $r_1 < x_2 < r_2$

is also true. Prove that $x_1 = x_2$. This strange looking result is actually quite useful — indeed, it will be used later in the text. *Hint:* It is easiest to use a *proof by contraposition*: begin by supposing $x_1 < x_2$, then prove there exist rational numbers r_1, r_2 such that $r_1 < x_1 < r_2$ and $r_2 \leq x_2$.

The remaining exercises examine the notion of *completeness* of the real number system.

Exercise 1.3.
Show that a nested sequence of bounded *open* intervals *might* or *might not* have a common point. *Hint:* Show that the nested sequence of bounded open intervals $A_n = (-1/n, 1/n)$ *does* have a common point (simply identify the point!), while the sequence $B_n = (0, 1/n)$ *does not* have a common point (show that any real number x must be excluded from at least some of the B_n intervals).

Exercise 1.4.
Show that $\mathbb{Q}$, the rational number system, is not complete. *Hint:* For any two real numbers $a < b$ define the *rational* closed interval $[a, b]_{\mathbb{Q}}$ to be the set of all rational numbers in $[a, b]$, i.e., $[a, b]_{\mathbb{Q}} = [a, b] \cap \mathbb{Q}$. Then consider the collection of rational bounded closed intervals $\{I_n\}_{n=1}^{\infty}$ where $I_n = [\sqrt{2}, \sqrt{2} + 1/n]_{\mathbb{Q}}$ for each positive integer n.

Exercise 1.5.

(a) Consider the sequence

$$\{\frac{1}{2}, -\frac{1}{2}, \quad \frac{2}{3}, -\frac{2}{3}, \quad \frac{3}{4}, -\frac{3}{4}, \quad \frac{4}{5}, -\frac{4}{5}, \ldots\}.$$

Show that this sequence does not converge in the real number system. However, illustrate the truth of Bolzano's Theorem for this sequence by giving examples of at least two subsequences that do converge in $\mathbb{R}$.

(b) Consider the infinite decimal expansion for the square root of 2:

$$\sqrt{2} = 1.41421356....$$

We use this expansion to define a sequence as follows:

$$\{1, -1, 1.4, -1.4, 1.41, -1.41, 1.414, -1.414, 1.4142, -1.4142, \ldots\}.$$

Show that this sequence does not converge in the real number system. However, illustrate the truth of Bolzano's Theorem for this sequence by giving examples of at least two subsequences that do converge in $\mathbb{R}$.

(c) The sequences in parts (a) and (b) are also sequences in $\mathbb{Q}$, the rational number system. However, show that only one of these sequences has a subsequence that converges in $\mathbb{Q}$. This shows that Bolzano's Theorem is true only for some sequences in $\mathbb{Q}$, but not all of them. What is the property that $\mathbb{R}$ possesses but $\mathbb{Q}$ lacks that allows this to happen?

Exercise 1.6. ***Infinite Decimal Expansions.***

What is the meaning of an infinite decimal such as .1121231234...? In this exercise you will see that any infinite decimal denotes a unique real number *because the real number system is complete!* Suppose x_1, x_2, x_3, $\ldots$ is a sequence of digits (integers between 0 and 9). For each positive integer n define the two finite decimals

$$a_n = .x_1x_2\ldots x_n \quad \text{and} \quad b_n = .x_1x_2\ldots x_n + 1/10^n.$$

This means that a_n and b_n are equal to

$$a_n = \frac{x_1}{10} + \frac{x_2}{100} + \cdots + \frac{x_n}{10^n} \quad \text{and} \quad b_n = \frac{x_1}{10} + \frac{x_2}{100} + \cdots + \frac{x_n+1}{10^n}.$$

(a) Show that a_n and b_n are both rational numbers.

(b) Show that $[a_1, b_1]$, $[a_2, b_2]$, $\ldots$ is a nested sequence of bounded closed intervals.

(c) Use completeness to show that there is a *unique* real number x in all of the intervals $[a_1, b_1]$, $[a_2, b_2]$, $\ldots$. This is the real number corresponding to the infinite decimal expansion given by the digits x_1, x_2, x_3, x_4, $\ldots$, i.e., $x = .x_1x_2x_3x_4\ldots$.
Hint: The existence of a real number x in all the intervals is immediate from completeness. Uniqueness is a little less trivial. Suppose y is a second number in all the intervals, and let n be a positve integer large enough so that $|x - y| > 1/10^n$. Could x and y both be in the interval $[a_n, b_n]$?

§I.2 The Incidence Axioms

A formal description of the real number system is often given using the *axiomatic method*. This begins with the real numbers as **undefined objects** and then states **axioms** (also called *postulates* or *assumptions*) which define the desired properties of real numbers. Commonly there are a set of *field axioms* (describing the basic properties of addition and multiplication), a set of *order axioms* (describing the behavior of $<$ and $>$), and a *completeness axiom* (equivalent to the nested intervals property of §1). Then all desired results about real numbers are proven from these axioms (or from results previously derived from the axions).

When using the axiomatic method for a mathematical theory, the desire is always to state the minimal number of axioms possible. In particular, if

Axiom D can be proven from Axioms A, B, and C, then Axiom D should be removed from the list of axioms and relabeled as a **theorem** or **proposition**.

It is also important to demonstrate that a proposed set of axioms is *consistent*, meaning that the axioms don't ultimately contradict one other. Verifying consistency generally requires exhibiting a concrete example, called a **model** for the axiom system, in which all the axioms are indeed satisfied.

An axiomatic approach is necessary for the rigorous development and understanding of geometry. In this chapter we therefore develop a set of axioms sufficient to support Euclidean geometry in the plane and explain the meaning and important consequences of the axioms. Variations of this axiom system will describe different geometries — that will be explained in Volume II of this text. The axiom system of this chapter will show Euclidean geometry developed in a logical order and provide the tools needed for the remainder of the book. We will, however, develop only those results which are necessary for subsequent work and will present only some of the proofs.[2]

The Undefined Objects.

The (**Euclidean**) **plane** is a set $\mathcal{E}$ consisting of **points**. There is also a collection $\mathcal{L}$ of special subsets of $\mathcal{E}$ called (**straight**) **lines**.

The points and lines are the *undefined objects* of our axiom system — their properties will be determined solely by the axioms we set down. Note, however, that our intuitive notion for a line is that of a straight line which is *infinitely long* in both of its directions.

We begin our list of axioms with those concerning *incidence*, properties about the intersection or containment of sets. These are not surprising.

Incidence Axioms.

I-1. The plane $\mathcal{E}$ contains at least three *non-collinear* points, i.e., three points which are not all contained on the same line.

I-2. Given two distinct points p and q, there is *exactly one* line $\overleftrightarrow{pq}$ containing both.

There's not much we can prove from just these two axioms. However, we do have the following simple proposition:

Proposition 2.1.

Two different lines can intersect in ***at most one*** *point.*

Proof. Suppose ℓ_1 and ℓ_2 are two different lines whose intersection contains two distinct points, p and q. Axiom I-2 states there is only *one* line which contains both p and q. Hence ℓ_1 and ℓ_2 must both equal that line, and hence

[2] A complete axiomatic development of Euclidean geometry is given in Edwin Moise, *Elementary Geometry from an Advanced Standpoint*, as cited at the start of this chapter.

the two lines are not different, contradicting our starting assumption. Thus we cannot have two distinct points in the intersection of ℓ_1 and ℓ_2. $\square$

Because our *Incidence Axioms* are so brief and simple there are a wide variety of wildly different *models* for this initial axiom system. We present several such models, all of which will have important uses later in the text and its sequel.

Example 2.2. $\mathbb{R}^2$***: The Real Cartesian Plane***.

The *points* of this model are all ordered pairs (x, y), where x and y are any two real numbers. Thus the *plane* in this model, which we denote as $\mathbb{R}^2$, is simply the set of all ordered pairs of real numbers:

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}.$$

A *line* in this model is the set of all points (x, y) that satisfy an equation of the form $ax + by + c = 0$ for some set of real numbers a, b, c, where at least one of the two numbers a or b is not zero. This definition of line includes those with well-defined "slopes," i.e., those whose equations can be put in the form $y = mx + B$ (the case when $b \neq 0$), as well as vertical lines of the form $x = C$ (the case when $b = 0$ but $a \neq 0$). Written formally, the set $\mathcal{L}$ of lines is given by

$$\mathcal{L} = \{L_{a,b,c} \mid a, b, c \in \mathbb{R} \text{ and } a^2 + b^2 \neq 0\}, \text{ where}$$
$$L_{a,b,c} = \{(x, y) \mid ax + by + c = 0\}.$$

In order for the set $\mathbb{R}^2$ with the specified lines $\mathcal{L}$ to be a model for the *Incidence Axioms*, we must verify each axiom for the pair $\mathbb{R}^2$ and $\mathcal{L}$. We must take care in our verifications: *we cannot appeal to "intuitive" understandings of the meaning of points and lines; we must only use the precise definitions of points and lines as given in the definitions of* $\mathbb{R}^2$ *and* $\mathcal{L}$. Although the tools needed for these verifications are simply high school algebra and basic logical thinking, the resulting arguments are surpisingly sophisticated.

I-1. We must show that there are at least three points in $\mathbb{R}^2$ which are not contained in any single line. For example, consider the points $(0, 0)$, $(1, 1)$, and $(-1, 1)$. Suppose they were all contained in one line $L_{a,b,c}$. Then an equation of the form $ax + by + c = 0$ is satisfied by all three points, giving

$$\begin{aligned} a\,0 + b\,0 + c &= 0, \\ a\,1 + b\,1 + c &= 0, \\ a\,(-1) + b\,1 + c &= 0. \end{aligned}$$

Hence $c = 0$, $a + b = 0$, and $-a + b = 0$. But these last two equations imply $a = b = 0$, which is not allowed in our definition of a line. This proves that our three points do not lie on the same line.

I-2. Given two distinct points $p = (x_1, y_1)$ and $q = (x_2, y_2)$, we must show there is exactly one line L in $\mathbb{R}^2$ containing both points.

First consider the case $x_1 = x_2$. We will show that the only line containing p and q is the line with equation $x = x_1$. Clearly this line does contain the two points, so now we must prove it to be unique. If the two points are also contained in another line $L_{a,b,c}$, then

$$ax_1 + by_1 + c = 0 \quad \text{and} \quad ax_2 + by_2 + c = 0.$$

Since x_1 equals x_2, this gives $by_1 = by_2$, which implies $y_1 = y_2$ *if* $b \neq 0$. But since p and q are distinct points with $x_1 = x_2$, then y_1 cannot equal y_2. Thus $b = 0$, and hence a cannot be zero. Thus the equation of $L_{a,b,c}$ must be of the form $ax + c = 0$ with $a \neq 0$, which can be rewritten as

$$x = -c/a.$$

Since $p = (x_1, y_1)$ satisfies this equation, this shows $x_1 = -c/a$, so that the equation for $L_{a,b,c}$ can indeed be rewritten as $x = x_1$, as we desired to show. Thus, in the case where x_1 and x_2 are equal, we have shown there exists exactly one line in $\mathcal{L}$ containing our two points.

Now consider the second case, when x_1 and x_2 are unequal. If $L_{a,b,c}$ is a line (that we must show exists and is unique) containing both points, then

$$ax_1 + by_1 + c = 0 \quad \text{and} \quad ax_2 + by_2 + c = 0.$$

We claim that in this case b cannot be zero. For suppose $b = 0$. Then

$$ax_1 + c = 0 \quad \text{and} \quad ax_2 + c = 0,$$

which gives either $x_1 = x_2$ or $a = 0$. But neither can be true (since $x_1 \neq x_2$ by assumption, and if $a = 0$, then both a and b are zero, contradicting our definition of a line in $\mathbb{R}^2$). Hence $b \neq 0$, and so dividing the equation $ax + by + c = 0$ by b shows that our line can be specified by an equation of the form $y = (-a/b)x + (-c/b)$. Since this linear equation is satisfied by both $p = (x_1, y_1)$ and $q = (x_2, y_2)$, we obtain a system of two equations:

$$\begin{aligned} y_1 &= (-a/b)x_1 + (-c/b), \\ y_2 &= (-a/b)x_2 + (-c/b). \end{aligned}$$

Solving for the two unknown quantities $-a/b$ and $-c/b$ yields

$$-a/b = \frac{y_2 - y_1}{x_2 - x_1}, \text{ the slope of the line,} \tag{2.3a}$$

and

$$-c/b = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}, \text{ the } y\text{-intercept of the line.} \tag{2.3b}$$

Using these equations we can now show the existence and uniqueness of the line $L_{a,b,c}$ containing the points p and q. For the existence of $L_{a,b,c}$ let $b = 1$ and define the necessary values of a and c from equations (2.3a) and (2.3b). The line so defined will contain p and q, as desired. For the uniqueness of $L_{a,b,c}$, notice that although b can have any non-zero value, equations (2.3a) and (2.3b) then show a and c will be fixed multiples of b. In other words, the linear equation $ax + by + c = 0$ is just a non-zero multiple of one fixed equation, and hence all of these equations define *just one unique line!*

Hence given two distinct points in $\mathbb{R}^2$, there is *exactly* one line containing both points. This verifies Axiom I-2 for $\mathbb{R}^2$, as desired. □

As you can see from Example 2.2, verifying axioms for a specific model can be an intricate process. However, once done for a specific model, the axiom system is seen to be consistent. Moreover, any theorem proven for the axiom system applies to the specific model. For example, since $\mathbb{R}^2$ has been shown to satisfy the *Incidence Axioms*, then Proposition 2.1 must apply to $\mathbb{R}^2$, i.e., any two distinct lines in $\mathbb{R}^2$ can intersect in at most one point.

Our next example, the *Poincaré disk*, will seem quite odd to those readers who have not encountered non-Euclidean geometric systems. The primary oddity is that what we call a "line" in the Poincaré disk is not always "straight" in the ordinary Euclidean sense. In fact, most of the Poincaré "lines" are actually parts of Euclidean circles. But though not "straight" in the ordinary sense, the collection of Poincaré lines, combined with the Poincaré disk, will satisfy the two incidence axioms!

This model will be important in Volume II of this text: it will provide the basis for a standard model of ***hyperbolic geometry***.

Example 2.4. ***$\mathbb{P}^2$: The Poincaré Disk.***

The *points* of this model are the ordered pairs of real numbers (x, y) that lie inside the unit disk, i.e., for which $x^2 + y^2 < 1$. Thus the "plane" in this model, which we denote as $\mathbb{P}^2$, is the open unit disk

$$\mathbb{P}^2 = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } x^2 + y^2 < 1\}.$$

There will be two types of *lines* in this model, both shown in Figure 2.5. One will be any ordinary straight line containing the origin $(0, 0)$ (restricted of course to the open disk $\mathbb{P}^2$). These lines can be described algebraically by taking any two real numbers a and b, at least one being non-zero, and defining the line $L_{a,b}$ to be the collection of points (x, y) in the open unit disk that satisfy the equation $ax + by = 0$, i.e.,

$$L_{a,b} = \{(x, y) \mid ax + by = 0 \text{ and } x^2 + y^2 < 1\}.$$

Such a line is ℓ_1 in Figure 2.5. The other type of "line" will be any ordinary circle in $\mathbb{R}^2$ (restricted to $\mathbb{P}^2$) which intersects the unit circle

$x^2 + y^2 = 1$ in a perpendicular fashion, i.e., the tangent lines to the two circles at a point of intersection must form a right angle. Examples are ℓ_2, ℓ_3, and ℓ_4 in Figure 2.5. In Exercise 2.4 you will show that such a "line" can be described algebraically by taking any two real numbers α and β, where $\alpha^2 + \beta^2 > 1$, and defining the "line" $L^\circ_{\alpha,\beta}$ (a circle in $\mathbb{R}^2$) to be the collection of points (x, y) in the open unit disk that satisfy the equation $(x - \alpha)^2 + (y - \beta)^2 = \alpha^2 + \beta^2 - 1$, i.e.,

$$L^\circ_{\alpha,\beta} = \{(x, y) \mid (x - \alpha)^2 + (y - \beta)^2 = \alpha^2 + \beta^2 - 1, x^2 + y^2 < 1\}.$$

The collection $\mathcal{L}$ of lines in the Poincaré disk is simply the collection of all the lines $L_{a,b}$ and $L^\circ_{\alpha,\beta}$ defined above, i.e.,

$$\mathcal{L} = \{L_{a,b} \mid a^2 + b^2 > 0\} \cup \{L^\circ_{\alpha,\beta} \mid \alpha^2 + \beta^2 > 0\}.$$

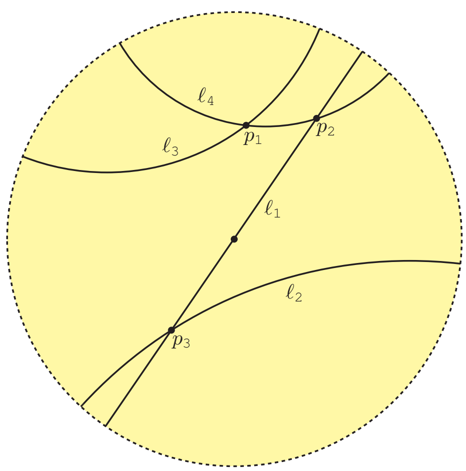


Figure 2.5. Lines and points in $\mathbb{P}^2$, the Poincaré disk.

In order for the set $\mathbb{P}^2$ with the specified lines $\mathcal{L}$ to be a model for the *Incidence Axioms*, we must demonstrate the truth of the two axioms I-1 and I-2 for the pair $\mathbb{P}^2$ and $\mathcal{L}$, just as we did for the real Cartesian plane $\mathbb{R}^2$. However, since the Poincaré disk will be new to most readers of this text, even intuitively it may not be clear that the two axioms are valid! Indeed they are — we will outline verifications below, leaving the details to Exercise 2.4.

I-1. We must show that there are at least three points in $\mathbb{P}^2$ which are not contained by any single "line." It turns out that any three points in the Poincaré disk which lie on a line *not* passing through the origin cannot lie on any one Poincaré "line." Draw some pictures to see why this must be true. You can easily verify this claim algebraically for the three points $(0, 0)$, $(0, 1/2)$, and $(1/2, 0)$, proving Axiom I-1 for $\mathbb{P}^2$.

I-2. Given any two points $p = (x_1, y_1)$ and $q = (x_2, y_2)$ in the Poincaré disk, we must show there exists one and only one "line" in $\mathcal{L}$ that contains both points. As with the real Cartesian plane $\mathbb{R}^2$ we have two cases to consider.

First consider the case where p and q both lie on an ordinary straight line containing the origin $(0, 0)$. Then the coordinates of p and q must satisfy an equation of the form $ax + by = 0$ for some real numbers a and b, not both zero. Moreover, the only linear equations of this form which are valid for both p and q are just constant multiples of $ax + by = 0$. This proves that p and q are contained in one unique line of the form $L_{a,b}$. Furthermore, one can show (Exercise 2.4) that the two points p and q, being collinear with the origin, cannot lie on any "line" (circle) of the form $L^\circ_{\alpha,\beta}$.

Now consider the case where p and q do not lie on an ordinary straight line containing the origin. This condition will be shown in Exercise 2.4 to be equivalent to the expression $x_1y_2 - x_2y_1$ not equaling zero. However, this condition is just what is needed to algebraically verify that there is exactly one pair of numbers (α, β) such that the equation

$$(x - \alpha)^2 + (y - \beta)^2 = \alpha^2 + \beta^2 - 1$$

is satisfied by the coordinates of both p and q — you will verify this in Exercise 2.4. Hence p and q are indeed contained in exactly one line of the form $L^\circ_{\alpha,\beta}$, and they cannot be on any line of the form $L_{a,b}$.

This shows that any two points of the Poincaré disk are members of exactly one Poincaré line in $\mathcal{L}$, finishing the verification of Axiom I-2. □

The Poincaré disk stretched our concept of "line." The next example, the *real projective plane*, will stretch our concept of "point" as well as line. In this model each "point" is a pair of antipodal (opposite) points on the unit sphere, and each "line" is the collection of "points" which lie along a great circle of the sphere!

Like the Poincaré disk, this model will reappear in Volume II of this text. When expanded, it will become a basic model for ***elliptic geometry***.

Example 2.6. $\mathbb{RP}^2$***: The Real Projective Plane***.

We start with the standard unit sphere S^2 in $\mathbb{R}^3$:

$$S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}.$$

If $P = (x_0, y_0, z_0)$ is a point on S^2, then the point on the sphere directly opposite P is its *antipodal* point $Q = (-x_0, -y_0, -z_0)$. The "points" of $\mathbb{RP}^2$, the *real projective plane*, are the *pairs* of antipodal points of the sphere. It is as though we simply consider a point and its antipodal point as "the same." Thus the set $\mathbb{RP}^2$ can be expressed as

$$\mathbb{RP}^2 = \{\{(x, y, z), (-x, -y, -z)\} \mid x^2 + y^2 + z^2 = 1\}.$$

A *great circle* on a sphere is a circle gotten by intersecting the sphere with a plane that passes through the center of the sphere. If a point P on a sphere lies on a great circle, then so does its antipodal point. The "lines" of the real projective plane are simply the collections of all pairs of antipodal points that lie on a given fixed great circle. Since any plane through the origin in $\mathbb{R}^3$ is given by an equation of the form $ax + by + cz = 0$ where at least one of the numbers a, b, c is non-zero, then any three such numbers determine a "line" $L_{a,b,c}$ in $\mathbb{RP}^2$ by

$$L_{a,b,c} = \{\{(x,y,z),(-x,-y,-z)\} \mid x^2+y^2+z^2 = 1 \text{ and } ax+by+cz = 0\}.$$

The collection of all such $L_{a,b,c}$ forms the collection $\mathcal{L}$ of "lines" in $\mathbb{RP}^2$:

$$\mathcal{L} = \{L_{a,b,c} \mid a^2 + b^2 + c^2 > 0\}.$$

Because there is only one type of line for $\mathbb{RP}^2$, verifying the *Incidence Axioms* for the real projective plane is actually easier than for the real Cartesian plane or the Poincaré disk. We leave the verification of these axioms to Exercise 2.5.

Notice that Axiom I-2 would be false had we not identified antipodal points. If we do not make this identification, then we would simply obtain the unit sphere S^2, with the lines being just the great circles. However, if we take two antipodal points on the sphere — which are distinct points on S^2 — then there are an infinite number of great circles which contain these two points. Thus Axiom I-2 does not hold for S^2.

The next example of a model for the *Incidence Axioms* is perhaps the strangest of all. The *Moulton plane* has some surprising properties that will be highly instructive when ***projective geometry*** is studied in Volume II.

Example 2.7. $\mathbb{MP}^2$***: The Moulton Plane.***

The points of the *Moulton plane* $\mathbb{MP}^2$ are the same as for the real Cartesian plane $\mathbb{R}^2$: all ordered pairs of real numbers. Thus

$$\mathbb{MP}^2 = \{(x,y) \mid x, y \in \mathbb{R}\}.$$

However the collection of lines in $\mathbb{MP}^2$ is quite odd. There are three kinds of lines, all illustrated in Figure 2.8:

(1) all vertical lines in $\mathbb{R}^2$, i.e., all collections of points (x,y) satisfying an equation of the form $x = a$ for a fixed constant $a \in \mathbb{R}$ (see ℓ_1 in Figure 2.8),

(2) all lines in $\mathbb{R}^2$ with non-positive slope, i.e., all collections of points (x,y) satisfying an equation of the form $y = mx + b$ where $m \leq 0$ and b is a fixed real constant (see ℓ_2 in Figure 2.8),

(3) all "bent" lines in $\mathbb{R}^2$ with a positive slope $m > 0$ when $x < 0$ and a positive slope of half that amount, $m/2 > 0$, when $x > 0$. Thus we have all collections of points (x, y) satisfying an equation

$$y = \begin{cases} mx + b & \text{when } x \leq 0, \\ \frac{1}{2}mx + b & \text{when } x > 0, \end{cases}$$

where $m > 0$ and b is a real constant (see ℓ_3 and ℓ_4 in Figure 2.8).

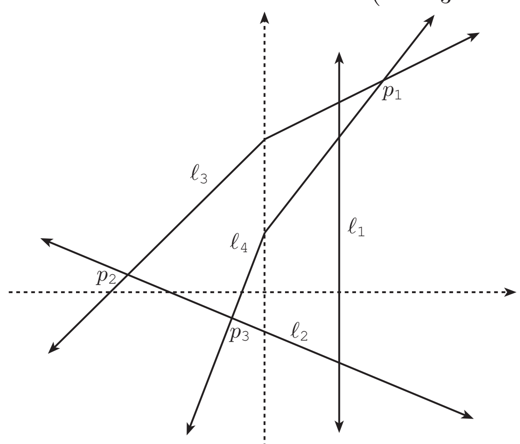


Figure 2.8. Lines and points in the Moulton plane.

It is not difficult to verify that the Moulton plane satisfies the *Incidence Axioms*. The details are left to Exercise 2.6.

The next and final model may be surprising in that it is "three dimensional." A little thought should make this understandable: the *Incidence Axioms* imply nothing about dimension. Further axioms will be needed to imply that our geometry is "two dimensional."

Example 2.9. $\mathbb{R}^3$***: Real Cartesian Space***.

The points in *real Cartesian space* $\mathbb{R}^3$ are all ordered triples of real numbers, i.e.,

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}.$$

Lines in $\mathbb{R}^3$ are most conveniently described via parametric equations. Given six fixed real constants $a_1, a_2, a_3, b_1, b_2, b_3$, where at least one of the first three is non-zero, a line L is defined as the collection of points (x, y, z) that can be expressed as

$$\begin{aligned} x =& a_1 t + b_1, \\ y =& a_2 t + b_2, \\ z =& a_3 t + b_3 \end{aligned}$$

for some real number t. Thus

$$L = \{ (x, y, z) \mid \text{there exists } t \in \mathbb{R} \text{ such that } x = a_1 t + b_1,\ y = a_2 t + b_2,\ z = a_3 t + b_3 \}.$$

There is a useful *vector* interpretation for L. Start with the point (b_1, b_2, b_3) and add to it all multiples of the vector (a_1, a_2, a_3).

The set $\mathcal{L}$ of lines in $\mathbb{R}^3$ is the collection of all subsets of $\mathbb{R}^3$ of this form.

Verifying the *Incidence Axioms* for $\mathbb{R}^3$ will be left to Exercise 2.7.

The *Incidence Axioms* provide a very modest start on developing geometry in the plane. We will make a quantum leap in the next section, postulating the existence of a well-behaved *distance* for any two points in the plane.

Exercises I.2

Exercise 2.1.

Suppose we have a system of *points* and *lines* that satisfy the *Incidence Axioms*. If r and s are distinct points on the line $\overleftrightarrow{pq}$, then what is the relationship between the two lines $\overleftrightarrow{pq}$ and $\overleftrightarrow{rs}$? Prove your claim using only the Incidence Axioms and/or Proposition 2.1.

Exercise 2.2.

Suppose we have a system of *points* and *lines* that satisfy the *Incidence Axioms*. Prove there exist at least three distinct lines that do not all intersect at one point. Use only the Incidence Axioms and/or Proposition 2.1.

Exercise 2.3.

The *Incidence Axioms* do not go very far in describing a geometric system. As an example, consider the following system. Let $\mathcal{E}_0$ be a set of three non-collinear points, and call any two-point subset of $\mathcal{E}_0$ a "line." Verify that the *Incidence Axioms* are valid for this system.

Exercise 2.4. ***The Poincaré Disk***

In this problem you will verify the various unproven claims made in the discussion of Example 2.4, the Poincaré disk. This will complete the proof that the Poincaré disk satisfies the *Incidence Axioms*.

(a) Suppose C is an ordinary circle in $\mathbb{R}^2$ with center (α, β) and radius $r > 0$, i.e.,

$$C = \{(x, y) \mid (x - \alpha)^2 + (y - \beta)^2 = r^2\}.$$

Using basic high school analytic geometry, show that C intersects the unit circle $x^2 + y^2 = 1$ in two right angles if and only if the radius r equals $\sqrt{\alpha^2 + \beta^2 - 1}$.

Hint: Let O be the origin, c the center of C, and p a point of intersection of C with the unit circle. For both directions of your verification consider properties of the triangle $\triangle Ocp$. The reverse implication requires some messy algebra.

(b) Show that a Poincaré "line" $L^{\circ}_{\alpha,\beta}$, as defined in Example 2.4, is part of an ordinary circle in $\mathbb{R}^2$ which intersects the unit circle $x^2+y^2=1$ in two right angles.

(c) Complete the proof of Axiom I-1 for the Poincaré disk by verifying that the three points $(0,0)$, $(0,1/2)$, and $(1/2,0)$ cannot lie on any Poincaré line. To do so, you must consider Poincaré lines of both forms $L_{a,b}$ and $L^{\circ}_{\alpha,\beta}$, showing contradictions occur if you assume either type contains the three given points.

(d) Show that the equation defining the Poincaré line $L^{\circ}_{\alpha,\beta}$ can be rewritten in the following useful form:

$$x^2 - 2\alpha x + y^2 - 2\beta y + 1 = 0. \tag{2.10}$$

(e) Suppose $p = (x_1, y_1)$ and $q = (x_2, y_2)$ are two distinct points in the Poincaré disk $\mathbb{P}^2$ which lie along an ordinary straight line containing the origin $(0,0)$. Show p and q cannot both lie on any Poincaré line of the form $L^{\circ}_{\alpha,\beta}$.

Outline: $p = (x_1, y_1)$ and $q = (x_2, y_2)$ will be collinear with the origin if and only if there exists a real number λ such that $x_2 = \lambda x_1$ and $y_2 = \lambda y_1$. Moreover, by interchanging p and q if necessary, we can assume q is no farther from the origin than p. This is equivalent to assuming $-1 \le \lambda < 1$.

To show p and q cannot both lie on a Poincaré line of the form $L^{\circ}_{\alpha,\beta}$, assume the opposite, i.e., that both do lie on such a line. Then (2.10) is valid for both $p = (x_1, y_1)$ and $q = (\lambda x_1, \lambda y_1)$. Subtract the resulting two equations from each other and use

$$\lambda^2 - 1 = (\lambda + 1)(\lambda - 1)$$

to obtain $\lambda(x_1^2 + y_1^2) = 1$. Show this contradicts the assumption that $p = (x_1, y_1)$ is in the open unit disk $\mathbb{P}^2$.

(f) If the Poincaré line $L^{\circ}_{\alpha,\beta}$ contains both $p = (x_1, y_1)$ and $q = (x_2, y_2)$, then show the following system of equations is valid:

$$\begin{aligned} x_1\alpha + y_1\beta &= (x_1^2 + y_1^2 + 1)/2, \\ x_2\alpha + y_2\beta &= (x_2^2 + y_2^2 + 1)/2. \end{aligned}$$

(g) Show that two points $p = (x_1, y_1)$ and $q = (x_2, y_2)$ are on an ordinary line in $\mathbb{R}^2$ that also contains the origin $(0,0)$ if and only if $x_1 y_2 = x_2 y_1$. Be sure to consider the cases when some of the coefficients are zero.

(h) Suppose $p = (x_1, y_1)$ and $q = (x_2, y_2)$ are *not* on an ordinary line in $\mathbb{R}^2$ that also contains the origin. In this case show there is exactly one pair of values for α and β which satisfy the system of equations in (f). This proves there exists exactly one Poincaré line $L^\circ_{\alpha,\beta}$ containing the points p and q.

Exercise 2.5. ***The Real Projective Plane.***

(a) Verify Axiom I-1 for the *real projective plane* $\mathbb{RP}^2$ (Example 2.6). *Hint:* Pick three simple points on the sphere S^2, none of which is an antipodal point for the remaining two. Show these points cannot lie on a plane in $\mathbb{R}^3$ that contains the origin. Then show why this means the corresponding pairs of antipodal points in $\mathbb{RP}^2$ cannot lie on one real projective line.

(b) Verify Axiom I-2 for $\mathbb{RP}^2$.

Exercise 2.6. ***The Moulton Plane.***

(a) Determine the equation of the line in the Moulton plane containing the points $(-1, 1)$ and $(2, -5)$.

(a) Determine the equation of the line in the Moulton plane containing the points $(-1, 1)$ and $(2, 7)$.

(c) Verify the *Incidence Axioms* for the Moulton plane $\mathbb{MP}^2$.

Exercise 2.7. ***Real Cartesian Space.***

Verify the *Incidence Axioms* for real Cartesian space $\mathbb{R}^3$.

§I.3 Distance and Coordinate Systems on Lines

In Euclidean geometry there exists a *distance* between any two points in the plane. We build this into our model via the assumption of a *coordinate system* for each line.

The desire is that each line ℓ appear to be a "copy" of the real number line $\mathbb{R}$, which at the very least requires the existence of a *one-to-one correspondence*[3] χ from ℓ to $\mathbb{R}$. Such a mapping χ is termed a *coordinate system* on ℓ.

Definition 3.1.

A **coordinate system** χ on a line ℓ is a *one-to-one correspondence* $\chi : \ell \to \mathbb{R}$.

We now assume, for each line ℓ in $\mathcal{E}$, the existence of a fixed coordinate system $\chi_\ell : \ell \to \mathbb{R}$. This adds the coordinate systems to our collection of *undefined objects* of the previous section; the properties of the coordinate systems will be specified by subsequent axioms.

[3] A *one-to-one correspondence* χ from a set A to a set B is a function $\chi : A \to B$ such that, for each $b \in B$, there is a *unique* $a \in A$ for which $\chi(a) = b$. This is equivalent to the mapping $\chi : A \to B$ being both *one-to-one* and *onto*. In other words, we have a "pairing" between elements of A and elements of B.

Notice a simple but important consequence of our assumption: *every line has an infinite number of distinct points*. For suppose ℓ is a line. Then the assumed coordinate system $\chi_\ell : \ell \to \mathbb{R}$ is a one-to-one correspondence between ℓ and $\mathbb{R}$. Since $\mathbb{R}$ is infinite, this means ℓ must also be infinite, as claimed. For this reason, the "three point geometry" of Exercise 2.3 will no longer satisfy the axiomatic system we are building.

Given a line ℓ, any two points p, q in ℓ are identified with two points $\chi_\ell(p)$, $\chi_\ell(q)$ in $\mathbb{R}$. But points in $\mathbb{R}$ have a ***distance*** defined between them; it is therefore natural to take the distance between $\chi_\ell(p)$, $\chi_\ell(q)$ in $\mathbb{R}$, which is $|\chi_\ell(p) - \chi_\ell(q)|$, and use it as the distance between p and q in ℓ, i.e.,

the ***distance*** $d(p,q)$ between p and q in ℓ equals $|\chi_\ell(p) - \chi_\ell(q)|$.

Since any two distinct points in the plane $\mathcal{E}$ are contained on exactly one line, we have therefore defined a distance $d(p,q)$ between any two points p, q in $\mathcal{E}$. This number is denoted by several different notations: $d(p,q)$, $|\overline{pq}|$, or simply pq.

Let $\mathcal{E} \times \mathcal{E}$ denote the collection of all *ordered pairs* (p,q) of two points in the plane. Then the distance d is a *function* assigning a real number to each pair (p,q) in $\mathcal{E} \times \mathcal{E}$, written as $d : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$. We summarize this definition of distance as follows:

Definition 3.2.

For each line ℓ in the plane fix a coordinate system $\chi_\ell : \ell \to \mathbb{R}$. Then the **distance function** on the plane $\mathcal{E}$ is the function $d : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ which assigns to any two points p, q a real number $d(p,q) = pq$ defined by

$$d(p,q) = pq = \begin{cases} |\chi_\ell(p) - \chi_\ell(q)| & \text{if } p \neq q \text{ where } \ell = \overleftrightarrow{pq}, \\ 0 & \text{if } p = q. \end{cases} \tag{3.3}$$

This number $d(p,q) = pq$ is called the **distance** between p and q.

All the expected elementary properties of distance for points *lying along a single line* are valid for our distance function. These are summarized in the next proposition.

Proposition 3.4. ***Distance Properties along a Line.***

Let d be the distance function (3.3) *on the plane $\mathcal{E}$. Then*

(a) $d(p,q) \geq 0$ *for any two points p and q,*

(b) $d(p,q) = 0$ *if and only if $p = q$,*

(c) $d(p,q) = d(q,p)$ *for any two points p and q,*

(d) $d(p,q) \leq d(p,r) + d(r,q)$ *for any three* ***collinear*** *points p, q, and r.*

Proof. All of these properties follow from (3.3) and the corresponding properties of distance in $\mathbb{R}$. In particular, (d) follows from the *triangle inequality* of $\mathbb{R}$: $|x + y| \leq |x| + |y|$ for any two real numbers x, y. $\square$

It is important to realize that property (d) is only valid (at this time) for *collinear* points. Subsequent axioms will extend (d) to all points in the plane, at which time it will become known as the **triangle inequality** for $\mathcal{E}$. The possible failure of the triangle inequality under our current small set of axioms is examined in Exercise 3.3 (see also Exercise 3.4).

Any line ℓ actually has an infinite number of coordinate systems. For example, if χ is a coordinate system on ℓ and a, b are fixed real numbers such that $a \neq 0$, then defining $\xi : \ell \to \mathbb{R}$ by $\xi(p) = a\,\chi(p) + b$ for all points p on ℓ produces a new coordinate system on ℓ (Exercise 3.1). However, two different coordinate systems on a line ℓ will not necessarily produce the same distance function on ℓ. If they do, the coordinate systems are said to be *equivalent*.

Definition 3.5.

Two coordinate systems $\chi : \ell \to \mathbb{R}$ and $\xi : \ell \to \mathbb{R}$ on a line ℓ are **equivalent** if they give the same distances between points on ℓ.

If χ_ℓ is the coordinate system we've fixed for a line ℓ, then we can replace χ_ℓ by any *equivalent* coordinate system without changing the distance function (3.3) defined on the plane $\mathcal{E}$. In fact, given a line ℓ, we will often desire a coordinate system with the origin (the point with coordinate zero) at a specified point $p \in \ell$ and the positive coordinates on a specified side of p. If the coordinate system χ_ℓ originally chosen for ℓ does not have these desired properties, it is not difficult to prove that there exists an *equivalent* coordinate system that does. This is the useful *Ruler Placement Theorem.*

Proposition 3.6. ***The Ruler Placement Theorem.***

Let ℓ be a line with the chosen coordinate system $\chi_\ell : \ell \to \mathbb{R}$ and let p and q be two distinct points of ℓ. Then ℓ has an **equivalent** *coordinate system ξ such that p is the origin and q has a positive coordinate, i.e.,*

$$\xi(p) = 0 \text{ and } \xi(q) > 0.$$

Proof. Exercise 3.1.

The Real Cartesian Plane. In the previous section we discussed a number of models for the *Incidence Axioms*. We now return to the first and most fundamental of these models, Example 2.2, the *real Cartesian plane* $\mathbb{R}^2$, and show that we can fix coordinate systems on the lines of $\mathbb{R}^2$ such that the distance function they generate via (3.3) is the standard distance function studied in analytic geometry:

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{3.7}$$

To verify our claim, for each line ℓ in $\mathbb{R}^2$ we must produce a coordinate system χ which is compatible with (3.7). Any line ℓ in $\mathbb{R}^2$ is defined to be the set of points (x, y) that satisfy an equation of the form $ax + by + c = 0$ for constants a, b, c where at least one of a, b is non-zero. If $b = 0$, then $a \neq 0$ and we can write $x = -c/a$ for every point in this vertical line. But if $b \neq 0$, then our line is given by the equation $y = -(a/b)x - c/b$. This allows the following definition of a coordinate system χ_ℓ on the line ℓ with equation $ax + by + c = 0$:

$$\chi_\ell((x, y)) = \begin{cases} y & \text{if } b = 0, \\ x\sqrt{1 + (\frac{a}{b})^2} & \text{if } b \neq 0. \end{cases}$$

As you will show in Exercise 3.2, χ_ℓ is a coordinate system on ℓ compatible with (3.7). Hence, choosing such a coordinate system for each line results in the standard distance function (3.7) on the real Cartesian plane $\mathbb{R}^2$. □

Exercises I.3

Exercise 3.1.

(a) Suppose χ is a coordinate system for a line ℓ and a, b are two real numbers such that $a \neq 0$. Define a new function ξ on ℓ by $\xi(p) = a\,\chi(p) + b$ for all points p on ℓ. Show that ξ is also a coordinate system for ℓ. *Hints:* You need to show ξ is *one-to-one* and *onto*. To show ξ is *one-to-one*, you assume $\xi(p) = \xi(q)$ and prove that $p = q$. To show ξ is *onto*, you take any real number x and show there exists a point p on ℓ such that $\xi(p) = x$.

(b) Suppose χ and ξ are coordinate systems for a line ℓ as given in part (a). For what values of a and b will these two coordinate systems be equivalent, i.e., yield the same distance function on ℓ?

(c) Prove the *Ruler Placement Theorem*. *Hint:* Starting with $\chi = \chi_\ell$ and the points $p, q \in \ell$, find values of a and b so that an equivalent coordinate system ξ is defined such that $\xi(p) = 0$ and $\xi(q) > 0$.

Exercise 3.2.

Suppose ℓ is a line in the *real Cartesian plane* defined by the equation $ax + by + c = 0$ and $\chi_\ell : \ell \to \mathbb{R}$ is the function given by

$$\chi_\ell((x, y)) = \begin{cases} y & \text{if } b = 0, \\ x\sqrt{1 + (\frac{a}{b})^2} & \text{if } b \neq 0. \end{cases}$$

(a) Prove that χ_ℓ is a coordinate system on ℓ.

(b) If every line ℓ in $\mathbb{R}^2$ is given the coordinate system χ_ℓ as defined in (a), prove that the distance function defined on $\mathbb{R}^2$ is the standard distance function studied in analytic geometry:

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Exercise 3.3.

In this problem we will ultimately show that the *triangle inequality* for *non-collinear points* is not a consequence of the *Incidence Axioms* and the existence of coordinate systems. To do so, suppose $\mathcal{E}$ is a set with a collection $\mathcal{L}$ of special subsets called *lines* that satisfy the *Incidence Axioms* and such that each line has a coordinate system. *Assume no further properties for this system!* Hence,

- for each line $\ell \in \mathcal{L}$ there is a coordinate system $\chi_\ell : \ell \to \mathbb{R}$, and
- the set $\{\chi_\ell \mid \ell \in \mathcal{L}\}$ generates a distance function $d : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$.

(a) Choose a line ℓ_0 from $\mathcal{L}$ and a positive real number λ. Then define a new coordinate system $\xi : \ell_0 \to \mathbb{R}$ by $\xi(p) = \lambda\, \chi_{\ell_0}(p)$ for all $p \in \ell_0$. If this is the only line whose coordinate system is changed, how does the new distance function d_λ compare to the original d? *Hint:* Compute $d_\lambda(p, q)$ when p and q are both points of ℓ_0 and when at least one of the points is not on ℓ_0.

(b) The *triangle inequality* states that for any three points p, q, and r we have $d(p, q) \leq d(p, r) + d(r, q)$. Prove that you can choose a positive real number λ in part (a) so that the triangle inequality will *not* always be valid for the the distance function d_λ.

(c) Explain why parts (a) and (b) show that the *Incidence Axioms* and a coordinate system for each line do not always imply the triangle inequality for non-collinear points.

Exercise 3.4. ***The Moulton Plane.***

Recall the Moulton plane $\mathbb{MP}^2$ from Example 2.7. To each Moulton line ℓ we can define a coordinate system $\chi_\ell : \ell \to \mathbb{R}$ such that the distance between any two points $p, q \in \ell$ will equal the ordinary $\mathbb{R}^2$ distance *as measured along the (possibly bent) Moulton line* ℓ. Hence, if ℓ is a Moulton line with a bend at $p_0 = (0, y)$ and p and q are on opposite sides of the bend, then the Moulton distance from p to q is given by $pp_0 + p_0q$, where pp_0 and p_0q are the ordinary $\mathbb{R}^2$ lengths of the two line segments $\overline{pp_0}$ and $\overline{p_0q}$.

(a) Prove the assertions just made by determining formulas for coordinate systems χ_ℓ for each type of Moulton line ℓ, verifying that the distance function on $\mathbb{MP}^2$ so produced agrees with the $\mathbb{R}^2$ distance in the sense stated above. (These formulas will *not* be needed for the other parts of this problem.)

(b) The points $p = (-3, 0)$, $r = (0, 3)$, and $q = (3, 6)$ are collinear in the real Cartesian plane $\mathbb{R}^2$. Are they collinear in the Moulton plane?

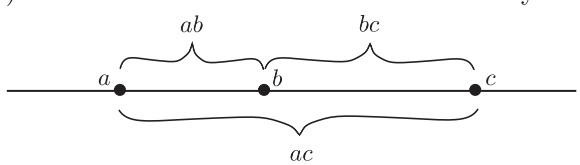
(c) Compare the Moulton distances $d(p, q)$ and $d(p, r) + d(r, q)$. What important fact does this tell you about the Moulton plane?

§I.4 Betweenness

In the previous section we used the coordinate systems $\chi_\ell : \ell \to \mathbb{R}$ for all lines ℓ in the plane to define a distance function on the collection of all pairs of points of the plane. We now use the coordinate systems and the distance function to define and analyze *betweenness:* given three distinct points on a line ℓ, how can we determine when one point is *between* the other two? We present an elegant method based on the distance function.

Definition 4.1.

Suppose a, b, and c are three distinct collinear points (i.e., they all lie on one line). Then b is **between** a and c if and only if $ab + bc = ac$:



Given a coordinate system χ_ℓ, the **coordinate** of a point $p \in \ell$ is the number x assigned by χ_ℓ to p, i.e., $x = \chi_\ell(p)$ is the coordinate of p on ℓ. The next result shows that *betweenness* of points on ℓ is mirrored by *betweenness* of the corresponding coordinates in $\mathbb{R}$.

Proposition 4.2.

Let ℓ be a line and let a, b, c be three distinct points of ℓ with coordinates x, y, z, respectively. Then the point b is between the points a and c if and only if the number y is between the numbers x and z.

Proof. First suppose the coordinate y is between the coordinates x and z. There are two possibilities: $x < y < z$ or $z < y < x$. Suppose the first. By the definition of the distance function (3.3) we have

$$ac = |\chi_\ell(a) - \chi_\ell(c)| = |x - z| = z - x,$$

the last equality following from $x < z$. Similarly

$$ab = y - x \quad \text{and} \quad bc = z - y.$$

Then simple addition gives the desired result:

$$ab + bc = (y - x) + (z - y) = z - x = ac,$$

proving point b is indeed between a and c. The second possibility, $z < y < x$, is handled in the same fashion, showing again that b is between a and c.

Showing that if b is between a and c, then y is between x and z, is left as Exercise 4.1. □

Because of Proposition 4.2, the properties of betweenness for points on a line can be carried over from the usual order properties for real numbers. What follows are some unsurprising results that will be useful in subsequent work. The proofs are left for Exercise 4.3.

Proposition 4.3.

(a) *For any three distinct collinear points, exactly one is between the other two.*

(b) *For any two distinct points a and c there exists a point b between a and c and a point d such that c is between a and d.*

(c) *For any two distinct points a and c there exists a unique* ***midpoint****, i.e., a point b which is between a and c such that $ab = bc = ac/2$.*

We define the concepts of *line segments* and *rays* by using betweenness:

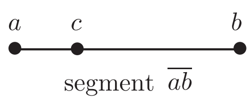
Definition 4.4.

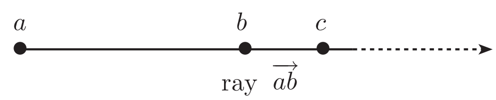
(a) Given two distinct points a and b, the **line segment** $\overline{ab}$ is the set consisting of a, b, and all the points c between a and b, i.e.,

$$\overline{ab} = \{\ c \in \overleftrightarrow{ab}\ \mid c = a,\ c = b, \text{ or } c \text{ is between } a \text{ and } b\}.$$

(b) Given two distinct points a and b, the **ray** from a through b, denoted by $\overrightarrow{ab}$, is the set of points c on the line $\overleftrightarrow{ab}$ such that a is *not* between b and c, i.e.,

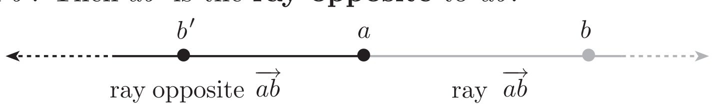
$$\overrightarrow{ab} = \{c \in \overleftrightarrow{ab}\ \mid a \text{ is } \textit{not} \text{ between } b \text{ and } c\}:$$





Definition 4.5.

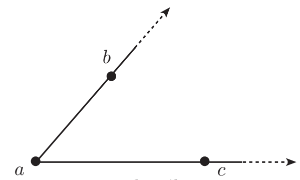
Given a ray $\overrightarrow{ab}$, let b' be collinear with a and b such that a is between b and b'. Then $\overrightarrow{ab'}$ is the **ray opposite** to $\overrightarrow{ab}$:



Given $\overrightarrow{ab}$, from Proposition 4.3b we know there exists a point b' such that a is between b and b'. This proves the ray opposite to $\overrightarrow{ab}$ does indeed exist. Furthermore, it is easy to show (Exercise 4.5) that the line $\overleftrightarrow{ab}$ is the union of $\overrightarrow{ab}$ and $\overrightarrow{ab'}$ and that these two rays intersect only at the point a. This is clearly seen in the figure above.

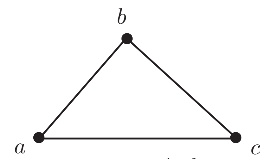
Definition 4.6.

(a) An **angle** with vertex at the point a is the union of two rays $\overrightarrow{ab}$ and $\overrightarrow{ac}$, both starting at a, where a, b, and c are not collinear:
$$\angle bac = \overrightarrow{ab} \cup \overrightarrow{ac}.$$



angle $\angle bac$

(b) The **triangle** $\triangle abc$ with non-collinear vertices a, b, c is the union of the three line segments $\overline{ab}$, $\overline{bc}$, and $\overline{ac}$:
$$\triangle abc = \overline{ab} \cup \overline{bc} \cup \overline{ac}.$$



triangle $\triangle abc$

We state some of the most useful consequences of these definitions in the next proposition, leaving the proofs for Exercise 4.8.

Proposition 4.7.

(a) *If a and b are distinct points, then $\overline{ab} = \overline{ba}$.*

(b) *If b_1 is a point of $\overrightarrow{ab}$ other than a, then $\overrightarrow{ab} = \overrightarrow{ab_1}$.*

(c) *$\overline{a_1b_1} = \overline{a_2b_2}$ if and only if the set of endpoints $\{a_1, b_1\}$ equals[4] the set of endpoints $\{a_2, b_2\}$.*

(d) *$\triangle a_1b_1c_1 = \triangle a_2b_2c_2$ if and only if the set of vertices $\{a_1, b_1, c_1\}$ equals the set of vertices $\{a_2, b_2, c_2\}$.*

Our concepts of line, line segment, and angle do not have "direction" associated with them. If direction is desired, we must specify further information to obtain *directed lines*, *directed line segments*, and *directed angles*.

Definition 4.8.

(a) A **directed line** ℓ is a line along with the choice of a ray $\overrightarrow{r}$ in ℓ that indicates the *positive direction* along ℓ.

(b) A **directed line segment** $\overline{ab}$ is a line segment along with the designation of a as the *initial* endpoint and b as the *terminal* endpoint. The ray $\overrightarrow{ab}$ indicates the *positive direction* along $\overline{ab}$.

(c) A **directed angle** $\measuredangle bac$ is the union of two rays, $\overrightarrow{ab} \cup \overrightarrow{ac}$, where $\overrightarrow{ab}$ is designated as the *initial* ray and $\overrightarrow{ac}$ as the *terminal* ray.

Although all ordinary (non-directed) angles must be constructed from non-collinear points, we drop that restriction when dealing with directed angles.

[4]Equality between the two sets $\{a_1, b_1\}$ and $\{a_2, b_2\}$ does not necessarily imply $a_1 = a_2$ and $b_1 = b_2$. We could also have the other match-up, i.e., $a_1 = b_2$ and $b_1 = a_2$. This same warning applies to part (d) of the proposition, where there are six possible match-ups between the two sets of triangle vertices.

Hence a directed angle $\measuredangle bac$ can simply be a ray (when $\overrightarrow{ab} = \overrightarrow{ac}$) or a line (when $\overrightarrow{ac}$ is the ray opposite to $\overrightarrow{ab}$).

The concepts of directed angles and directed lines will be particularly important when we construct isometries of the plane in Chapter II.

We end this section on the distance function and its immediate consequences with the definition of **congruence** for line segments. Intuitively two geometric figures are *congruent* if one can be "moved" (without altering size or shape) so as to exactly coincide with the other. Making the concept of "congruence via movement" rigorous is one of the important goals of this book. However, we start initially by defining congruence for particular types of figures by using properties that do not involve "movement." Later all these *ad hoc*, figure-specific definitions for congruence will be shown to be part of a single, unified concept (Definition V.1.1).

Definition 4.9. ***Line Segment Congruence.***

Two line segments, $\overline{ab}$ and $\overline{cd}$, are **congruent**, written $\overline{ab} \cong \overline{cd}$, if and only if the segments have the same length. Thus

$$\overline{ab} \cong \overline{cd} \quad \text{if and only if} \quad ab = cd.$$

It should be intuitively reasonable that two segments can be moved so as to exactly coincide if and only if they have the same length.

Congruence is an **equivalence relation** on the collection of all line segments in the plane. This means that congruence has the following three properties:

(1) *Reflexivity.* $\overline{ab} \cong \overline{ab}$ for every line segment $\overline{ab}$.

(2) *Symmetry.* If $\overline{ab} \cong \overline{cd}$, then $\overline{cd} \cong \overline{ab}$.

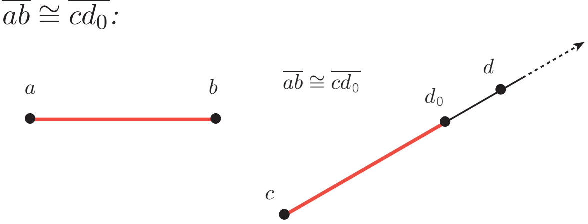
(3) *Transitivity.* If $\overline{ab} \cong \overline{cd}$ and $\overline{cd} \cong \overline{ef}$, then $\overline{ab} \cong \overline{ef}$.

These easily verified properties make congruence between line segments act like *equality*. That makes sense since if two line segments are congruent, then one can be moved so that it "is" equal to the other! *Equivalence relations* will play a central role in Chapter V (see Definition V.1.8).

Given a line segment $\overline{ab}$, it is easy to construct congruent copies wherever needed by appealing to the next result. The proof, which depends on Proposition 3.6, the *Ruler Placement Theorem,* is left to Exercise 4.9.

Proposition 4.10. ***Segment Construction.***

Given a line segment $\overline{ab}$ and a ray $\overrightarrow{cd}$, there is exactly one point $d_0 \in \overrightarrow{cd}$ such that $\overline{ab} \cong \overline{cd_0}$:



Exercises I.4

Exercise 4.1.

Prove the remaining direction of Proposition 4.2: Let ℓ be a line and let a, b, and c be three distinct points of ℓ with coordinates x, y, and z, respectively. If the point b is between the points a and c, then the number y is between the numbers x and z.

Exercise 4.2.

Suppose χ and ξ are *equivalent* coordinate systems on line ℓ. If $a, b, c \in \ell$, then prove

b is between a and c according to the coordinate system χ

if and only if

b is between a and c according to the coordinate system ξ.

Hence any two *equivalent* coordinate systems on ℓ will determine betweenness among points on ℓ in exactly the same way. Hence, when considering questions of betweenness on a line ℓ you can change the given coordinate system to any *equivalent* system. This is often convenient, especially in view of Proposition 3.6, the *Ruler Placement Theorem*.

Exercise 4.3.

Prove the results in Proposition 4.3. *Hints:* Use Proposition 4.2 in all parts. It is smart to apply the *Ruler Placement Theorem* to obtain and use a coordinate system χ that is equivalent to χ_ℓ but more convenient for the computations. Changing coordinate systems in this manner is a legitimate technique because of Exercise 4.2.

Exercise 4.4.

(a) Show that, given a ray $\overrightarrow{ab}$, there exists a coordinate system χ on $\ell = \overleftrightarrow{ab}$ equivalent to χ_ℓ but for which

$$\overrightarrow{ab} = \{c \in \overleftrightarrow{ab} \mid 0 \leq \chi(c)\}.$$

(b) Given a line segment $\overline{ab}$ of length y, show there exists a coordinate system χ on $\ell = \overleftrightarrow{ab}$ equivalent to χ_ℓ but for which $\chi(a) = 0$ and

$$\overline{ab} = \{c \in \overleftrightarrow{ab} \mid 0 \leq \chi(c) \leq y\}.$$

Exercise 4.5.

(a) Suppose $\overrightarrow{ab}$ is a ray and χ the coordinate system on $\overleftrightarrow{ab}$ from Exercise 4.4a, i.e., $\overrightarrow{ab} = \{c \in \overleftrightarrow{ab} \mid 0 \leq \chi(c)\}$. If a is between b and b', so that $\overrightarrow{ab'}$ is the *ray opposite* $\overrightarrow{ab}$, prove

$$\overrightarrow{ab'} = \{c' \in \overleftrightarrow{ab} \mid 0 \geq \chi(c')\}.$$

(b) Show that $\overrightarrow{ab}$ and $\overrightarrow{ab'}$ intersect only in the one point a and that their union is the line $\overleftrightarrow{ab}$. This verifies the figure for Definition 4.5.

Exercise 4.6.

Suppose $\chi = \chi_\ell$ is the chosen coordinate system on a line ℓ and $a, b \in \ell$ are any two distinct points on ℓ. Then prove

$$\overrightarrow{ab} = \begin{cases} \{c \in \ell \mid \chi(c) \geq \chi(a)\} & \text{if } \chi(b) > \chi(a), \\ \{c \in \ell \mid \chi(c) \leq \chi(a)\} & \text{if } \chi(b) < \chi(a). \end{cases}$$

Exercise 4.7.

Verify that the ray $\overrightarrow{ab}$ equals the union of the line segment $\overline{ab}$ with the set of points c such that b is between a and c, i.e.,

$$\overrightarrow{ab} = \overline{ab} \cup \{c \in \overleftrightarrow{ab} \mid b \text{ is between } a \text{ and } c\}.$$

Hint: Use Exercise 4.4.

Exercise 4.8.

Prove the four parts of Proposition 4.7. (*Hints:* Exercise 4.4 may prove useful in some of the verifications. The final part of the proposition is the most difficult. For this it might be helpful to first show that the only points of the line $\overleftrightarrow{ab}$ which lie on the triangle $\triangle abc$ are the points of the line segment $\overline{ab}$.)

Exercise 4.9.

Prove Proposition 4.10, *Segment Construction*. (*Hint:* Apply Proposition 3.6, the *Ruler Placement Theorem*, to the line $\overleftrightarrow{cd}$. Then show there is one and only one coordinate for the desired point d_0.)

§I.5 The Plane Separation Axiom

The axiomatic system we've developed so far is satisfied not only by all lines and points *in the plane* but also by all lines and points *in three-dimensional space*. In this section we introduce another axiom which will force us into an essentially two-dimensional situation.

Definition 5.1.

A subset A of the plane $\mathcal{E}$ is **convex** if, whenever p and q are two points of A, then the line segment $\overline{pq}$ joining p to q is also contained in A.

For a set A to be convex, it must contain the line segments $\overline{ab}$ for ***all*** pairs of points $p, q \in A$. This is true for all the sets shown in Figure 5.2.

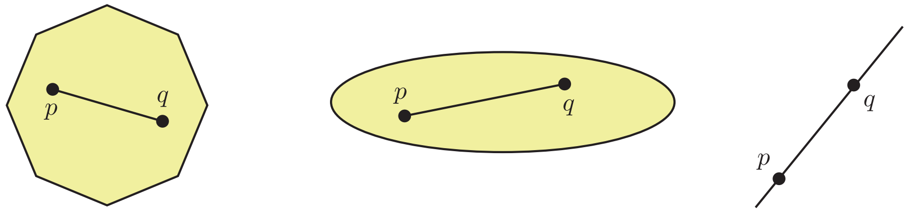


Figure 5.2. Three examples of convex sets in the plane.

Thus a set A is *not* convex if there is at least one pair of points $p, q \in A$ such that some portion of the segment $\overline{pq}$ fails to lie in A. This is seen in the sets of Figure 5.3.

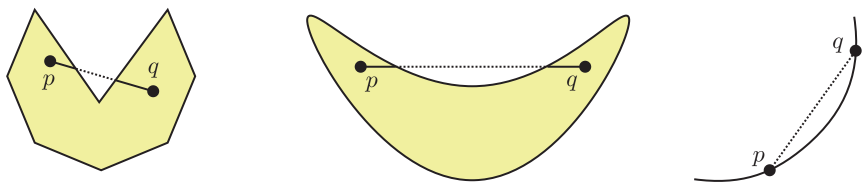
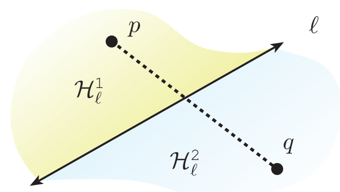


Figure 5.3. Three examples of non-convex sets in the plane.

Now consider the situation of a line ℓ in the plane $\mathcal{E}$. Intuitively the line will divide $\mathcal{E}$ into three disjoint pieces: the line ℓ itself and the two parts of the plane on either side of ℓ. These two sets are called *half planes*, and it should be intuitively clear that they are both convex (if p and q both lie on the same side of ℓ, then all the points between p and q should also lie on this same side of ℓ). Moreover, if p and q lie on opposite sides of ℓ, then the line segment $\overline{pq}$ will intersect ℓ. These "facts," however, do not follow from our previous axioms — they form the substance of our *Plane Separation Axiom:*

The Plane Separation Axiom.

PS. If a line ℓ is removed from the plane $\mathcal{E}$, the result is a disjoint union of two non-empty convex sets $\mathcal{H}_\ell^1$ and $\mathcal{H}_\ell^2$ such that if $p \in \mathcal{H}_\ell^1$ and $q \in \mathcal{H}_\ell^2$, then the line segment $\overline{pq}$ intersects ℓ:



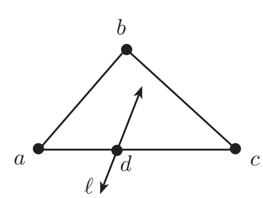
Definition 5.4.

The two non-empty convex sets $\mathcal{H}_\ell^1$ and $\mathcal{H}_\ell^2$ formed by removing the line ℓ from the plane are called **half planes**, and the line ℓ is the **edge** of each half plane.

The *Plane Separation Axiom* has important consequences, many of which we will use in subsequent work almost without thinking. Here is an example of such a result.

Proposition 5.5. ***Pasch's Axiom.***

Suppose the line ℓ and triangle $\triangle abc$ both lie in the plane $\mathcal{E}$, with ℓ intersecting $\overline{ac}$ at some point d between a and c. Then ℓ also intersects either $\overline{ab}$ or $\overline{bc}$.

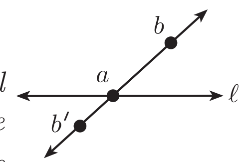


Proof. If ℓ contains either point a or c, then we are done. So assume that ℓ and $\overline{ac}$ only intersect at d.

In that case, the *Plane Separation Axiom* implies a and c are on opposite sides of ℓ, since otherwise $\overline{ac}$ could not intersect ℓ. However, if ℓ does not intersect either $\overline{ab}$ or $\overline{bc}$, then the *Plane Separation Axiom* also implies a and b are on the same side of ℓ, and that b and c are also on the same side of ℓ. Oops! This means that all three points a, b, and c must be on the same side of ℓ, contradicting our initial observation that a and c are on opposite sides. Hence we must have ℓ intersecting either $\overline{ab}$ or $\overline{bc}$, as desired. □

The *Plane Separation Axiom* will allow us to define the important concepts of *interior of an angle* and *interior of a triangle*. However, in order to justify these definitions, we need the following simple and intuitive result.

Proposition 5.6.



Suppose a is on line ℓ and b is not on ℓ. Then all the points (other than a) of ray $\overrightarrow{ab}$ lie on the same side of ℓ, and all the points (other than a) of $\overrightarrow{ab'}$, the ray opposite to $\overrightarrow{ab}$, lie on the opposite side of ℓ.

Proof. Suppose c is a point of $\overrightarrow{ab}$ which lies on the side of ℓ which is opposite to the side containing b. Then the *Plane Separation Axiom* tells us that the line segment $\overline{bc}$ must intersect the line ℓ. However, this intersection must be the point a (otherwise ℓ and $\overleftrightarrow{ab}$ would have two points of intersection, giving that they are equal by Axiom I-2, and hence b would be on ℓ, which is false). Hence a is between b and c. But this contradicts the definition of c being on the ray $\overrightarrow{ab}$ since by definition a cannot be between b and c. Hence our initial assumption that b and c are on opposite sides of ℓ cannot be true, proving that all points of $\overrightarrow{ab}$ (other than a) are on the same side of ℓ.

Now let $\overrightarrow{ab'}$ be the ray opposite to $\overrightarrow{ab}$, and let c' be any point of $\overrightarrow{ab'}$ other than a. Suppose c' is on the same side $\mathcal{H}$ of ℓ as b — we need to show that this is impossible. Since c' is not on $\overrightarrow{ab}$, then a is between b and c'. But since $\mathcal{H}$ is convex and contains both b and c', it must also contain a. Oops! This is not possible since a is on the line ℓ which has no intersection with $\mathcal{H}$. Hence c' must be on the opposite side of ℓ as b, as we desired. □

Now consider an angle $\angle abc$. By definition this angle is the union of two rays, $\overrightarrow{ba} \cup \overrightarrow{bc}$, and the three points a, b, c are non-collinear. Hence a lies in one of the two half planes determined by $\overleftrightarrow{bc}$ — denote this half plane as $\mathcal{H}_a$. Similarly c lies in one of the two half planes determined by $\overleftrightarrow{ab}$ — denote this half plane as $\mathcal{H}_c$. The *interior of angle* $\angle abc$ is the intersection of $\mathcal{H}_a$ and $\mathcal{H}_c$, as shown in Figure 5.8.

Definition 5.7.

The **interior of angle** $\angle abc$, denoted int $\angle abc$, is the intersection

$$\text{int}\,\angle abc = \mathcal{H}_a \cap \mathcal{H}_c$$

where $\mathcal{H}_a$ is the half plane with edge $\overleftrightarrow{bc}$ containing a, and
$\mathcal{H}_c$ is the half plane with edge $\overleftrightarrow{ab}$ containing c.

From Proposition 5.6 we know the half plane $\mathcal{H}_a$ contains the full ray $\overrightarrow{ba}$ (except for the point b). Similarly $\mathcal{H}_c$ contains $\overrightarrow{bc}$ (except for b).

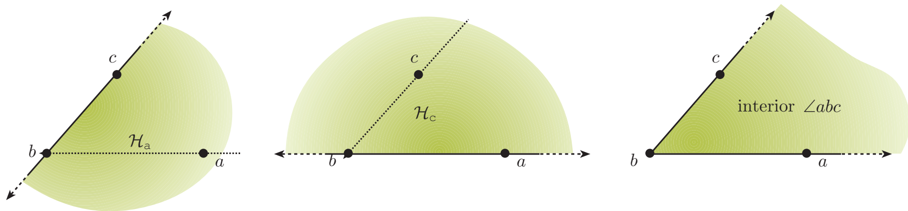
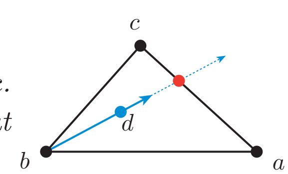


Figure 5.8. An angle interior is the intersection of half planes $\mathcal{H}_a$ and $\mathcal{H}_c$.

The following result, the *Crossbar Theorem*, is highly intuitive and quite important. It is also not easy to prove! However, it is a consequence of the three axioms we have given thus far, and we provide an outline of the steps of the proof in Exercise 5.10.

Theorem 5.9. ***The Crossbar Theorem.***

Suppose d is in the interior of the angle $\angle abc$. Then the ray $\overrightarrow{bd}$ intersects the line segment $\overline{ac}$ at a point between a and c.



Here is an important application of the *Crossbar Theorem.*

Definition 5.10.

Suppose a, b, c, d are four non-collinear points in the plane such that the four line segments $\overline{ab}$, $\overline{bc}$, $\overline{cd}$, $\overline{da}$ intersect only at their endpoints.

(a) The **quadrilateral** $\square abcd$ is the union $\overline{ab} \cup \overline{bc} \cup \overline{cd} \cup \overline{da}$. These four line segments are the **sides** of the quadrilateral. The two line segments $\overline{ac}$ and $\overline{bd}$ are the **diagonals** of the quadrilateral.

(b) The quadrilateral is **convex** if each side lies entirely in one of the half planes determined by (the line containing) the opposite side.

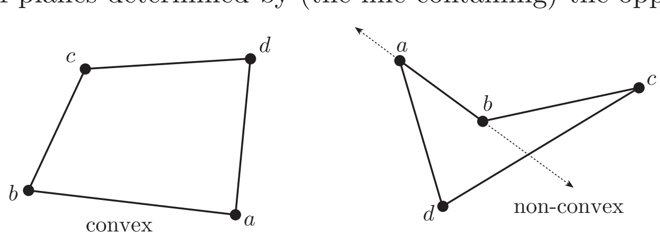


Figure 5.11. Two quadrilaterals.

The second quadrilateral in Figure 5.11 is not convex since the side $\overline{cd}$ does *not* lie in just one of the half planes determined by the opposite line $\overleftrightarrow{ab}$.

The picture of the convex quadrilateral in Figure 5.11 should make the following proposition intuitively believable.

Proposition 5.12.
Each vertex of a convex quadrilateral is in the opposite angle's interior.

Proof. Consider vertex d in the first (convex) quadrilateral shown in Figure 5.11. We will show that d is in the interior of angle $\angle abc$, i.e., that d is in the two half planes $\mathcal{H}_a$ and $\mathcal{H}_c$ whose intersection gives the interior of $\angle abc$. Since the quadrilateral is convex, we know $\overline{cd}$ is on one side of the line $\overleftrightarrow{ab}$, i.e., in the half plane $\mathcal{H}_c$. In particular, d is in $\mathcal{H}_c$. Similarly, $\overline{ad}$ is on one side of $\overleftrightarrow{bc}$, i.e., in the half plane $\mathcal{H}_a$. Thus d is in $\mathcal{H}_a$. Hence d is in $\mathcal{H}_a \cap \mathcal{H}_c$, the interior of $\angle abc$, as desired. □

As can be seen in the second quadrilateral in Figure 5.11, the two diagonals $\overline{ac}$ and $\overline{bd}$ do not always intersect. However, we claim the diagonals for any *convex* quadrilateral always intersect. The proof, however, will require the *Crossbar Theorem.*

Proposition 5.13.
The diagonals of a convex quadrilateral always intersect each other.

Proof. Suppose $\square abcd$ is a convex quadrilateral as shown on the left in Figure 5.11. From Proposition 5.12 we know that d is in the interior of $\angle abc$. Thus, by Theorem 5.9 — the *Crossbar Theorem* — the ray $\overrightarrow{bd}$ intersects the line segment $\overline{ac}$ at some point p. This is shown on the left half of Figure 5.14.

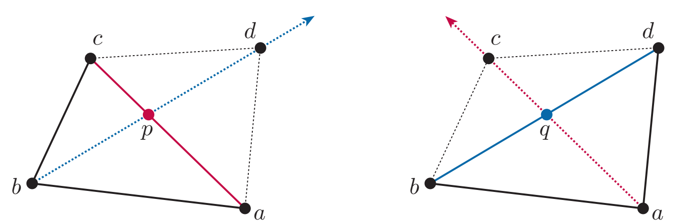


Figure 5.14. Two applications of the *Crossbar Theorem.*

Another application of the *Crossbar Theorem* to the vertex c (which is in the interior of $\angle dab$) shows that the ray $\overrightarrow{ac}$ intersects the line segment $\overline{bd}$ at some point q. This is shown in the right half of Figure 5.14. Now suppose p and q are not the same point. Then p and q would be two distinct points on the line $\overleftrightarrow{ac}$ as well as the line $\overleftrightarrow{bd}$. Hence, since there is only one line containing two distinct points (Axiom I-2), then $\overleftrightarrow{ac}$ and $\overleftrightarrow{bd}$ would be the same line, and hence a, b, c, d would all be collinear. This is not possible for a quadrilateral, and hence $p = q$. But since p lies on the diagonal line segment $\overline{bd}$ and q lies on the diagonal line segment $\overline{ac}$, the two diagonals intersect, as desired. □

We define the *interior of a triangle* in a manner similar to the definition of the interior of an angle. The **interior of a triangle** $\triangle abc$ is merely the intersection of three half planes: the side of $\overline{ab}$ containing c, the side of $\overline{bc}$ containing a, and the side of $\overline{ac}$ containing b.

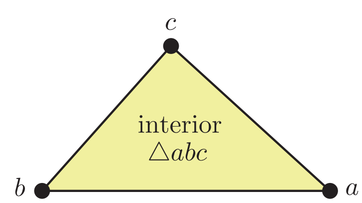
Definition 5.15.

The **interior of triangle** $\triangle abc$, denoted $\operatorname{int} \triangle abc$, is the intersection

$$\operatorname{int} \angle abc = \mathcal{H}_a \cap \mathcal{H}_b \cap \mathcal{H}_c$$

where $\mathcal{H}_a$ is the half plane with edge $\overleftrightarrow{bc}$ containing a,
$\mathcal{H}_b$ is the half plane with edge $\overleftrightarrow{ac}$ containing b,
$\mathcal{H}_c$ is the half plane with edge $\overleftrightarrow{ab}$ containing c.

Thus the interior consists of all the points on the "inside" of the triangle. In particular, points on the sides of the triangle are *not* part of the interior:



The following properties of the interior of a triangle are easily established. Their proofs are left for Exercise 5.6.

Proposition 5.16.

(a) *The interior of a triangle is a convex set.*

(b) *The interior of a triangle is the intersection of the interiors of its three angles.*

(c) *The interior of a triangle is the intersection of the interiors of* ***any two*** *of its angles.*

Exercises I.5

Exercise 5.1.

(a) Suppose A and B are convex subsets of the plane $\mathcal{E}$. Prove the intersection $A \cap B$ is also convex. What about the union, $A \cup B$?

(b) Let A be any set of points in the plane and let B be the union of all the line segments $\overline{pq}$ where p and q are both points of A. Is the set B convex? Either prove this is true or produce a counterexample.

Exercise 5.2.

Suppose the line ℓ does not contain any of the vertices of the triangle $\triangle abc$. Prove ℓ cannot intersect all three of the sides of the triangle. (*Hint:* Apply the *Plane Separation Axiom* to ℓ and the three line segments that comprise the sides of $\triangle abc$.)

Exercise 5.3.

Prove any half plane contains at least three non-collinear points. Be sure to cite all necessary axioms and propositions.

Exercise 5.4.

In a triangle $\triangle abc$ the angle which is **opposite** to the side $\overline{bc}$ is the interior angle with vertex at a, i.e., $\angle bac$. Prove every side of a triangle (except for its vertices) is in the interior of its opposite angle. (*Hint:* Proposition 5.6.)

Exercise 5.5.

(a) Suppose x_1 and x_2 lie on one side of $\overleftrightarrow{ab}$. If a, x_1, and x_2 are not collinear, prove either x_1 is in the interior of $\angle bax_2$ or x_2 is in the interior of $\angle bax_1$.

(b) Suppose x_1 and x_2 lie on opposite sides of $\overleftrightarrow{ab}$ and a, x_1, and x_2 are not collinear. Let b' be a point of $\overleftrightarrow{ab}$ such that a is between b and b'. Prove either b or b' is in the interior of $\angle x_1ax_2$.

Exercise 5.6.

Prove the three parts of Proposition 5.16:

(a) The interior of a triangle is a convex set.

(b) The interior of a triangle is the intersection of the interiors of its three angles.

(c) The interior of a triangle is the intersection of the interiors of ***any two*** of its angles.

Exercise 5.7.

Prove that if a line intersects the interior of a triangle, then the line must intersect at least one side of the triangle.

Exercise 5.8.

Prove the converse of Proposition 5.13: if the diagonals of a quadrilateral intersect, then the quadrilateral is convex.

Exercise 5.9.

Determine if the *Plane Separation Axiom* is valid for the following examples from §2. Explain your reasoning in each case.

(a) The real Cartesian plane $\mathbb{R}^2$, Example 2.2.

(b) The Poincaré disk $\mathbb{P}^2$, Example 2.4.

(c) The real projective plane $\mathbb{RP}^2$, Example 2.6.

(d) The Moulton plane $\mathbb{MP}^2$, Example 2.7.

(e) Real Cartesian space $\mathbb{R}^3$, Example 2.9.

Exercise 5.10. ***The Crossbar Theorem.***
This exercise leads you through a proof of Theorem 5.9, the *Crossbar Theorem*. For that reason you cannot use any results of the text past Theorem 5.9 nor any exercise that depends on this theorem.

(a) Suppose $\ell = \overleftrightarrow{x_1x_2}$ with y_1 and y_2 points on opposite sides of ℓ. Prove the rays $\overrightarrow{x_1y_1}$ and $\overrightarrow{x_2y_2}$ do not intersect.

(b) Given a triangle $\triangle abc$, let d be a point between a and c, and let e be a point on the same side of $\ell = \overleftrightarrow{ac}$ as b. Prove the ray $\overrightarrow{de}$ intersects either the line segment $\overline{ab}$ or the line segment $\overline{bc}$.
Outline. First draw a picture! With $\overrightarrow{de'}$ as the ray opposite to $\overrightarrow{de}$ and with $\ell = \overleftrightarrow{ac}$, use (a) to show $\overrightarrow{de'}$ does not intersect $\overline{ab}$ or $\overline{bc}$. Proposition 5.5 will then complete the proof.

(c) Prove Theorem 5.9, the *Crossbar Theorem*.
Outline. Let $\overrightarrow{ba'}$ be the ray opposite to $\overrightarrow{ba}$. Then use (b) to show $\overrightarrow{bd}$ intersects either $\overline{ca'}$ or $\overline{ca}$. However, use (a) with $\ell = \overleftrightarrow{bc}$ to show that $\overrightarrow{ca'}$ and $\overrightarrow{bd}$ cannot intersect. Conclude that $\overrightarrow{bd}$ must intersect $\overline{ca}$, and show that the point of intersection cannot be a or c. This will prove the *Crossbar Theorem*.

§I.6 The Angular Measure Axioms

Let $\mathcal{A}$ denote the collection of all angles in the plane. In this section we assume a new *undefined object* for our axiomatic system: an **angle measure function** $m : \mathcal{A} \to \mathbb{R}$ that assigns to each angle $\angle A$ a real number $m\angle A$, understood to be its measure in degrees.[5]

The basic properties of this function are specified by four *Angle Measure Axioms*. None of these axioms will surprise you — they are all intuitively clear from our usual notion of angle measurement. What is less obvious is that this list of axioms, when combined with our other axioms for Euclidean geometry as summarized in §15, is sufficient to completely characterize angular measure, i.e., no other notion of angular measure can satisfy this complete set of axioms.

[5] For geometry any "unit of angular measure" may be employed — we initially choose a *degree* since that is the most universally understood. *Radian* measure, while necessary to produce good calculus properties with the trigonometric functions, is not required for purely geometric goals. We will use both measures in this text, choosing the most convenient for the task at hand.

Since one of the axioms concerns *linear pairs* of angles, we preface our listing of the *Angle Measure Axioms* with the definition of a linear pair.

Definition 6.1.

Two angles $\angle bac$ and $\angle cad$ sharing a common vertex a and a common side $\overrightarrow{ac}$ form a **linear pair** if their non-common sides $\overrightarrow{ab}$ and $\overrightarrow{ad}$ are *opposite rays*, i.e., the union of the non-common sides $\overrightarrow{ab}$ and $\overrightarrow{ad}$ is the full line $\overleftrightarrow{bd}$.

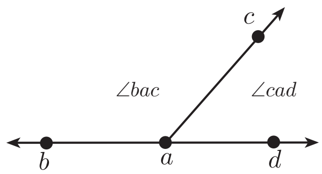


Figure 6.2. $\angle bac$ and $\angle cad$ form a linear pair.

Angle Measure Axioms.

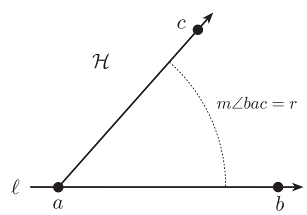
Let $\mathcal{A}$ denote the collection of all angles in the plane. Then there exists an **angle measure** function $m : \mathcal{A} \to \mathbb{R}$ with the following properties.

M-1. For every angle $\angle A$, its measure $m\angle A$ is between 0 and 180°:

$$0 < m\angle A < 180°.$$

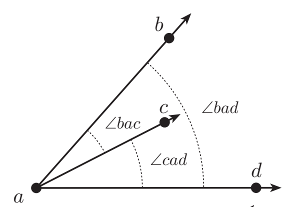
M-2. ***Angle Construction***. Suppose $\overrightarrow{ab}$ is a ray on the line ℓ and $\mathcal{H}$ is one of the two half planes with edge ℓ. Then for every number $0 < r < 180°$ there is a unique ray $\overrightarrow{ac}$, with c in $\mathcal{H}$, such that

$$m\angle bac = r.$$



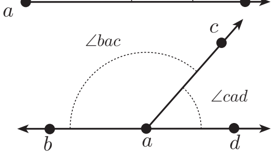
M-3. ***Angle Addition***. If c is a point in the interior of $\angle bad$, then

$$m\angle bad = m\angle bac + m\angle cad.$$



M-4. ***Supplements***. If two angles $\angle bac$ and $\angle cad$ form a *linear pair*, then they are *supplementary*, i.e.,

$$m\angle bac + m\angle cad = 180°.$$



For notational convenience the relationship $m\angle abc < m\angle def$ will often be written simply as $\angle abc < \angle def$.

Angle measure now allows us an easy way to define *congruence* of angles.[6]

[6]Recall the general discussion of congruence of geometric figures prior to Definition 4.9.

Definition 6.3. ***Angle Congruence.***

Two angles, $\angle A$ and $\angle B$, are **congruent**, written $\angle A \cong \angle B$, if and only if they have the same angular measure. Thus

$$\angle A \cong \angle B \quad \text{if and only if} \quad m\angle A = m\angle B.$$

This is a reasonable definition since, intuitively, an angle $\angle A$ can be "moved onto" an angle $\angle B$ if and only if the two angles have the same measure.

Results that prove two angles congruent are important in geometry. What follows — *the Vertical Angle Theorem* — is such a result.

Suppose two distinct lines ℓ and m intersect at a point a, and let b, b' be points of ℓ on opposite sides of a, and let c, c' be points of m on opposite sides of a. This is shown in Figure 6.4. Then the two angles $\angle bac$ and $\angle b'ac'$ are said to form a **vertical pair**.

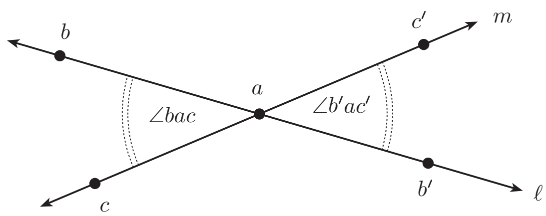


Figure 6.4. Angles $\angle bac$ and $\angle b'ac'$ form a *vertical pair*.

Theorem 6.5. ***The Vertical Angle Theorem***

If two angles form a vertical pair, then they are congruent.

Proof. Since the point a is between b and b', $\overrightarrow{ab}$ and $\overrightarrow{ab'}$ are opposite rays. Hence $\angle bac$ and $\angle cab'$ form a linear pair, so that the two angles are supplementary by Axiom M-4. Hence $m\angle bac + m\angle cab' = 180°$, so that

$$m\angle bac = 180° - m\angle cab'.$$

The same argument applies to the angles $\angle b'ac'$ and $\angle cab'$, yielding

$$m\angle b'ac' = 180° - m\angle cab'.$$

Combining our two equalities shows $m\angle bac = m\angle b'ac'$, proving the two angles $\angle bac$ and $\angle b'ac'$ are congruent, as desired. □

Angle Bisectors. An important consequence of the *Angle Measure Axioms* is the existence and uniqueness of an *angle bisector* for every angle.

Definition 6.6.

A ray $\overrightarrow{ad}$ is an **angle bisector** of angle $\angle bac$ if

(a) d is in the interior of $\angle bac$ and

(b) the angles $\angle bad$ and $\angle dac$ are congruent.

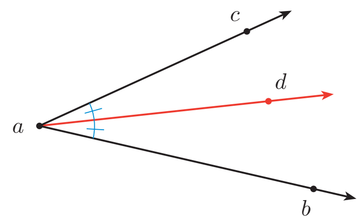


Figure 6.7. The ray $\overrightarrow{ad}$ bisects the angle $\angle bac$.

Proposition 6.8.

Every angle has a unique angle bisector.

Proof. We sketch the proof and leave the details to Exercise 6.2. Let $\ell = \overleftrightarrow{ab}$. Then by Angle Axiom M-2 there exists a unique ray $\overrightarrow{ad}$ with d on the same side of ℓ as c such that $m\angle bad = \frac{1}{2}m\angle bac$. We must show the following.

(a) The point d is in the interior of $\angle bac$. If this is not true, then you can show that point c would have to be in the interior of $\angle bad$, and this will lead to a contradiction of Angle Axiom M-3.

(b) $m\angle bad = m\angle dac$. This follows from (a) and Angle Axiom M-3.

(c) The ray $\overrightarrow{ad}$ is the *only* angle bisector for $\angle bac$. Just apply Angle Axiom M-2. $\square$

Perpendicularity. By Angle Axiom M-4 the angle measures of a linear pair of angles sum to 180°. Hence if linear pair angles are congruent to each other, then each has measure 90°. We call such angles *right angles*.

Definition 6.9.

(a) A **right angle** is an angle whose measure is 90° (equivalently, an angle that is a member of a linear pair of congruent angles).

(b) An angle is **acute** if its measure is less than 90°.

(c) An angle is **obtuse** if its measure is greater than 90°.

(d) Two angles are **complementary** if their measures add to 90°.

Perpendicularity is now easily defined for rays, lines, and line segments. Two such objects are perpendicular if their union produces a right angle, as formalized in the next definition.

Definition 6.10.

(a) Two rays, $\overrightarrow{ab}$ and $\overrightarrow{ac}$, with the same initial point are **perpendicular**, written $\overrightarrow{ab} \perp \overrightarrow{ac}$, if their union $\angle bac$ is a right angle. Thus

$$\overrightarrow{ab} \perp \overrightarrow{ac} \quad \text{if and only if} \quad m\angle bac = 90^\circ.$$

(b) Two lines with intersection point a, which can therefore be expressed in the forms $\overleftrightarrow{ab}$ and $\overleftrightarrow{ac}$, are **perpendicular** if the corresponding rays are perpendicular, i.e.,

$$\overleftrightarrow{ab} \perp \overleftrightarrow{ac} \quad \text{if and only if} \quad \overrightarrow{ab} \perp \overrightarrow{ac}.$$

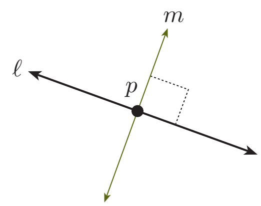
(c) Two line segments, $\overline{ab}$ and $\overline{ac}$, with a common endpoint a are **perpendicular** if the corresponding lines are perpendicular, i.e.,

$$\overline{ab} \perp \overline{ac} \quad \text{if and only if} \quad \overleftrightarrow{ab} \perp \overleftrightarrow{ac}.$$

The existence of many perpendicular lines follows from the next result.

Proposition 6.11.

Suppose ℓ is a line in the plane $\mathcal{E}$ and p is any point of ℓ. Then there exists a unique line m containing p which is perpendicular to ℓ.



Proof. Pick a point b on ℓ which is *not* equal to p. According to Angle Axiom M-2 we can find a point c such that $m\angle bpc = 90°$ (Figure 6.12.). Thus $m = \overleftrightarrow{pc}$ is the desired perpendicular to ℓ.

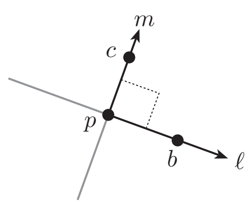


Figure 6.12. Pick c so that $\angle bpc$ is a right angle.

This proves *existence* of the perpendicular line m. We now prove *uniqueness*. Suppose m' is another line through p which is perpendicular to ℓ. Then pick a point c' on m' on the same side of ℓ as the point c, as in Figure 6.13.

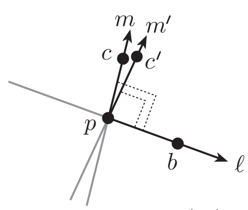


Figure 6.13. Suppose $m = \overleftrightarrow{pc}$ and $m' = \overleftrightarrow{pc'}$ are both perpendicular to ℓ.

Consider the angle $\angle bpc'$. This must be a right angle, a consequence of Exercise 6.3. Hence, since $\angle bpc$ is also a right angle by construction, then Angle Axiom M-2 gives that the two rays $\overrightarrow{pc}$ and $\overrightarrow{pc'}$ must be equal. But this means the two lines $m = \overleftrightarrow{pc}$ and $m' = \overleftrightarrow{pc'}$ are equal, as desired. □

Directed Angle Measure at a Point. Sometimes a concept can be more complicated than it appears on the surface. This is the case for "clockwise" and "counterclockwise" rotation, or the more general procedure of defining **directed angular measure**. This development illuminates the subtleties inherent in the dichotomy of clockwise/counterclockwise.

Why does the concept of "clockwise rotation" around a point seem so *intuitively* clear? Simple: because we are all familiar with clocks, having *seen* them all our lives. And since all clocks "rotate in the same direction," we have no question as to what is meant by clockwise rotation. However, suppose you had to describe, solely in words, the rotational direction specified by "clockwise" to someone who had never seen a clock? You would most certainly draw a picture or wave your hand in an appropriate way. But suppose you could not use pictures or hand gestures or anything other than words?! Try to do this! *Do you see the problem?* [7]

The difficulty is that clockwise and counterclockwise are merely the mirror images of each other, and there is no inherent way to distinguish one rotational direction from the other except that each is the opposite of the other! Unless you can actually "show" someone the clockwise direction (which could mean simply identifying something with clockwise rotation which is known to the person), there is no fundamental way to distinguish one rotational direction from the other.

So we are forced to pick some object with a fixed rotational direction (we use a clock), arbitrarily designate its rotational direction as "clockwise," and then classify all other rotations as *clockwise* or *counterclockwise* via comparison with the fixed, "standard" clock.

We now formalize our concept of directed angle measure. Let a_0 be any chosen point in the plane $\mathcal{E}$. We will show there are exactly two inherently different and natural choices for directed angle measure at the point (or vertex) a_0. To do so, we will use the concept of *equality modulo* 360:

Definition 6.14.

Two numbers, θ_1 and θ_2, are **equal modulo 360**,

$$\theta_1 = \theta_2 \bmod 360,$$

if they differ by a multiple of 360, i.e., $\theta_1 = \theta_2 + 360k$ for an integer k.

The basic properties of equality modulo 360 will be developed in Exercise 6.6.

We use the notation $\measuredangle A = \measuredangle bac$ to denote the **directed angle** with **initial ray** $\overrightarrow{ab}$ and **terminal ray** $\overrightarrow{ac}$, as given in Definition 4.8. A directed angle is **trivial** if it equals just one ray, i.e., if the two rays comprising the angle are the same. A **straight angle** is a directed angle in which the two rays are the opposites of each other. If a directed angle $\measuredangle A$ is not trivial or straight, then the two rays comprising $\measuredangle A$ form an ordinary angle which we naturally denote as $\angle A$.

[7] The phenomenal success of digital watches may result in a crisis of rotational confusion: the loss of the one universally recognized standard for classifying the two rotational directions.

To each directed angle $\measuredangle A$ with vertex at a_0 we will associate a *directed angle measure*, i.e., a real number $m\measuredangle A = \theta$ in the range $-180° < \theta \le 180°$ with the following properties:

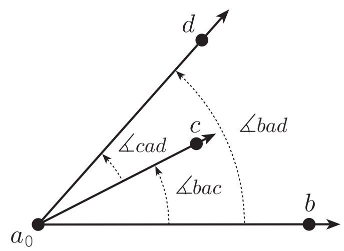
(1) $\measuredangle A$ is trivial if and only if $m\measuredangle A = 0°$.

(2) $\measuredangle A$ is a straight angle if and only if $m\measuredangle A = 180°$.

(3) If $\measuredangle A$ is not trivial or a straight angle, then $|m\measuredangle A| = m\angle A$.

(4) If b, c, d are *any* three points other than a_0, then

$$m\measuredangle ba_0d = m\measuredangle ba_0c + m\measuredangle ca_0d \mod 360:$$



Hence directed angle measure must agree up to sign with ordinary angle measure when the latter is defined, and the choices of the signs for the directed angle measure are to be made so that directed angle addition is valid in all cases.

Restricting the values of directed angle measure to $-180° < \theta \le 180°$ is logically complete and straightforward in that all directed angles are assigned a measure and this measure is natural and unique. However, there is a practical, interpretive problem: our initial formulation of directed angle measure does not convey a *dynamic* notion of rotation.

For example, consider a directed angle $\measuredangle ba_0c$ with directed measure 90°. This provides a *static* interpretation for the angle: there is a *positive* angle of 90° separating the initial ray $\overrightarrow{a_0b}$ from the terminal ray $\overrightarrow{a_0c}$. However, if we really wish to consider a *dynamic*, rotatory process which takes $\overrightarrow{a_0b}$ to $\overrightarrow{a_0c}$, then we have a problem: *there are an infinite number of different "rotations" which could move $\overrightarrow{a_0b}$ onto $\overrightarrow{a_0c}$!* Only one of these rotations is accurately modeled by the measure 90°, and that is a "quarter turn in the positive direction." However, what about a "three-quarter turn in the negative direction"? Or a "one-and-a-quarter turn in the positive direction"?

These other rotatory movements are best described by angle measurements which indicate the number of *total revolutions* about the vertex, as well as the direction of the revolutions. A "three-quarter turn in the negative direction" is best described by the measure $-270°$, while a "one-and-a-quarter turn in the positive direction" is described by the measure 450°.

Thus a fixed directed angle $\measuredangle A$ represents an infinite number of rotatory movements, each with its own angle measurement. Any two of these measurements will differ by an integer multiple of 360°; hence *all measurements associated with a given directed angle will be equal modulo* 360.

For this reason we revise the definition of a directed angle measure $m\measuredangle A$ to be a *collection* of numbers, described in the next definition.

Definition 6.15.

A **directed angle measure** m at a point a_0 is an assignment to each directed angle $\measuredangle A$ with vertex at a_0 of a collection $m\measuredangle A$ of all those real numbers which are equal modulo 360 and have the following properties:

(1) $\measuredangle A$ is trivial if and only if $m\measuredangle A = 0^\circ \mod 360$.

(2) $\measuredangle A$ is a straight angle if and only if $m\measuredangle A = 180^\circ \mod 360$.

(3) If $\measuredangle A$ is not trivial or a straight angle, then $|\theta| = m\angle A$, where θ is that value of $m\measuredangle A$ such that $-180^\circ < \theta < 180^\circ$.

(4) If b, c, d are *any* three points other than a_0, then
$$m\measuredangle ba_0d = m\measuredangle ba_0c + m\measuredangle ca_0d \mod 360.$$

For each directed angle $\measuredangle A$ there is a unique θ in the collection $m\measuredangle A$ which falls in the range $-180^\circ < \theta \leq 180^\circ$. This is the directed angle measure we defined initially. We call this θ the **standard measure** of $\measuredangle A$. In what follows, we will often consider the standard measure θ as *the* directed angle measure of $\measuredangle A$. Although confusing at first, this convention is convenient.

See Exercise 6.7 for properties of directed angle measure at a point a_0.

Existence of Directed Angle Measure at a Point. Although we have defined the meaning of directed angle measure at a point, that does not prove a directed angle measure actually exists. Fortunately directed angle measures do exist — exactly two of them:

Proposition 6.16.

Exactly two directed angle measures exist at each point a_0 in the Euclidean plane $\mathcal{E}$. Each measure is the negative of the other.

This distinguishes between the two possible directions for rotation at a point. It may seem so obvious a result that it needs no proof. However, it is "obvious" only since our intuitive picture for the plane has two rotational directions at any point. A proof is needed to show that this property actually follows from our axioms for the Euclidean plane as thus far developed.

Since the statement of Proposition 6.16 is so simple and intuitively believable, the proof may be skipped by readers not interested in so formal a development of directed angle measure.

Proof of Proposition 6.16. (*Optional.*) We first verify that there are *at most two* possible directed angle measures at a_0. To see this, choose a right angle $\angle b_0 a_0 c_0$. Then, if m is a directed angle measure at a_0, there are two possible standard values for m on $\measuredangle b_0 a_0 c_0$: $90°$ or $-90°$.

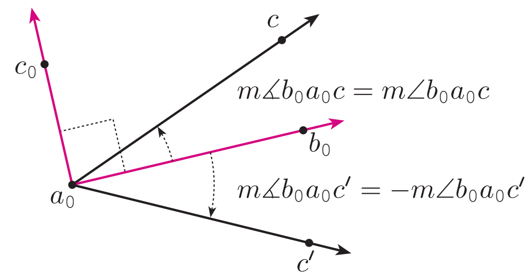


Figure 6.17. One *positive* right angle fixes all directed angle measures.

We claim the standard value chosen for $m\measuredangle b_0a_0c_0$ determines the measures of all the other directed angles with vertex at a_0. To see this, suppose c is any point on the same side of $\overleftrightarrow{a_0b_0}$ as c_0, as shown in Figure 6.17. Then the standard value of $m\measuredangle b_0a_0c$ must be positive for otherwise additivity could not be true (see Exercise 6.7c). Hence, in view of property (3), the standard value of $m\measuredangle b_0a_0c$ must equal the ordinary angle measure $m\angle b_0a_0c$. This completely determines the value set of $m\measuredangle b_0a_0c$.

Now consider any point c' on the opposite side of $\overleftrightarrow{b_0a_0}$ from c_0, also shown in Figure 6.17. Then the standard measure of the directed angle $\measuredangle b_0a_0c'$ must be negative, which implies that the standard measure of $m\measuredangle b_0a_0c'$ equals the negative of the ordinary angle measure $m\angle b_0a_0c'$. Thus the value set of $m\measuredangle b_0a_0c'$ is completely determined.

Finally, the measure of any directed angle $\measuredangle ca_0d$ with vertex a_0 is given by

$$m\measuredangle ca_0d = m\measuredangle ca_0b_0 - m\measuredangle da_0b_0 \mod 360.$$

Thus specifying one of the two possible standard values for $m\measuredangle b_0a_0c_0$ determines the measures of all other directed angles at a_0, proving there are no more than two distinct directed angle measure functions at a point a_0.

We now show by construction that two directed angle measure functions do indeed exist. Fix a ray $\overrightarrow{a_0b_0}$, and let $\ell_0 = \overleftrightarrow{a_0b_0}$ be the directed line with positive direction given by $\overrightarrow{a_0b_0}$. Choose one of the two sides of ℓ_0, denoted by $\mathcal{H}_0^+$, as the *positive* (or *counterclockwise*) side. Let $\mathcal{H}_0^-$ denote the opposite, or *negative*, side. For any point x in the plane other than a_0, assign a number to the ray $\overrightarrow{a_0x}$ in the following way (illustrated in Figure 6.19):

$$\begin{aligned} m(\overrightarrow{a_0x}) = &\quad 0^\circ \;\text{ if } \overrightarrow{a_0x} = \overrightarrow{a_0b_0} \text{ (example is } x = x_0 \text{ in Figure 6.19)}, \\ = &\quad 180^\circ \;\text{ if } \overrightarrow{a_0x} \text{ is opposite to } \overrightarrow{a_0b_0} \text{ (e.g., } x = x_2), \\ = &\quad m\angle b_0a_0x \text{ if } x \text{ is in } \mathcal{H}_0^+, \text{ the positive side of } \ell_0 \text{ (e.g., } x = x_1), \\ = &-m\angle b_0a_0x \text{ if } x \text{ is in } \mathcal{H}_0^-, \text{ the negative side of } \ell_0 \text{ (e.g., } x = x_3). \end{aligned}$$

Using these numerical assignments for rays beginning at a_0, we can define a measure m for the directed angles at a_0 by the following formula:

$$m\measuredangle ca_0d = m(\overrightarrow{a_0d}) - m(\overrightarrow{a_0c}) \quad \text{mod } 360. \tag{6.18}$$

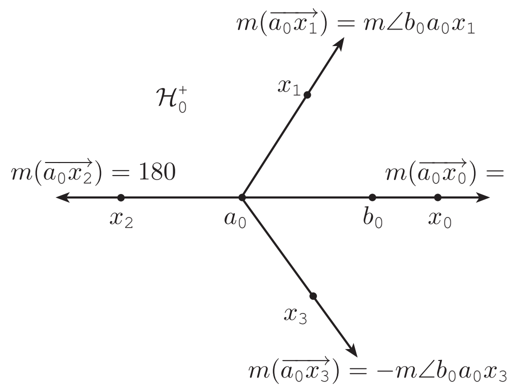


Figure 6.19. Defining a directed angle measure from rays.

An elementary but meticulous verification is required to show that m does indeed satisfy the four desired properties for a directed angle measure at a_0. This is outlined in Exercise 6.8. Hence, since there are two choices for $\mathcal{H}_0^+$, the positive side of the directed line $\ell_0 = \overleftrightarrow{a_0b_0}$, we have constructed both possible directed angle measures at a_0. $\square$

Exercises I.6

Exercise 6.1.

Show that congruence of angles is an equivalence relation as defined in §4 following Definition 4.9.

Exercise 6.2.

Supply the details for the proof of Proposition 6.8:

Every angle has exactly one angle bisector.

Exercise 6.3.

According to our definition, two lines are said to be perpendicular if at least one of the four angles formed at their intersection is a right angle. In such a case prove that *all four angles* must be right angles.
Hint: The *Vertical Angle Theorem.*

Exercise 6.4.

(a) Suppose x_1 and x_2 lie on one side of $\overleftrightarrow{ab}$. Prove that x_1 is in the interior of $\angle bax_2$ if and only if $0 < \angle bax_1 < \angle bax_2$.

Hint: The proof of the reverse direction is tricky. A proof by contradiction seems advisable, and you may find it helpful to use Proposition 5.6 or Exercise 5.4.

(b) Suppose x_1 and x_2 lie on opposite sides of $\overleftrightarrow{ab}$. Prove that b is in the interior of $\angle x_1ax_2$ if and only if $m\angle x_1ab + m\angle x_2ab < 180°$.

Exercise 6.5.

Suppose c and c' are points on opposite sides of a line ℓ and b, a, b' are three points on ℓ such that a is between b and b' — see Figure 6.20. Prove that c, a, c' are collinear if and only if $\angle bac \cong \angle b'ac'$. (This exercise will be used in Chapter II when proving reflections are isometries.) *Hint:* For the reverse implication, pick c'' to be a point such that $\overrightarrow{ac''}$ is the ray opposite to $\overrightarrow{ac}$. Then c, a, c'' are collinear; show that this implies $\angle bac \cong \angle b'ac''$. Then show that $\overrightarrow{ac''} = \overrightarrow{ac'}$ by application of Angle Axiom M-2, *Angle Construction*.

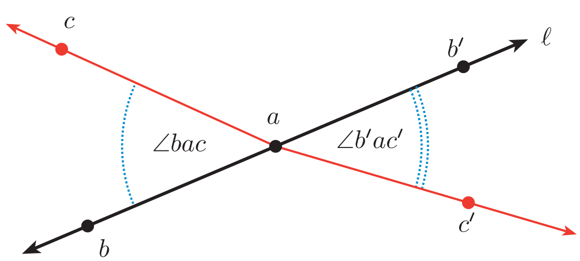


Figure 6.20.

Exercise 6.6.

The algebraic properties of equality modulo 360 are as expected for addition and subtraction but are less well-behaved for multiplication.

(a) If $\theta = \alpha$ mod 360, then prove $-\theta = -\alpha$ mod 360.

(b) If $\theta = \alpha$ mod 360 and $\psi = \beta$ mod 360, then prove

$$\theta + \psi = \alpha + \beta \text{ mod } 360 \quad \text{and} \quad \theta - \psi = \alpha - \beta \text{ mod } 360.$$

(c) If $\theta = \psi + \alpha$ mod 360, then prove $\theta - \psi = \alpha$ mod 360.

(d) If $\theta = \alpha$ mod 360 and k is any real number, then is $k\theta = k\alpha$ mod 360? What if k is an *integer*? Prove those properties which are true and give counterexamples for those which are false.

(e) If $\theta = \alpha$ mod 360 and $\psi = \beta$ mod 360, then is $\theta\psi = \alpha\beta$ mod 360? Either prove this is true or give a counterexample.

Exercise 6.7.

Verify the following basic properties of a directed angle measure m at a point a_0. (Assume all points listed below are distinct from a_0.)

(a) Prove that $m\measuredangle ba_0c = -m\measuredangle ca_0b \mod 360$.
Hint: What can you say about $m\measuredangle ba_0c + m\measuredangle ca_0b$?

(b) Suppose c_1 and c_2 are points on opposite sides of $\ell_0 = \overleftrightarrow{a_0b}$ such that $\angle ba_0c_1 \cong \angle ba_0c_2$. Prove $m\measuredangle ba_0c_1 = -m\measuredangle ba_0c_2 \mod 360$.

(c) Suppose $m\measuredangle b_0a_0c_0 = 90° \mod 360$ and c is on the same side of the line $\overleftrightarrow{a_0b_0}$ as c_0. Prove that the standard value of $m\measuredangle b_0a_0c$ is positive.
Hint: Consider the cases when (i) $\overrightarrow{a_0c_0} = \overrightarrow{a_0c}$, (ii) c and b_0 are on the same side of $\overleftrightarrow{a_0c_0}$, and (iii) c and b_0 are on opposite sides of $\overleftrightarrow{a_0c_0}$. In each situation analyze the consequences of
$$m\measuredangle b_0a_0c_0 = m\measuredangle b_0a_0c + m\measuredangle ca_0c_0 \mod 360.$$

Exercise 6.8.

Suppose $\ell_0 = \overleftrightarrow{a_0b_0}$, with $\mathcal{H}_0^+$ chosen as the positive side of ℓ_0 and $\mathcal{H}_0^-$ chosen as the negative side. Let m be defined by (6.18) on directed angles with vertex at a_0. In this exercise you will verify that m is indeed a directed angle measure at a_0.

(a) Verify conditions (1) and (2) of Definition 6.15. These require simple case-by-case analyses.

(b) Verify condition (4) of Definition 6.15. This is easy.

(c) Proof of condition (3) of Definition 6.15 requires a case-by-case analysis. Verify the condition in the following cases:
  (i) c, d in $\mathcal{H}_0^+$, with c in the interior of $\angle b_0a_0d$.
  (ii) c, d in $\mathcal{H}_0^+$, with d in the interior of $\angle b_0a_0c$.
  (iii) c in $\mathcal{H}_0^+$, d in $\mathcal{H}_0^-$, and b_0 in the interior of $\angle ca_od$.
  (iv) c in $\mathcal{H}_0^+$, d in $\mathcal{H}_0^-$, and b_0' in the interior of $\angle ca_od$, where $\overrightarrow{a_0b_0'}$ is the ray opposite to $\overrightarrow{a_0b_0}$.

(d) The four subcases of (c) yield four more subcases when $\mathcal{H}_0^+$ and $\mathcal{H}_0^-$ are interchanged. Show that condition (3) for each of the new subcases can be deduced from the old by using $c' = \sigma_{\ell_0}(c)$ and $d' = \sigma_{\ell_0}(d)$ in conjuction with Exercise 6.7b.

(e) The remaining (singular) cases for condition (3) are those in which at least two of the points b_0, c, d are collinear with a_0. Verify condition (3) in these cases.

§I.7 Triangles and the SAS Axiom

We assume a coordinate system for each line in the plane and use these coordinate systems to define a distance function for $\mathcal{E}$. However, among our current axioms, only the *Plane Separation Axiom* requires any "compatibility" between coordinate systems on different lines. This results from relationships established by the axiom linking *betweenness* on different lines. We thus have compatibility restrictions between the distance functions along distinct lines, even if the nature of these restrictions is somewhat obscure.

In this section we add an axiom that explicitly involves the side lengths of triangles — it will result in greatly strengthen "global" distance relationships between coordinate systems on different lines.

The axiom gives conditions that imply *congruence* between two triangles. As discussed prior to Definition 4.9, the intuitive meaning of congruence for two geometric figures is that one of the figures can be "moved" onto the other so that all parts match in an identical fashion, i.e., when "moved," the two figures are "identical." For two triangles to be congruent, there should be a correspondence between the sides (and hence the angles) so that corresponding sides have the same length and corresponding angles have the same angular measure. We formalize this notion in the next definition.

Definition 7.1. ***Triangle Congruence.***

(a) A **congruence** between two triangles, written $\triangle abc \cong \triangle ABC$, means that corresponding sides are congruent and corresponding angles are congruent, i.e.,

$$ab = AB, \quad bc = BC, \quad ac = AC$$

and

$$m\angle a = m\angle A, \quad m\angle b = m\angle B, \quad m\angle c = m\angle C.$$

(b) Triangles are **congruent** if there exists a congruence between them.

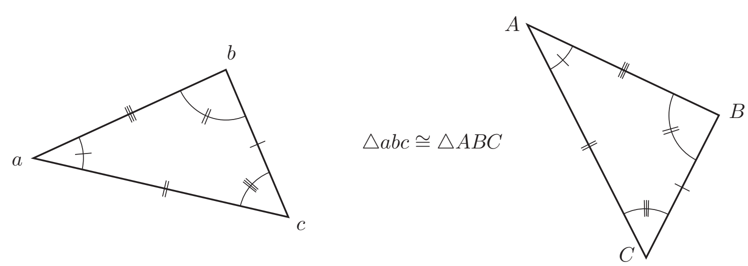


Figure 7.2. $\triangle abc \cong \triangle ABC$ via the correspondence $abc \leftrightarrow ABC$.

We need two parts for Definition 7.1 for the following reason. To say that the two triangles $\triangle abc$ and $\triangle ABC$ are congruent means that *one of the six possible correspondences* between the two triangles must be a congruence, i.e., one the following six possible congruences must be valid:

$$\triangle abc \cong \triangle ABC, \quad \triangle abc \cong \triangle ACB,$$
$$\triangle abc \cong \triangle BAC, \quad \triangle abc \cong \triangle BCA,$$
$$\triangle abc \cong \triangle CAB, \quad \triangle abc \cong \triangle CBA.$$

For example, in Figure 7.3 the triangles $\triangle abc$ and $\triangle ABC$ are congruent, but not via the "standard" correspondence $abc \leftrightarrow ABC$, i.e., it is *not* true that $\triangle abc \cong \triangle ABC$. The correspondence which *does* work is $abc \leftrightarrow BCA$, so the congruence which *is* valid is $\triangle abc \cong \triangle BCA$.

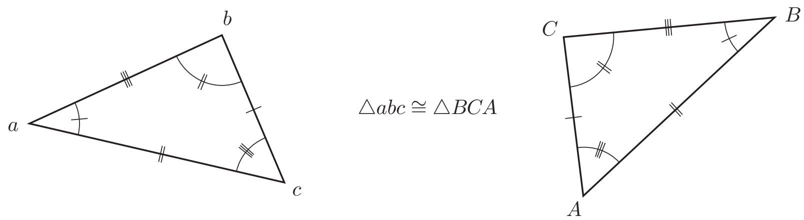


Figure 7.3. $\triangle abc \cong \triangle ABC$ via the correspondence $abc \leftrightarrow BCA$.

To prove two triangles are congruent from the definition, one needs to verify that some correspondence between vertices yields a congruence. This means verifying six equalities as given in Definition 7.1a. In fact, there are several standard results in Euclidean geometry that state that a congruence between triangles can be established by verifying merely three of the six desired equalities. In other words, if certain collections of three of the equalities are true, then the other three equalities will automatically be true.

However, these results — known as the *Basic Congruence Theorems* — are not consequences of the seven axioms we have assumed thus far — this will be shown in Exercise 7.3. We must add another axiom. Our method is to assume one of the desired congruence results as the new axiom.

The SAS (Side-Angle-Side) Axiom.

SAS. Suppose a correspondence between two triangles is such that two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle. Then the correspondence is a congruence between the two triangles.

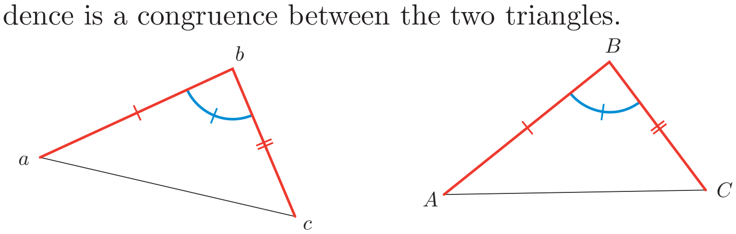


Figure 7.4. The *SAS Axiom* states that in this situation $\triangle abc \cong \triangle ABC$.

Stated in symbols, suppose $\triangle abc$ and $\triangle ABC$ are two triangles such that $\overline{ab} \cong \overline{AB}$, $\angle b \cong \angle B$, and $\overline{bc} \cong \overline{BC}$, as shown in Figure 7.4. Then the

SAS Axiom states that the two triangles are congruent via the congruence $\triangle abc \cong \triangle ABC$.

Given the *SAS Axiom*, we can establish many of the elementary geometry results concerning triangles. One of the most basic (and important!) is Euclid's *Pons Asinorum*.[8]

Proposition 7.5. ***Pons Asinorum*.**

If two sides of a triangle are congruent, then the angles opposite them are congruent.

Proof. Suppose $\triangle abc$ is such that $\overline{ab} \cong \overline{bc}$, as shown in Figure 7.6. We must verify $\angle a \cong \angle c$. The trick is to apply *SAS* to the correspondence $abc \leftrightarrow cba$ of the triangle $\triangle abc$ with itself. In other words, we will use *SAS* to verify the congruence $\triangle abc \cong \triangle cba$. To do so, notice that we have the necessary ingredients for the *SAS Axiom:* $\overline{ab} \cong \overline{bc}$, $\overline{cb} \cong \overline{ba}$, and $\angle b \cong \angle b$.

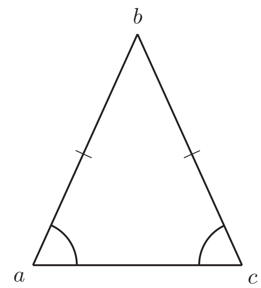


Figure 7.6

Hence $\triangle abc \cong \triangle cba$ by the *SAS Axiom,* and thus the corresponding angles $\angle a$ and $\angle c$ must indeed be congruent, as desired. □

Pons Asinorum concerns *isosceles* triangles, as we now define.

Definition 7.7.

(a) A triangle is **isosceles** if at least two sides are congruent.

(b) A triangle is **equilateral** if all three sides are congruent.

Thus *Pons Asinorum* says that not only are two sides of an isosceles triangle congruent, but the two corresponding angles must also be congruent. The converse is also true: congruence of two angles implies congruence of the corresponding sides (Exercise 7.1a).

Theorem 7.8. ***ASA (Angle-Side-Angle)*.**[9]

Suppose a correspondence between two triangles is such that two angles and the included side of the first triangle are congruent to the corresponding parts of the second triangle. Then the correspondence is a congruence between the two triangles.

Stated in symbols, suppose $\triangle abc$ and $\triangle ABC$ are two triangles such that $\angle a \cong \angle A$, $\overline{ac} \cong \overline{AC}$, and $\angle c \cong \angle C$, as shown in Figure 7.9. Then *ASA* states that the two triangles are congruent via the congruence $\triangle abc \cong \triangle ABC$.

[8]The name *Pons Asinorum* means "asses' bridge," a name derived from a particular diagram Euclid used in his proof of the result. The proof we give is shorter and does not require Euclid's diagram. So we are left with the strange name but without its motivation!

[9]See the generalization of *ASA* to be established in Exercise 8.2.

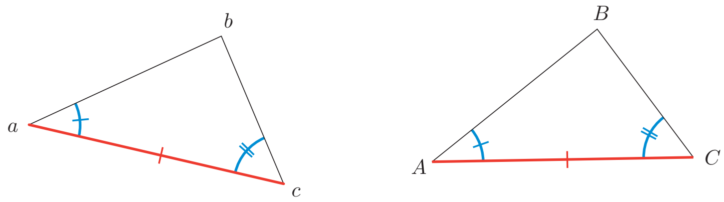


Figure 7.9. ASA implies $\triangle abc \cong \triangle ABC$.

Proof of ASA. From Proposition 4.10 there exists a point B_0 on ray $\overrightarrow{AB}$ such that $\overline{ab} \cong \overline{AB_0}$ (shown in Figure 7.10). SAS proves $\triangle abc$ and $\triangle AB_0C$ are congruent. Thus, to prove $\triangle abc \cong \triangle ABC$, we must show B_0 equals B.

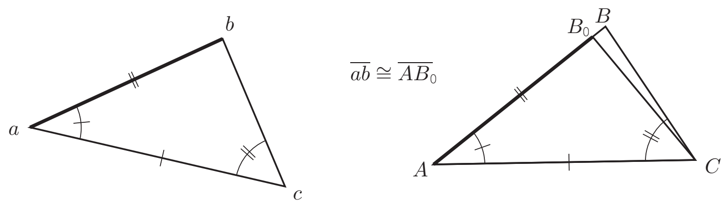


Figure 7.10. SAS implies $\triangle abc \cong \triangle AB_0C$.

From the congruence $\triangle abc \cong \triangle AB_0C$ we know $\angle ACB_0 \cong \angle acb$. But by assumption we also know $\angle acb \cong \angle ACB$. Hence $\angle ACB_0 \cong \angle ACB$. Therefore, by Angle Axiom M-2, the rays $\overrightarrow{CB_0}$ and $\overrightarrow{CB}$ are equal, and hence the lines $\overleftrightarrow{CB_0}$ and $\overleftrightarrow{CB}$ are the same. But this one line therefore intersects the (different) line $\overleftrightarrow{AB}$ in two points, B and B_0. Since two different lines can intersect in at most one point by Proposition 2.1, we conclude $B = B_0$. $\square$

The third basic congruence result, the *SSS Theorem*, takes some effort to establish. An outline of its proof is given in Exercise 7.2.

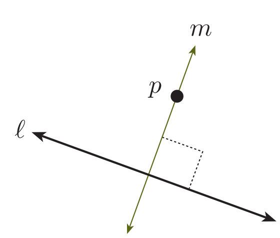
Theorem 7.11. ***SSS* (*Side-Side-Side*).**

Suppose a correspondence between two triangles is such that the three sides of the first triangle are congruent to the corresponding sides of the second triangle. Then the correspondence gives a triangle congruence.

We can now use SAS to generalize Proposition 6.11 to apply to any point p in the plane. This will be an important result for our subsequent work.

Theorem 7.12.

Suppose ℓ is a line in the plane $\mathcal{E}$ and p is any point in $\mathcal{E}$. Then there exists a unique line m containing p which is perpendicular to ℓ.



Proof. If p lies on ℓ, then the desired result is merely Proposition 6.11. So assume p does not lie on ℓ.

Pick distinct points a and c on ℓ and let $\mathcal{H}$ be the half plane with edge ℓ containing p. Let $\mathcal{H}'$ be the half plane opposite to $\mathcal{H}$. From Angle Axiom M-2 there exists a point q in $\mathcal{H}'$ such that $\angle caq \cong \angle cap$ (see Figure 7.13).

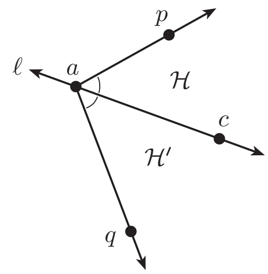


Figure 7.13. Pick $q \in \mathcal{H}'$ such that $\angle caq \cong \angle cap$.

From Proposition 4.10, *Segment Construction,* there exists a unique point p' on the ray $\overrightarrow{aq}$ such that $\overline{ap'} \cong \overline{ap}$. We claim $m = \overleftrightarrow{pp'}$ is perpendicular to ℓ, which will prove the *existence* of the desired perpendicular (see Figure 7.14).

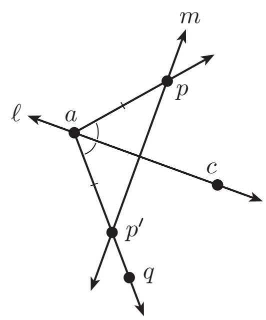


Figure 7.14. Pick $p' \in \overrightarrow{aq}$ such that $ap' = ap$.

To prove that m is perpendicular to ℓ, note that p and p' are on opposite sides of ℓ. Hence, from the *Plane Separation Axiom,* the line segment $\overline{pp'}$ intersects the line ℓ at some point d. We now have two cases to consider.

Case (1): The points a and d are distinct (Figure 7.15). We therefore have two triangles, $\triangle adp$ and $\triangle adp'$, with congruences $\overline{ap} \cong \overline{ap'}$, $\angle pad \cong \angle p'ad$, and $\overline{ad} = \overline{ad}$. Hence *SAS* applies to give $\triangle adp \cong \triangle adp'$.

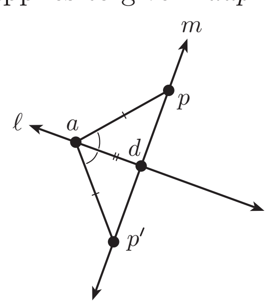


Figure 7.15. Case (1): $a \neq d$.

In particular, the two angles $\angle adp$ and $\angle adp'$ are congruent. However, these two angles form a linear pair and hence are supplementary by Angle Axiom M-4. But two angles can be supplementary and congruent only if they both have angle measure 90°, i.e, if they are right angles. Aha! This means that $m = \overleftrightarrow{dp}$ and $\ell = \overleftrightarrow{da}$ are perpendicular, as desired.

Case (2): The points a and d are the same (Figure 7.16). Since $\angle cap \cong \angle cap'$ by construction, then we have $\angle cdp \cong \angle cdp'$.

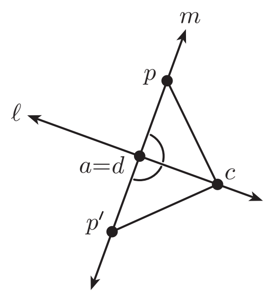


Figure 7.16. Case (2): $a = d$.

As in Case (1) we therefore have two angles that are both supplementary (since they are a linear pair) and congruent, which means they are both right angles. Hence $m = \overleftrightarrow{dp}$ and $\ell = \overleftrightarrow{dc}$ must be perpendicular, as desired.

To finish, we must verify the *uniqueness* of the line m which contains p and is perpendicular to ℓ. Suppose m and m_0 are two such lines (Figure 7.17).

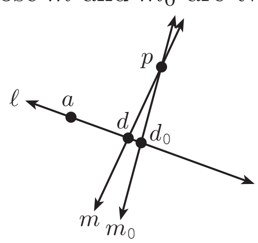


Figure 7.17. Suppose m and m_0 are both perpendicular to ℓ.

We have to show that $m = m_0$. Let d and d_0 be the intersection points of m and m_0 with ℓ, respectively. By Proposition 4.3b we can pick a third point a on ℓ which is *not* between d and d_0. Hence rays $\overrightarrow{ad}$ and $\overrightarrow{ad_0}$ are equal.

From Exercise 6.3 all four angles formed at d by m and ℓ are right angles (Figure 7.18). Now pick $p' \in m$ on the opposite side of ℓ from p such that $\overline{dp'} \cong \overline{dp}$ — this is possible by Proposition 4.10, *Segment Construction*. Then consider triangles $\triangle adp$ and $\triangle adp'$. These must be congruent by the *SAS Axiom* since $\overline{dp} \cong \overline{dp'}$, $\angle adp \cong \angle adp'$, and $\overline{da} = \overline{da}$.

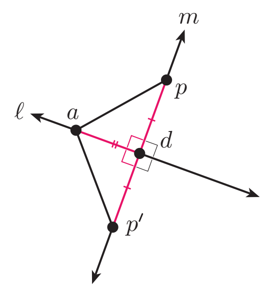


Figure 7.18. *SAS* implies $\triangle adp \cong \triangle adp'$.

Hence, as shown on the left in Figure 7.20, from $\triangle adp \cong \triangle adp'$ we obtain

$$\angle dap' \cong \angle dap \qquad \text{and} \qquad \overline{ap'} \cong \overline{ap}. \tag{7.19a}$$

Repeat the construction of the previous paragraph for m_0, giving $p'_0 \in m_0$ such that p and p'_0 are on opposite sides of ℓ and $\overline{d_0p'_0} \cong \overline{d_0p}$. Then, as above, we obtain the triangle congruence $\triangle ad_0p \cong \triangle ad_0p'_0$, which implies

$$\angle d_0ap'_0 \cong \angle d_0ap \qquad \text{and} \qquad \overline{ap'_0} \cong \overline{ap}, \tag{7.19b}$$

as shown on the right in Figure 7.20. We now show that the points p' and p'_0 are the same. This will give the desired equality of m and m_0.

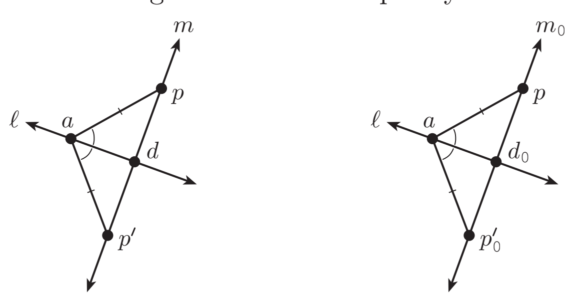


Figure 7.20. We find that $\angle dap' = \angle d_0ap'_0$ and $ap' = ap'_0$.

Since the point $a \in \ell$ does *not* lie between d and d_0, rays $\overrightarrow{ad}$ and $\overrightarrow{ad_0}$ are equal, and we obtain the following congruences:

$$\begin{aligned} \angle dap' &\cong \angle dap \text{ from (7.19a)}, \\ &= \angle d_0ap \text{ since } \overrightarrow{ad} = \overrightarrow{ad_0}, \\ &\cong \angle dap'_0 \text{ from (7.19b)}, \end{aligned}$$

and

$$\overline{ap'} \cong \overline{ap} \cong \overline{ap'_0} \text{ from (7.19ab)}.$$

Since both p' and p'_0 lie on the same side of line $\ell = \overleftrightarrow{da}$, the congruences $\angle dap' \cong \angle dap'_0$ and $\overline{ap'} \cong \overline{ap'_0}$ prove $p' = p'_0$ by invoking the uniqueness condition of Proposition 4.10, *Segment Construction*. But this gives $m = \overleftrightarrow{pp'} = \overleftrightarrow{pp'_0} = m_0$. Hence there is only one line through p which is perpendicular to ℓ, as desired. $\square$

Exercises I.7

Exercise 7.1.

(a) Prove the converse of *Pons Asinorum:* If two angles of a triangle are congruent, then the sides opposite them are congruent.

(b) As in Definition 7.7 a triangle is called **equilateral** if all three of its sides are congruent. Similarly a triangle is called **equiangular** if all three of its angles are congruent. Prove that a triangle is equilateral if and only if it is equiangular.

Exercise 7.2. ***SSS (Side-Side-Side).***

In this exercise you will prove Theorem 7.11, *SSS*:

> *Suppose a correspondence between two triangles is such that the three sides of the first triangle are congruent to the corresponding sides of the second triangle. Then the correspondence is a congruence between the two triangles.*

Stated in symbols, suppose $\triangle abc$ and $\triangle ABC$ are two triangles such that $\overline{ab} \cong \overline{AB}$, $\overline{bc} \cong \overline{BC}$, and $\overline{ac} \cong \overline{AC}$, as shown in Figure 7.21. Then $\triangle abc \cong \triangle ABC$.

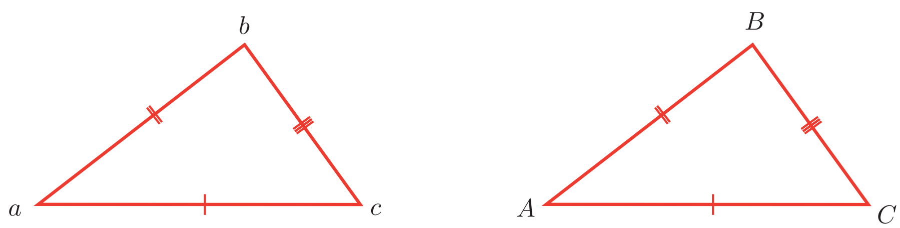


Figure 7.21. *SSS* implies $\triangle abc \cong \triangle ABC$.

You will construct a third triangle $\triangle ab'c$ on the side $\overline{ac}$ of $\triangle abc$ that you will prove to be congruent to both of the original triangles.

(a) Prove that there exists a point b' on the opposite side of $\ell = \overleftrightarrow{ac}$ from b such that $\triangle ab'c \cong \triangle ABC$. *Hint:* Construct b' so that $\angle cab' \cong \angle CAB$ as in Figure 7.22.

(b) Prove that the line segment $\overline{bb'}$ must intersect the line $\ell = \overleftrightarrow{ac}$ at some point e.

There are now several cases that must be handled separately, depending on the location of the point e relative to a and c.

(c) Suppose e is strictly between a and c, as shown in Figure 7.22. Verify that e must be in the interiors of both $\angle abc$ and $\angle ab'c$ (see Exercise 5.4). Then use this to prove $\angle abc \cong \angle ab'c$. *Hint:* What type of triangles are $\triangle bab'$ and $\triangle bcb'$?

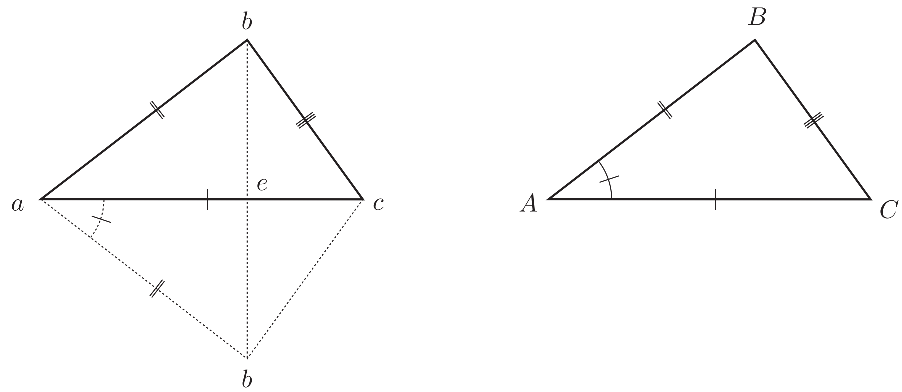


Figure 7.22. The construction of $\triangle ab'c$.

(d) Continuing with the assumption that e is strictly between a and c, conclude that $\triangle abc \cong \triangle ab'c$, and hence $\triangle abc \cong \triangle ABC$, as desired.

(e) Now suppose a is strictly between e and c (the case for c strictly between a and e is identical). Produce an argument similar to (c) and (d) to prove $\triangle abc \cong \triangle ABC$, as desired.

(f) Now suppose $e = a$ (the case $e = c$ is identical). Prove $\triangle abc \cong \triangle ABC$, as desired. This completes the proof of *SSS*.

Exercise 7.3.

In this exercise you will show that the *SAS Axiom* is *independent* of the previous seven axioms, i.e., that there is a system in which the first seven axioms are valid but the *SAS Axiom* is false. We define such an "aberrant" system as follows. Suppose $\mathcal{E}$ is a set, $\mathcal{L}$ a collection of special subsets of $\mathcal{E}$ called *lines*, each line $\ell \in \mathcal{L}$ having a coordinate system χ_ℓ, and m an angle measure function. Let d be the corresponding distance function, and assume our axiomatic system satisfies all of the eight axioms up to and including *SAS*.

We define a new axiomatic system by altering just one coordinate system as follows. Pick a specific line ℓ_0 in $\mathcal{E}$ with coordinate system $\chi = \chi_{\ell_0}$ and define a new coordinate system on ℓ_0 by $\eta(p) = 2\chi(p)$ for every point p on ℓ_0 (see Exercise 3.1a). Let ρ denote the new distance function that results on $\mathcal{E}$ by replacing χ with η. You first studied such a distance function alteration in Exercise 3.3.

(a) Show that, for any two points p and q in $\mathcal{E}$, we have

$$\rho(p,q) = \begin{cases} 2d(p,q) & \text{if } p \text{ and } q \textit{ both} \text{ lie on } \ell_0, \\ d(p,q) & \text{in all other cases.} \end{cases}$$

and that the first seven axioms are valid for our new axiomatic system where η replaces χ and consequently ρ replaces d.

(b) Show that the *SAS Axiom* is *not* always valid for the new distance function ρ and the original angle measure function m on the set $\mathcal{E}$. *Hint:* Consider the triangles $\triangle abc$ and $\triangle ABC$ shown in Figure 7.23. The side $\overline{ac}$ lies on the line ℓ_0 while none of the other five sides lie on ℓ_0. Assume that $\overline{ab} \cong \overline{AB}$, $\overline{bc} \cong \overline{BC}$, and $\angle b \cong \angle B$. If the *SAS Axiom* were valid, then we would conclude that $\overline{ac} \cong \overline{AC}$ *with respect to the new distance function* ρ. Show that this is *not* true!

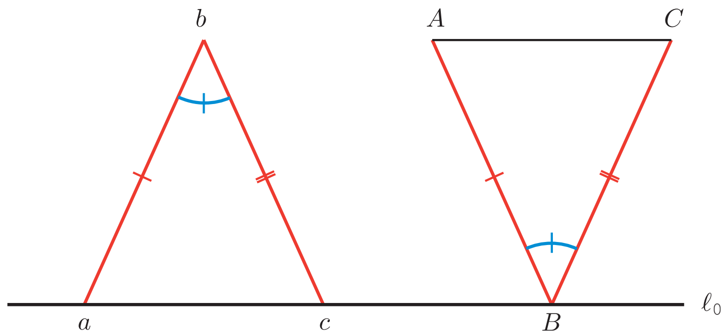


Figure 7.23. With the distance function of Exercise 7.3, $\overline{ac} \ncong \overline{AC}$.

Exercise 7.4.

Consider any line segment $\overline{ab}$. A line ℓ is a **perpendicular bisector** for $\overline{ab}$ if ℓ is perpendicular to $\overleftrightarrow{ab}$ at the midpoint of $\overline{ab}$.

(a) Prove that every line segment has a unique perpendicular bisector.

(b) Prove that the perpendicular bisector of $\overline{ab}$ consists of all the points c in the plane which are *equidistant* from a and b, i.e., $ac = bc$. *Hint:* You are asked to prove the equality of two *sets*. You must therefore prove that each set is a subset of the other.

(c) Suppose A, B, C are three *non-collinear* points in the plane and p, q are two points which are equidistant from A, B, and C, i.e.,
$$pA = qA, \quad pB = qB, \quad pC = qC.$$
Prove $p = q$.

Exercise 7.5.

(a) Suppose A, B, C are three distinct points in the plane for which there exists another point p that is equidistant from A, B, C, i.e., $pA = pB = pC$. Prove that A, B, C are non-collinear. *Hint:* Suppose A, B, C are collinear. Then use Exercise 7.4b and Theorem 7.12 to obtain a contradiction.

A **circle** $\mathcal{C}$ is a collection of all points equidistance from one fixed point, i.e., there exists a point c and a real number $r > 0$ such that

$$\mathcal{C} = \mathcal{C}_r[c] = \{x \in \mathcal{E} \mid cx = r\}.$$

The point c is the **center** of the circle $\mathcal{C}_r[c]$ and r is its **radius**. A **diameter** is any line segment joining two points on the circle and passing through its center.

(b) Show that two distinct circles cannot intersect at more than two distinct points. *Hint:* Suppose $\mathcal{C}_{r_1}[c_1]$ and $\mathcal{C}_{r_2}[c_2]$ intersect at the three distinct points A, B, C. Prove that the centers c_1 and c_2 are equal by considering the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$.

§I.8 Geometric Inequalities

Any triangle has three sides. Moreover, each side has two possible directions — hence any triangle has six *oriented* sides. For each oriented side we will obtain an *exterior angle* for the triangle as shown in Figure 8.1 and formally defined in Definition 8.2.

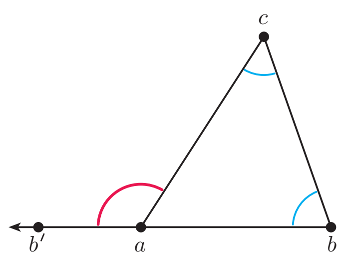


Figure 8.1. $\angle cab'$ is an *exterior angle* with *remote interior angles* $\angle b$, $\angle c$.

Definition 8.2.

Consider a triangle $\triangle abc$ and let $\overrightarrow{ab'}$ be the ray opposite to $\overrightarrow{ab}$. The angle $\angle cab'$ is the **exterior angle** of $\triangle abc$ associated with oriented side $\overline{ab}$, oriented from a to b. The two angles $\angle b$ and $\angle c$ are the **remote interior angles** for the exterior angle $\angle cab'$.

Exterior angles will be useful tools in our development of geometric inequalities, due primarily to the following result.

Theorem 8.3. ***The Exterior Angle Theorem.***

An exterior angle of a triangle is always larger than its two remote interior angles.

Proof. Consider a triangle $\triangle abc$. We will first prove that the exterior angle $\angle cab'$ is greater than the remote interior angle $\angle c = \angle acb$. The remote interior angle $\angle b = \angle abc$ will be handled afterwards.

Let d be the midpoint of $\overline{ac}$ (which exists by Proposition 4.3c), and let $\overrightarrow{de}$ be the ray opposite to $\overrightarrow{db}$, as shown in Figure 8.4. By Proposition 4.10, *Segment Construction,* we can choose the point e such that $de = db$.

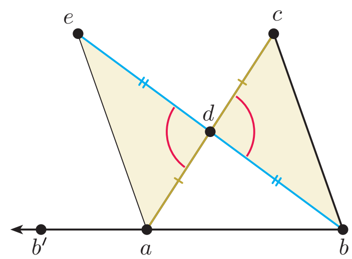


Figure 8.4. d is the midpoint of $\overline{ac}$, $\overrightarrow{de}$ is opposite to $\overrightarrow{db}$, and $de = db$.

By *SAS* we have $\triangle cdb \cong \triangle ade$. This follows since $\overline{cd} \cong \overline{ad}$ (d is the midpoint of $\overline{ac}$), $\overline{db} \cong \overline{de}$ (by definition of e), and $\angle cdb \cong \angle ade$ (from Theorem 6.5, the *Vertical Angle Theorem*). Hence $\angle c \cong \angle cae$, as shown in Figure 8.5.

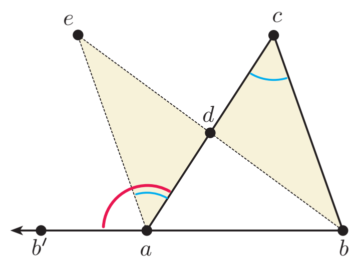


Figure 8.5. The angles $\angle c$ and $\angle cae$ are congruent.

However, the point e is in the interior of the angle $\angle cab'$.[10] Hence, by Angle Axiom M-3, *Angle Addition*, we see that $m\angle cab' = m\angle cae + m\angle eab'$. Thus the exterior angle $\angle cab'$ is greater than the remote interior angle $\angle c$.

Now consider the other remote angle $\angle b$. By a direct application of the result established above, $\angle b$ is less than the exterior angle $\angle bac'$, where $\overrightarrow{ac'}$ is the ray opposite to $\overrightarrow{ac}$. This is shown in Figure 8.6.

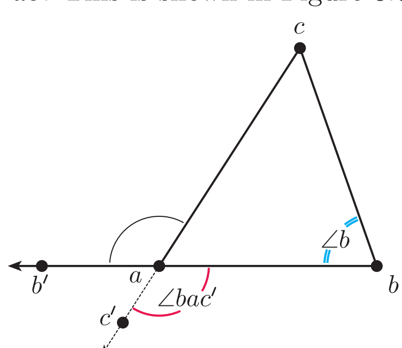


Figure 8.6. $\angle b$ is a remote internal angle for exterior angle $\angle bac'$.

[10] To rigorously prove that e is in the interior of $\angle cab'$, first note that b' and e are both on the side of $\overleftrightarrow{ac}$ opposite the point b. Then note that the points c and e are both on the same side of $\overleftrightarrow{ab'}$ — this follows from observing that all the points (other than a) of $\overrightarrow{ac}$ lie on one side of $\overleftrightarrow{ab'}$ (Proposition 5.6), all the points (other than b) of $\overrightarrow{be}$ lie on one side of $\overleftrightarrow{ab'}$ (Proposition 5.6), and $\overrightarrow{ac}$ and $\overrightarrow{be}$ intersect at the point d. Hence c, d, and e are all on the same side of $\overleftrightarrow{ab'}$, as desired.

But $\angle bac'$ and $\angle cab'$ form a vertical pair and hence are congruent by the *Vertical Angle Theorem* (Theorem 6.5). Thus $\angle b$ is less than $\angle cab'$. □

Here is another simple but important property relating inequalities between side lengths of a triangle with inequalities between corresponding angles.

Theorem 8.7.

Consider any triangle. Then the length of one side is less than the length of a second side if and only if the measure of the angle opposite the first side is less than the measure of the angle opposite the second side.

Proof. For any triangle $\triangle abc$ we will prove $ab < bc$ if and only if $\angle c < \angle a$.

First assume $ab < bc$ and let d be that point on the ray $\overrightarrow{ba}$ such that $bd = bc$ (see Figure 8.8). The point d exists by Proposition 4.10, *Segment Construction*. We will prove that $\angle c < \angle d < \angle a$.

Since $ab < bc$, we know that a is between b and d. Therefore a is in the interior of $\angle bcd$ by Exercise 5.4, proving that $\angle c = \angle bca < \angle bcd$ by Angle Axiom M-3, *Angle Addition*. But since $\overline{bc} \cong \overline{bd}$, Proposition 7.5 (*Pons Asinorum*) gives $\angle bcd \cong \angle d$, proving $\angle c < \angle d$.

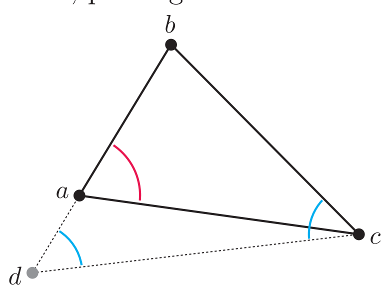


Figure 8.8. Point d is picked so that $bd = bc$.

We have only to show $\angle d < \angle a$. Aha! Since $\angle a$ is an exterior angle of $\triangle acd$ and $\angle d$ is a remote interior angle, Theorem 8.3 gives $\angle d < \angle a$.

We now establish the converse direction, i.e., if $\angle c < \angle a$, then $ab < bc$. Suppose this is *not* the case, i.e., suppose $ab \geq bc$. If $ab = bc$, then *Pons Asinorum* would give $\angle c \cong \angle a$, a contradiction. Moreover, if $ab > bc$, then by what we have just shown, we would have $\angle a < \angle c$, another contradiction. Hence we are left with $\angle c < \angle a$, as desired. □

We now can prove the most famous and important geometric inequality.

Theorem 8.9. ***The (Strict) Triangle Inequality.***

The sum of the lengths of any two sides of a triangle is greater than the length of the remaining side.

Proof. Consider triangle $\triangle abc$ in Figure 8.10. We will show $ac < ab + bc$.

By Proposition 4.10, *Segment Construction*, let d be that point on the ray opposite to $\overrightarrow{bc}$ such that $bd = ba$. We claim that the angle $\angle dac$ is greater than the angle $\angle adc$. To show this, note that $\angle adc \cong \angle dab$ by *Pons Asinorum* since $bd = ba$. But since b is between c and d, b is in the interior of $\angle dac$ by Exercise 5.4. Hence

$$\angle adc \cong \angle dab < \angle dac$$

by Angle Axiom M-3, *Angle Addition*. Thus $\angle adc < \angle dac$, as we claimed.

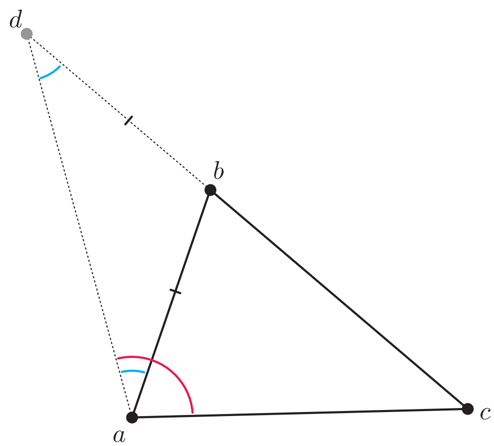


Figure 8.10. Point d is picked so that $bd = ba$.

Hence, by Theorem 8.7 applied to $\triangle acd$, we obtain $ac < dc$. This implies $dc = db + bc$ and $db = ab$, and hence $ac < ab + bc$, as desired. $\square$

Corollary 8.11. ***The Triangle Inequality.***
For any three points a, b, c in the plane, $ac \leq ab + bc$.

Proof. If the points a, b, c are non-collinear, then they form the vertices of a triangle $\triangle abc$ and the desired inequality follows from Theorem 8.9.

Now suppose a, b, c are collinear. If ℓ is the line containing the three points, let χ be a coordinate system on ℓ and let x, y, z be the coordinates of a, b, c, respectively. Then

$$|z - x| \leq |y - x| + |z - y|$$

is true by the properties of real numbers. However, by the definition of distance in Definition 3.2 this translates into $ac \leq ab + bc$, as desired. $\square$

Exercises I.8

Exercise 8.1.
Use Theorem 8.3 to give an alternate proof for the uniqueness claim of Theorem 7.12: *Given a point p and a line m there can be at most one line m containing p that is perpendicular to ℓ.*

Exercise 8.2. ***SAA (Side-Angle-Angle).***

Prove the *SAA Theorem,* a generalization of *ASA* (Theorem 7.8):

> *Suppose a correspondence between two triangles is such that one side and two angles of the first triangle are congruent to the corresponding side and angles of the second. Then the correspondence is a congruence between the two triangles.*

Hints: If the side is between the two angles, then the desired result is simply *ASA*, Theorem 7.8. So assume the side is *not* between the two angles. In particular, assume we have two triangles, $\triangle abc$ and $\triangle ABC$ as shown in Figure 8.12, where $\overline{ab} \cong \overline{AB}$, $\angle b \cong \angle B$, and $\angle c \cong \angle C$. Now choose the point C' on $\overrightarrow{BC}$ such that $BC' = bc$. Prove *SAA* by considering the triangle $\triangle ABC'$ and using the *Exterior Angle Theorem*.

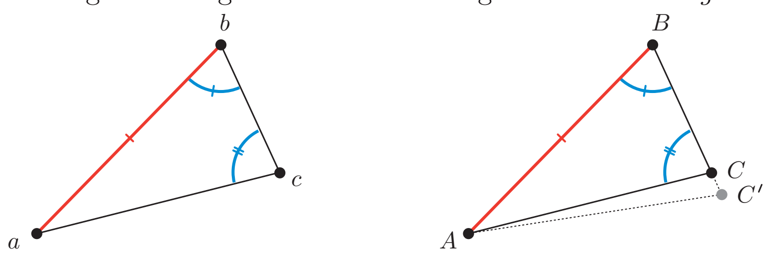


Figure 8.12. Point C' is picked so that $bc = BC'$.

Exercise 8.3. ***A Qualified SSA (Side-Side-Angle).***

(a) Consider the *SSA Conjecture:*

> *Suppose a correspondence between two triangles is such that two sides and one angle of the first triangle are congruent to the corresponding sides and angle of the second. Then the correspondence is a congruence between the two triangles.*

Disprove this conjecture by giving a counterexample, i.e., by producing a pair of *non-congruent* triangles that satisfy the hypotheses.

(b) Although the general *SSA* is false, prove *SSA* to be true when the angle under consideration is a right angle or an obtuse angle, i.e.,

> *Suppose* $\angle bac \cong \angle BAC$ *are right or obtuse angles. If* $\overline{ab} \cong \overline{AB}$ *and* $\overline{bc} \cong \overline{BC}$*, then* $\triangle abc \cong \triangle ABC$.

Hint: First show that we can assume $ac \leq AC$ by relabeling (if necessary) the vertices of the two triangles. Then consider $c' \in \overrightarrow{ac}$ such that $ac' = AC$. What can you say about $\triangle cbc'$?

Exercise 8.4.

Show that the shortest distance between a point p and a line ℓ not containing p is given by pd, where d is the intersection of ℓ with the line m containing p and perpendicular to ℓ. *Hints:* Let b be any point of ℓ other than d and let $\overrightarrow{dc}$ be the ray opposite $\overrightarrow{db}$, as shown in Figure 8.13.

How do angles $\angle pdc$ and $\angle pbd$ relate to each other? What does this say about angles $\angle pbd$ and $\angle pdb$?

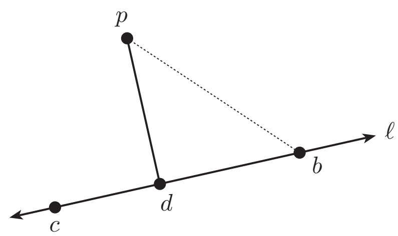


Figure 8.13. $\overrightarrow{dc}$ is the ray opposite to $\overrightarrow{db}$.

Exercise 8.5.

Show the angle bisector $\overrightarrow{bd}$ (Definition 6.6) of an angle $\angle abc$ consists precisely of the point b along with all points in the interior of $\angle abc$ which are equidistant from the lines $\overleftrightarrow{ba}$ and $\overleftrightarrow{bc}$. (The *distance from a point to a line* is defined to be the length of the perpendicular line segment joining the point to the line.)

Exercise 8.6.

(a) Given any triangle $\triangle abc$ and a point $e \in \overline{ac}$ that is between a and c as shown in Figure 8.14, prove $ae + eb < ac + cb$.
Hint: Use the *Strict Triangle Inequality* (Theorem 8.9).

(b) Given any triangle $\triangle abc$ and a point d in the interior as shown in Figure 8.14, prove $ad + db < ac + cb$.
Hint: Apply the *Crossbar Theorem* (Theorem 5.9) to show that the ray $\overrightarrow{bd}$ intersects $\overline{ac}$ at a point e between a and c.

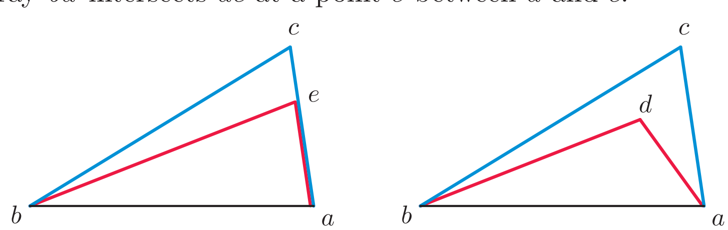


Figure 8.14. The blue lengths are greater than the red lengths.

Exercise 8.7. ***The Hinge Theorem***.

In this exercise you will prove the *Hinge Theorem:*

> *If two sides of one triangle are congruent to two sides of a second triangle, and the included angle of the first triangle is larger than the included angle of the second triangle, then the opposite side of the first triangle is larger than the opposite side of the second triangle.*

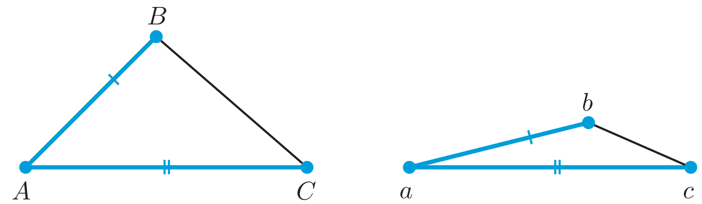


Figure 8.15. $\angle A$ greater than $\angle a$ implies BC greater than bc.

You will prove this result in a series of steps. Suppose $\triangle ABC$ and $\triangle abc$ are as in Figure 8.15: $\overline{AB} \cong \overline{ab}$, $\overline{AC} \cong \overline{ac}$, $\angle A$ greater than $\angle a$.

(a) Prove there exists a point B' such that B' is in the interior of $\angle BAC$ and $\triangle AB'C \cong \triangle abc$.

(b) Prove there exists D between B and C such that $\overrightarrow{AD}$ bisects $\angle BAB'$. *Hint:* Use the *Crossbar Theorem*. It is helpful to obtain a point D' between B and C such that $\overrightarrow{AD'} = \overrightarrow{AB'}$. See Figure 8.16.

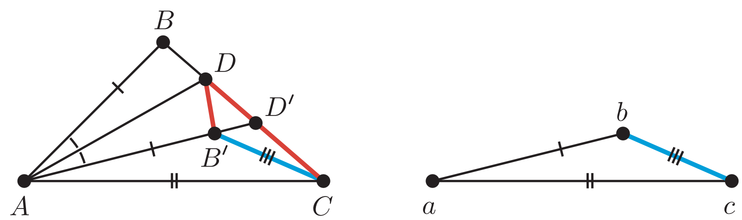


Figure 8.16. $\triangle AB'C$ is congruent to $\triangle abc$, and $\overrightarrow{AD}$ bisects $\angle BAB'$.

(c) Prove $\overline{DB} \cong \overline{DB'}$. *Hint:* Consider triangles $\triangle ADB$ and $\triangle ADB'$.

(d) Use (c) to prove $bc < BC$, thereby verifying the *Hinge Theorem*.

§I.9 Parallelism

A central concept in Euclidean geometry is that of *parallel lines.*

Definition 9.1.

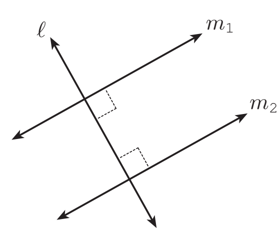
(a) Two lines ℓ_1 and ℓ_2 in the plane $\mathcal{E}$ are **parallel lines**, written $\ell_1 \parallel \ell_2$, if they do not intersect or (for convenience) if the two lines are equal.

(b) Two line segments $\overline{ab}$ and $\overline{cd}$ are **parallel line segments** if the two lines $\overleftrightarrow{ab}$ and $\overleftrightarrow{cd}$ determined by the segments are parallel.

Here is one simple method to verify parallelism.

Proposition 9.2.

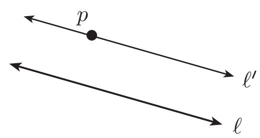
Suppose lines m_1 and m_2 are both perpendicular to a line ℓ. Then m_1 and m_2 are parallel.



Proof. If m_1 and m_2 are not disjoint, then they intersect in at least one point p. From Theorem 7.12 we know there is *only one* line m containing p which is perpendicular to ℓ. Hence m_1 would have to equal m_2, implying m_1 and m_2 are parallel. We have thus shown that m_1 and m_2 are either disjoint or equal; this proves they are parallel, as desired. □

Proposition 9.3.

Suppose ℓ is a line and p is a point. Then there exists at least one line ℓ' containing p which is parallel to ℓ:



Proof. From Theorem 7.12 there exists a line m perpendicular to ℓ and containing p (see Figure 9.4). Moreover, again by Theorem 7.12 (or Proposition 6.11), there exists a line ℓ' perpendicular to m and containing p.

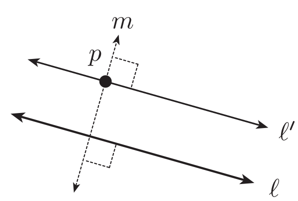


Figure 9.4. Line m is perpendicular to ℓ, and line ℓ' is perpendicular to m.

Thus ℓ and ℓ' are both perpendicular to m. Hence, by Proposition 9.2, ℓ and ℓ' are parallel, as desired. (Notice that if p is a point of ℓ, then the line ℓ' will have to equal ℓ.) □

Proposition 9.3 is an *existence* result: given a point p and a line ℓ, *there exists* a line ℓ' containing p that is parallel to ℓ. However, for centuries it was believed a *uniqueness* result must also be true, that there is a *unique* line ℓ' containing p which is parallel to ℓ. It was a major revelation when, in the nineteenth century, it was shown that such a uniqueness result does *not* follow from the previous axioms (nor any similar variants of the previous axioms).

In particular, there are geometric systems that satisfy all the "usual" axioms for standard Euclidean geometry except that through a point p not on a line ℓ there are *many* lines ℓ' which are parallel to ℓ. One such system is known as **hyperbolic geometry**. A model for hyperbolic geometry can be obtained from the *Poincaré disk,* Example 2.4. It is not too hard to see that through any point p not on a Poincaré line ℓ there are an infinite number of Poincaré lines ℓ' which do not intersect ℓ. (We will consider this topic in more depth in Volume II.)

Thus, in order for our set of axioms to be complete enough to yield the desired behavior for parallel lines, we need to add another axiom, known traditionally as the *Parallel Postulate*. This simply states the uniqueness of the line ℓ' through p parallel to the given line ℓ. However, before stating and using this postulate in the next section, we continue to derive those results on parallelism which do *not* require the *Parallel Postulate*. In this way you will better appreciate those results which *cannot* be obtained independent of the *Parallel Postulate*.

Alternate Interior Angles. Suppose m_1 and m_2 are two distinct lines in the plane. A third distinct line ℓ is said to be a **transversal** to m_1 and m_2 if ℓ intersects m_1 and m_2 in two (different) points p_1 and p_2, respectively.

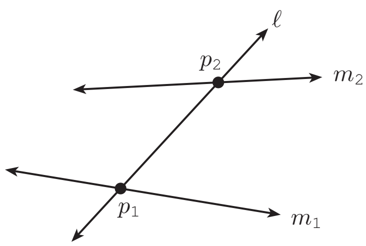


Figure 9.5. ℓ is transversal to m_1 and m_2.

In such a case we obtain four **interior angles**, i.e., the two angles formed by ℓ and m_1 which lie on the side of m_1 containing p_2 and the two angles formed by ℓ and m_2 which lie on the side of m_2 containing p_1.

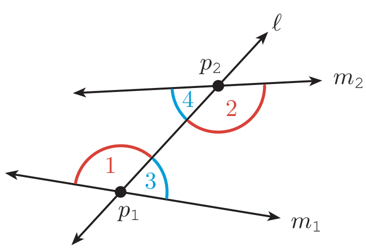


Figure 9.6. The transversal ℓ produces four interior angles.

We further organize these four angles into two pairs of *alternate interior angles* as follows. Let a_1 and b_1 be two points of m_1 which lie on opposite sides of ℓ, and let a_2 and b_2 be two points of m_2 which lie on opposite sides of ℓ, with a_1 and a_2 on the same side of ℓ. Then the four interior angles come in two pairs of **alternate interior angles**:

$$\angle 1 = \angle a_1p_1p_2 \quad \text{and} \quad \angle 2 = \angle b_2p_2p_1,$$
$$\angle 3 = \angle b_1p_1p_2 \quad \text{and} \quad \angle 4 = \angle a_2p_2p_1.$$

The first pair of alternate interior angles listed above is shown in Figure 9.7. As can be seen in the figure, the two angles $\angle a_1p_1p_2$ and $\angle b_2p_2p_1$ form a pair of alternate interior angles if and only if a_1 and b_2 lie on opposite sides of the line $\ell = \overleftrightarrow{p_1p_2}$.

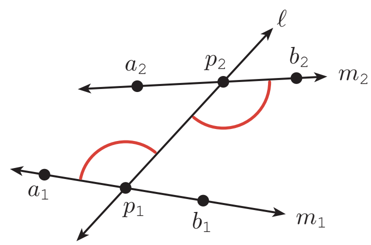


Figure 9.7. $\angle a_1p_1p_2$ and $\angle b_2p_2p_1$ are alternate interior angles.

We summarize these new concepts as follows.

Definition 9.8.

Suppose ℓ, m_1, m_2 are three distinct lines such that ℓ intersects m_1 and m_2 in the two distinct points p_1 and p_2, respectively. Let a_1 and b_1 be two points of m_1 which lie on opposite sides of ℓ, and let a_2 and b_2 be two points of m_2 which lie on opposite sides of ℓ, with a_1 and a_2 on the same side of ℓ. Then

$$\angle a_1p_1p_2 \quad \text{and} \quad \angle b_2p_2p_1 \quad (\angle 1 \text{ and } \angle 2)$$

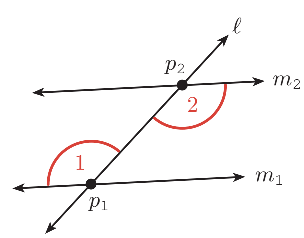
form a pair of **alternate interior angles** for the **transversal** ℓ of the lines m_1, m_2. The other pair of alternate interior angles is comprised of

$$\angle b_1p_1p_2 \quad \text{and} \quad \angle a_2p_2p_1 \quad (\angle 3 \text{ and } \angle 4).$$

We now can state our desired result, a sufficient condition for two lines to be parallel given in terms of the alternate interior angles of a transversal.

Proposition 9.9.

Suppose ℓ is a transversal for two distinct lines m_1 and m_2. If two alternate interior angles are congruent, then m_1 and m_2 are parallel.



Proof. Suppose the two lines m_1 and m_2 are not parallel. Then they intersect in some point p_3.

Let p_1 and p_2 be the intersection points of ℓ with m_1 and m_2, respectively, and consider the triangle $\triangle p_1p_2p_3$ as shown in Figure 9.10.

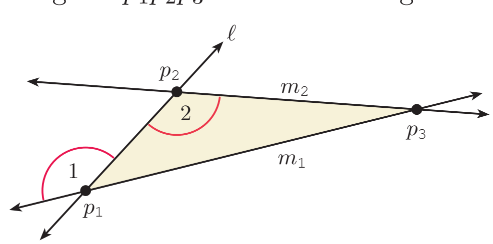


Figure 9.10. Suppose $\angle 1 \cong \angle 2$ but m_1 and m_2 intersect at p_3.

If the two congruent alternate interior angles are denoted by $\angle 1$ and $\angle 2$, then one of these angles — say $\angle 1$ — will be an *exterior* angle for $\triangle p_1p_2p_3$ and $\angle 2$ will be a remote interior angle for $\angle 1$. Hence $\angle 2 < \angle 1$ by Theorem 8.3, the *Exerior Angle Theorem*. Oops! This is a contradiction since by assumption we have $\angle 1 \cong \angle 2$. □

The converse of Proposition 9.9 can be proven only after we assume the *Parallel Postulate*.

Saccheri Quadrilaterals. A quadrilateral $\square abcd$, as defined in Definition 5.10, is called a **rectangle** if the four interior angles $\angle a$, $\angle b$, $\angle c$, $\angle d$ are all right angles. It may be surprising to learn that our current collection of axioms does *not* guarantee that any rectangles exist! This will require the *Parallel Postulate*. However, if we apply a rather natural procedure to construct a rectangle, we will obtain a specialized form of quadrilateral known as a *Saccheri quadrilateral*. Here is the procedure.

Begin with any line segment $\overline{ad}$. At the endpoints a and d use Proposition 6.11 to construct lines m_1 and m_2 which are perpendicular to $\overline{ad}$ and contain a and d, respectively. On m_1 pick a point b distinct from a, and on m_2 pick a point c distinct from d which is on the same side of $\overleftrightarrow{ad}$ as b. By Proposition 4.10, *Segment Construction*, we can choose c so that $\overline{ab} \cong \overline{cd}$. What results is called a *Saccheri quadrilateral,* as shown in Figure 9.12.

Definition 9.11.

(a) A quadrilateral $\square abcd$ is a **rectangle** if the four interior angles $\angle a$, $\angle b$, $\angle c$, $\angle d$ are all right angles.

(b) A quadrilateral $\square abcd$ is a **Saccheri quadrilateral** if $\angle a$ and $\angle d$ are right angles, points b and c lie on the same side of $\overleftrightarrow{ad}$, and $\overline{ab} \cong \overline{cd}$. The side $\overline{ad}$ is called the **lower base** and the side $\overline{bc}$ is called the **upper base**.

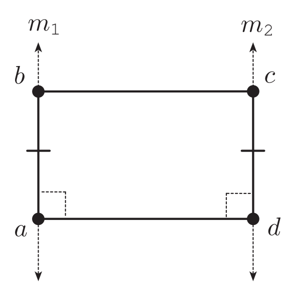


Figure 9.12. A Saccheri quadrilateral

In "ordinary" Euclidean geometry, i.e., when we complete our set of axioms, the collection of all Saccheri quadrilaterals in the plane will be identical to the collection of all rectangles in the plane — this will be shown in Exercise 10.5. However, this is *not* true without the *Parallel Postulate*, and in such a context the Saccheri quadrilaterals are important objects. This is particularly true in hyperbolic geometry.

Properties of Saccheri quadrilaterals will be derived in Exercise 9.2. They will then be used in Exercise 9.3 to prove the following result:

Let $\triangle abc$ be any triangle with interior angles $\angle a$, $\angle b$, and $\angle c$. Then
$$m\angle a + m\angle b + m\angle c \leq 180^\circ.$$

We will prove in Theorem 10.3 that, given the *Parallel Postulate*, this inequality is actually an equality.

Exercises I.9

Exercise 9.1.

Suppose $\square abcd$ is a rectangle, i.e., a Saccheri quadrilateral such that all four interior angles are right angles. Then prove that opposite sides of the rectangle are congruent, i.e., $\overline{ab} \cong \overline{cd}$ and $\overline{bc} \cong \overline{ad}$.
Hint: Split the quadrilateral into two triangles and apply Exercise 8.3.

Exercise 9.2. ***Saccheri Quadrilaterals***.

In this exercise we develop the important properties of Saccheri quadrilaterals. In particular, do ***not*** assume the *Parallel Postulate*!

(a) Prove that a Saccheri quadrilateral is a *convex* quadrilateral (Definition 5.10b). *Hints:* If $\square abcd$ is a Saccheri quadrilateral with lower base $\overline{ad}$, then the only potentially tricky part of verifying that $\square abcd$ is convex is proving that a and d are on the same side of the line $\overleftrightarrow{bc}$. To do so, first show that all points between a and d lie on the same side of $\overleftrightarrow{cd}$ as b and on the same side of $\overleftrightarrow{ab}$ as c. Then if a and d are on opposite sides of $\overleftrightarrow{bc}$, prove that b and c are on opposite sides of $\overleftrightarrow{ad}$, a contradiction.

(b) Prove that the diagonals of a Saccheri quadrilateral are congruent.

(c) In any Saccheri quadrilateral the lower base angles are congruent since they are both right angles. Prove that the upper base angles are also congruent (though, in the absence of the *Parallel Postulate*, they might not be right angles). *Hints:* If $\overline{ad}$ is the lower base of a Saccheri quadrilateral $\square abcd$, show that $\triangle bad \cong \triangle cda$. Use this to further show that $\angle bac \cong \angle cdb$, and then obtain $\triangle bac \cong \triangle cdb$.

(d) In any Saccheri quadrilateral, prove that the upper base is never shorter than the lower base. *Outline:* This is a challenging exercise! If $\square a_1b_1b_2a_2$ is your Saccheri quadrilateral with lower base $\overline{a_1a_2}$, then for each positive integer n build a collection of n copies of the quadrilateral, $\square a_kb_kb_{k+1}a_{k+1}$, $k = 1, \ldots, n$, each sharing a side with the previous figure. The points $a_1, \ldots, a_{n+1}$ are all collinear and separated by the distance a_1a_2. Show that all the segments $\overline{b_kb_{k+1}}$, $k = 1, \ldots, n$, are congruent. Use this to further show that

$$n\ a_1a_2 = a_1a_{n+1} \leq a_1b_1 + n\ b_1b_2 + a_1b_1.$$

This gives $a_1a_2 \leq b_1b_2 + (2/n)a_1b_1$ for each positive integer n. Use this to show that $a_1a_2 \leq b_1b_2$, as desired.

(e) If $\square abcd$ is a Saccheri quadrilateral with lower base $\overline{ad}$, prove that $\angle bdc \geq \angle abd$. *Hint:* Use the *Hinge Theorem* of Exercise 8.7.

Exercise 9.3.

You will use the properties of Saccheri quadrilaterals developed in Exercise 9.2 to prove that the sum of the measures of the interior angles of *any* triangle must be less than or equal to 180°. We *cannot* prove equality in this relationship; this requires the *Parallel Postulate*, as will be evident in the next section (also see Exercise 9.4).

(a) Show that the sum of the measures of the interior angles of any *right* triangle must be less than or equal to 180°. Further conclude that the hypotenuse of a right triangle (the side opposite the right angle) must be longer than either of the other sides. *Hints:* Suppose $\triangle abd$ is the right triangle with right angle $\angle dab$. Show that there exists a point c such that $\square abcd$ is a Saccheri quadrilateral with lower base $\overline{ad}$. Then apply Exercise 9.2e. The statement about the length of the hypotenuse will follow from a result in §8.

(b) Suppose $\triangle abc$ has $\overline{ac}$ as a longest side. If $d \in \overleftrightarrow{ac}$ is the foot of the perpendicular from b, prove that d lies between a and c.
Hint: Show that the other possibilities lead to contradictions. The first to consider is $d = a$, which can be handled by (a).

(c) Generalize (a) by showing that the sum of the measures of the interior angles of *any* triangle must be less than or equal to 180°.
Hint: Let $d \in \overleftrightarrow{ac}$ be the foot of the perpendicular from b. Then $\triangle abc$ is divided into two right triangles.

Exercise 9.4.

Consider the following two statements. Both are “well known” in elementary geometry, but neither has yet been established in our work.

Parallel Postulate.

Suppose ℓ is a line and p is a point not on ℓ.
Then there is a *unique* line ℓ' parallel to ℓ and containing p.

Triangle Sum Hypothesis.

Let $\triangle abc$ be any triangle with interior angles $\angle a$, $\angle b$, and $\angle c$.
Then $m\angle a + m\angle b + m\angle c = 180°$.

We have not established either of these results because *neither one follows from the axioms we have thus far developed for Euclidean geometry in the plane.*[11] However, the two results are *equivalent* in that assuming either statement allows us to prove the other. We will show that the *Parallel Postulate* implies the *Triangle Sum Hypothesis* in Theorem 10.3. You will establish the other implication in this exercise.

(a) Suppose ℓ is a line, p a point not on ℓ, and r a fixed real number, $0 < r < 90°$. Assuming the *Triangle Sum Hypothesis*, prove there exist points $q_0, q \in \ell$ (where q can be chosen on either side of q_0) such that $\triangle pq_0q$ is a right triangle with right angle at q_0 and $m\angle pqq_0 < r$.
Outline: Show there exists $q_0 \in \ell$ such that $\overline{pq_0}$ is perpendicular to ℓ. Then choosing a side of ℓ, define the sequence of points $q_1, q_2, \cdots \in \ell$ on the chosen side of ℓ in the following manner (see Figure 9.13):
Choose q_1 so that $q_0q_1 = pq_0$.
Choose q_2 so that $q_1q_2 = pq_1$ and q_1 is between q_0 and q_2.
If $k > 1$, choose q_k so that $q_{k-1}q_k = pq_{k-1}$ and
q_{k-1} is between q_{k-2} and q_k.
With $\theta = 90°$, show that $m\angle pq_1q_0 = \theta/2$, $m\angle pq_2q_0 = \theta/4$, and in general $m\angle pq_kq_0 = \theta/2^k$. Define $q = q_k$ for a k such that $\theta/2^k < r$.

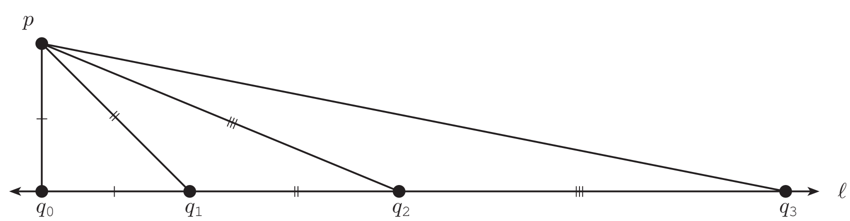


Figure 9.13. The construction for Exercise 9.4a.

(b) Prove the *Triangle Sum Hypothesis* implies the *Parallel Postulate*.
Outline: Suppose ℓ is a line and p is a point not on ℓ. Let $q_0 \in \ell$ be the point such that $\overline{pq_0}$ is perpendicular to ℓ (see Figure 9.14). If ℓ_0 is the line perpendicular to $\overleftrightarrow{pq_0}$ at p, then ℓ_0 is parallel to ℓ by Proposition 9.2. So suppose ℓ' is any other line containing p. Since it cannot be perpendicular to $\overleftrightarrow{pq_0}$, there exists a point a on ℓ' such that $0 < m\angle q_0pa < 90°$. Let $r = 90° - m\angle q_0pa$, so that $0 < r < 90°$. From (a) pick $q \in \ell$ (with q and a on the same side of $\overleftrightarrow{pq_0}$) such that $\triangle pq_0q$ is a right triangle with right angle at q_0 and $m\angle pqq_0 < r$. Show that a is in the *interior* of $\angle q_0pq$ and use this to prove that ℓ' must intersect ℓ.

[11]For many centuries scholars believed that the *Parallel Postulate* should follow from the more basic axioms of geometry. The history of the unsuccessful attempts to establish the result is filled with brilliant mathematics, stupendous errors, and fascinating personalities. We refer the interested reader to Chapter 23 of George Martin's *The Foundations of Geometry and the Non-Euclidean Plane*, Springer-Verlag, 1982.

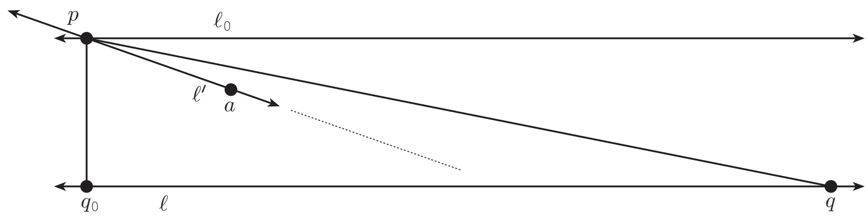
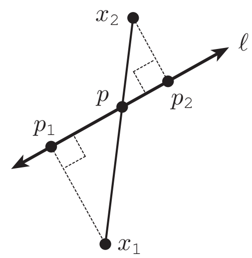


Figure 9.14. The construction for Exercise 9.4b.

Exercise 9.5.



Suppose x_1 and x_2 are points on opposite sides of a line ℓ such that $\overleftrightarrow{x_1x_2}$ is not perpendicular to ℓ. Let $\overline{x_1x_2}$ intersect ℓ at p, and let p_1 and p_2 be those points of ℓ such that $\overline{x_1p_1}$ and $\overline{x_2p_2}$ are perpendicular to ℓ.

(a) Why do points p_1 and p_2 exist and why are they unique?

(b) Why are the three points p_1, p, p_2 distinct?

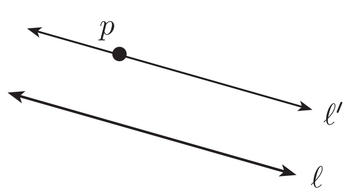
(c) Prove that p is between p_1 and p_2. *Hint:* Let m be the line through p which is perpendicular to ℓ. Then $\overline{x_1p_1}$ and $\overline{x_2p_2}$ are parallel to m (why?) and x_1 and x_2 are on opposite sides of m (why?).

§I.10 The Parallel Postulate

We now introduce the *Parallel Postulate*. This merely states the **uniqueness** of the parallel line whose existence was proven in Proposition 9.3.

The Parallel Postulate.

PP. Suppose ℓ is a line and p is a point not on ℓ.
Then there exists a *unique* line ℓ' parallel to ℓ and containing p:

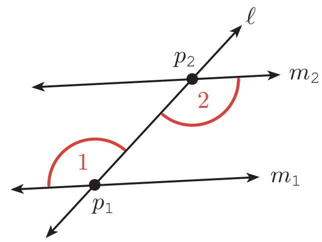


If p is a point *on* the line ℓ, then we don't need the *Parallel Postulate* to assert the existence of a *unique* line ℓ' containing p which is parallel to ℓ: the line ℓ itself is the unique ℓ' that we desire. Hence we state the new axiom only for points p not on the line ℓ since the result is trivially true if p is on ℓ. We don't include as axioms results that follow from other axioms!

Numerous important consequences follow from the *Parallel Postulate*, as we now show. The first is the converse to Proposition 9.9.

Proposition 10.1.

Suppose ℓ is a transversal for two distinct lines m_1 and m_2. If m_1 and m_2 are parallel, then alternate interior angles are congruent.



Proof. Let p_1 and p_2 be the intersection points of m_1 and m_2 with the line ℓ, respectively.

An interior angle at p_1 will be of the form $\angle a_1p_1p_2$ for a_1 a point of m_1 other than p_1, as shown in Figure 10.2. From Angle Axiom M-2, *Angle Construction*, there exists a point b_2 on the side of ℓ opposite to a_1 such that the angle $\angle p_1p_2b_2$ is congruent to $\angle a_1p_1p_2$.

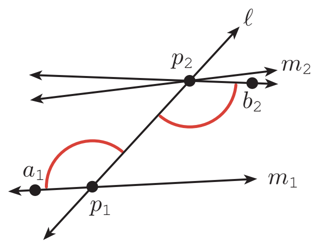


Figure 10.2. b_2 is picked so that $\angle p_1p_2b_2 \cong \angle a_1p_1p_2$.

But $\angle p_1p_2b_2$ and $\angle a_1p_1p_2$ are congruent alternate interior angles for the two lines m_1 and $\overleftrightarrow{p_2b_2}$ with respect to the transversal ℓ. Hence by Proposition 9.9 the line $\overleftrightarrow{p_2b_2}$ is parallel to m_1, and thus *both* lines $\overleftrightarrow{p_2b_2}$ and m_2 are parallel to m_1 and contain p_2. Aha! By the *Parallel Postulate* m_2 and $\overleftrightarrow{p_2b_2}$ must be the same line! In particular, the alternate interior angles for m_1 and m_2 and the transversal ℓ must be congruent, as desired. □

As promised in the previous section, we now show the *Parallel Postulate* implies that the interior angles of a triangle add up to 180°. (These are actually *equivalent statements*. See Exercise 9.4.)

Theorem 10.3.

Let $\triangle abc$ be any triangle with interior angles $\angle a$, $\angle b$, and $\angle c$. Then

$$m\angle a + m\angle b + m\angle c = 180^\circ.$$

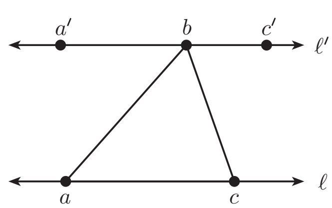


Figure 10.4. ℓ' is the line through b parallel to ℓ.

Proof. Let $\ell = \overleftrightarrow{ac}$. Referring to Figure 10.4, pick ℓ' through b such that ℓ' and ℓ are parallel. Let a' and c' be points of ℓ' such that a' and a are on the same side of the line $\overleftrightarrow{bc}$ and c' and c are on the same side of the line $\overleftrightarrow{ba}$. Then $\angle a'ba$ and $\angle a$ ($= \angle bac$) are alternate interior angles for the parallel lines ℓ and ℓ'. Hence, by Proposition 10.1 (a consequence of the *Parallel Postulate*) $\angle a'ba \cong \angle a$. By an identical argument, we also have the angle congruence $\angle c'bc \cong \angle c$ — see Figure 10.5.

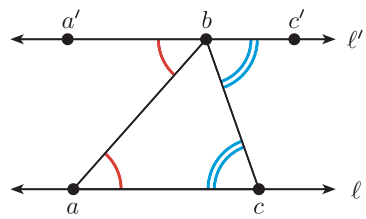


Figure 10.5. $\angle a'ba \cong \angle a$ and $\angle c'bc \cong \angle c$.

By Angle Axiom M-4, *Supplements*, $m\angle a'bc + m\angle c'bc = 180°$. But a is in the interior of $\angle a'bc$, and hence Angle Axiom M-3, *Angle Addition*, applied to the first of these angles, gives that $m\angle a'bc = m\angle a'ba + m\angle abc$. Combining these two equalities gives

$$m\angle a'ba + m\angle abc + m\angle c'bc = 180°.$$

However, from the previous paragraph we know that the first angle is congruent to $\angle a$ and the third angle is congruent to $\angle c$. Since the second angle is $\angle b$, we have obtained the desired equality of

$$m\angle a + m\angle b + m\angle c = 180°. \qquad \square$$

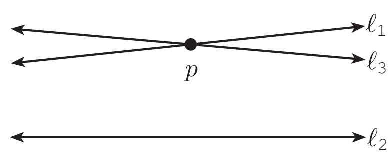
A commonly used property of parallel lines in the plane is the *transitivity* of parallelism: *if ℓ_1 is parallel to ℓ_2 and ℓ_2 is parallel to ℓ_3, then ℓ_1 is parallel to ℓ_3.* It might come as a surprise that the truth of this "basic fact" requires the *Parallel Postulate!* The proof makes clear why we have this dependence.

Theorem 10.6.

Parallelism between lines is transitive.

Proof. Suppose ℓ_1 is parallel to ℓ_2 and ℓ_2 is parallel to ℓ_3. We need to show ℓ_1 parallel to ℓ_3, i.e., ℓ_1 and ℓ_3 are either *disjoint* or *equal*.

If ℓ_1 and ℓ_3 are *not* disjoint, then the two lines will intersect, say at p:



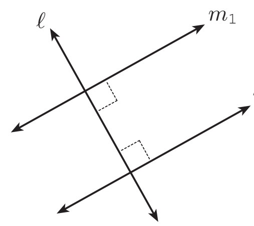
If p is on ℓ_2, then ℓ_1 equals ℓ_2 since these lines are parallel. The line ℓ_3 equals ℓ_2 for the same reason. Thus $\ell_1 = \ell_3$, proving them to be parallel.

If p is *not* on ℓ_2, then ℓ_1 and ℓ_3 are both parallel to ℓ_2 and contain the point p. The *Parallel Postulate* then gives $\ell_1 = \ell_3$, so the lines are again trivially parallel! Thus in all cases ℓ_1 and ℓ_3 are parallel lines, as desired. □

The next result is often useful, especially as it applies to both types of parallelism: m_1 and m_2 disjoint or equal. The proof is left to Exercise 10.3.

Proposition 10.7.

Two lines m_1 and m_2 are parallel if and only if any line ℓ perpendicular to m_1 is also perpendicular to m_2:



The existence of plenty of rectangles is another consequence of the *Parallel Postulate* — it follows from proving that all Saccheri quadrilaterals are actually rectangles. The proof of this result is left as Exercise 10.5.

Proposition 10.8.

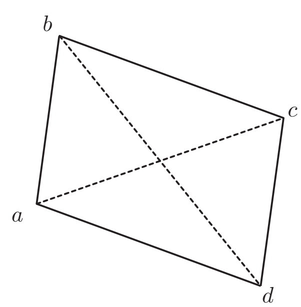
Every Saccheri quadrilateral is a rectangle.

Parallelograms. Parallelograms will be important in our subsequent work, especially when developing the important basic properties of *translations* in Chapter II. We now establish the basic results concerning parallelograms. Most of these results require the *Parallel Postulate*.

From Definition 5.10 a *quadrilateral* $\square abcd$ is the union of the four line segments $\overline{ab}$, $\overline{bc}$, $\overline{cd}$, and $\overline{da}$, where a, b, c, and d are four non-collinear points such that the four lines segments just listed intersect only at their endpoints. A *parallelogram* is merely a special type of quadrilateral.

Definition 10.9.

A **parallelogram** $\square abcd$ is a quadrilateral such that each pair of opposite sides are parallel line segments, i.e., $\overline{ab}$ and $\overline{cd}$ are parallel and $\overline{bc}$ and $\overline{da}$ are parallel. The line segments $\overline{ac}$ and $\overline{bd}$ are the **diagonals** of $\square abcd$.



Proposition 10.10.

(a) *A diagonal divides a parallelogram into two congruent triangles. In particular, given a parallelogram $\square abcd$, $\triangle abd \cong \triangle cdb$.*

(b) *In a parallelogram each pair of opposite sides are congruent.*

(c) *The diagonals of a parallelogram bisect each other.*

Proof of (a): Given a parallelogram $\square abcd$, let $m_1 = \overleftrightarrow{ab}$, $m_2 = \overleftrightarrow{cd}$, $\ell_1 = \overleftrightarrow{ad}$, and $\ell_2 = \overleftrightarrow{bc}$. Then $m_1 \parallel m_2$ and $\ell_1 \parallel \ell_2$. The diagonal $\overline{bd}$ forms two triangles, $\triangle abd$ and $\triangle cdb$, that we claim are congruent. We will prove this with *ASA,* Theorem 7.8, by verifying the following (see labels in Figure 10.11):

$$\overline{bd} \cong \overline{db},$$
$$\angle 1 \cong \angle 2,$$
$$\angle 3 \cong \angle 4.$$

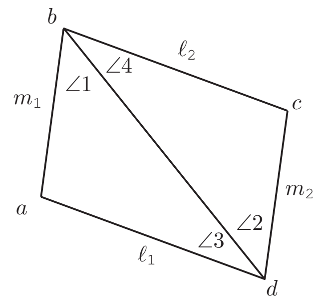


Figure 10.11.

- Segment congruence is trivial since $bd = db$.
- The angles $\angle 1$ and $\angle 2$ are alternate interior angles for the transversal $\overleftrightarrow{bd}$ of the parallel lines m_1 and m_2. Thus these angles are congruent by Proposition 10.1.
- Similarly, the angles $\angle 3$ and $\angle 4$ are alternate interior angles for the transversal $\overleftrightarrow{bd}$ of the parallel lines ℓ_1 and ℓ_2. Thus these angles are also congruent by Proposition 10.1.

Proof of (b): Given a parallelogram $\square abcd$, we wish to show that $\overline{ab} \cong \overline{cd}$ and $\overline{ad} \cong \overline{cb}$. However, these congruences follow immediate from the triangle congruence $\triangle abd \cong \triangle cdb$ established in part (a).

Proof of (c): This is Exercise 10.10b. $\square$

Future developments — in particular, the definition of *translations* in Chapter II — will depend on the following important results for parallelograms.

Proposition 10.12. ***Parallelogram Uniqueness.***

Suppose $\square abcd$ and $\square abcd_0$ are both parallelograms. Then $d = d_0$.

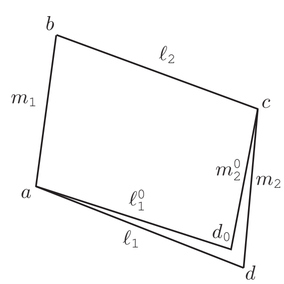


Figure 10.13. The points d and d_0 must be the same.

Proof. Given the parallelogram $\square abcd$, let $m_1 = \overleftrightarrow{ab}$, $m_2 = \overleftrightarrow{cd}$, $\ell_1 = \overleftrightarrow{ad}$, and $\ell_2 = \overleftrightarrow{bc}$. However, we have two additional line segments from the parallelogram $\square abcd_0$: $m_2^0 = \overleftrightarrow{cd_0}$ and $\ell_1^0 = \overleftrightarrow{ad_0}$, as shown in Figure 10.13.

The two lines m_2 and m_2^0 are both parallel to m_1 and contain the point c. Thus $m_2 = m_2^0$ by the *Parallel Postulate*. Similarly, the two lines ℓ_1 and ℓ_1^0 are both parallel to ℓ_2 and contain the point a. Thus $\ell_1 = \ell_1^0$.

The two points d and d_0 are therefore on both lines m_2 and ℓ_1. If the points are not equal, then m_2 and ℓ_1 would have to be the same line. But then a would be a point of intersection of the parallel lines m_1 and $m_2 = \ell_1$, giving $m_1 = m_2$. This would mean that all four points a, b, c, and d would be collinear, which is not possible. Thus d and d_0 cannot be distinct points, giving $d = d_0$, the desired result. $\square$

Theorem 10.14. ***Parallelogram Existence.***
Given three non-collinear points a, b, d in the plane, there exists a unique point c such that $\square abcd$ is a parallelogram.

Proof. Let $m_1 = \overleftrightarrow{ab}$ and $\ell_1 = \overleftrightarrow{ad}$. Since the points a, b, and d are non-collinear, the two lines m_1 and ℓ_1 are distinct and intersect only in the point a. By Proposition 9.3 there exist two other lines m_2 and ℓ_2 where m_2 contains the point d and is parallel to m_1 and ℓ_2 contains the point b and is parallel to ℓ_1. All this is shown in Figure 10.15.

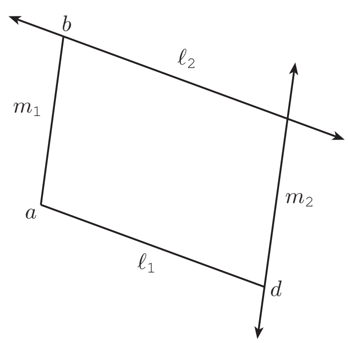


Figure 10.15. The point c will be at the intersection of m_2 and ℓ_2.

We claim that the two new lines m_2 and ℓ_2 must intersect in a point c. For suppose no such point exists. Then m_2 and ℓ_2 would be parallel. However, ℓ_2 is parallel to ℓ_1 so the transitivity of parallelism (Theorem 10.6) would imply that $m_2 \parallel \ell_1$. But this cannot happen since d is on both lines and a is only on ℓ_1 (if a were on m_2, then m_2 would have to equal m_1 and the points a, b, d would be collinear, contradicting our initial assumptions). Hence m_2 and ℓ_2 do indeed intersect in a point c which will therefore give the existence of a parallelogram $\square abcd$. The uniqueness of the point c follows from *Parallelogram Uniqueness* (Proposition 10.12). $\square$

The point c of Theorem 10.14 that completes the parallelogram $\square abcd$ can be fully characterized in a simple fashion. This is done in the next result, *Parallelogram Construction*. Besides the inherent interest of the result itself, it is a major step of our proof of *Desargues' Little Theorem* (Theorem 10.18), which in turn furnishes a major step in justifying some important results concerning translations in Chapter II.

Theorem 10.16. ***Parallelogram Construction.***

Given three non-collinear points a, b, d in the plane, the unique point c that produces a parallelogram $\square abcd$ is characterized as follows: c is that point such that $\overleftrightarrow{cd}$ is parallel to $\overleftrightarrow{ab}$, $cd = ab$, and b and c are on the same side of $\overleftrightarrow{ad}$.

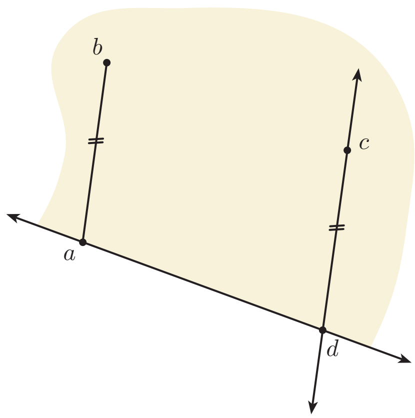


Figure 10.17. The point c will give a parallelogram $\square abcd$.

Proof. There exists a unique point c that makes $\square abcd$ a parallelogram from *Parallelogram Existence* (Theorem 10.14). That this point c has all the stated properties is easy to verify: $\overleftrightarrow{cd}$ is parallel to $\overleftrightarrow{ab}$ from the definition of parallelogram, $cd = ab$ is just Proposition 10.10b, and since $\overleftrightarrow{ad}$ and $\overleftrightarrow{bc}$ are parallel from the definition of a parallelogram, b and c must lie on the same side of $\overleftrightarrow{ad}$.

Now suppose c_0 is a point in the plane with all the stated properties: $\overleftrightarrow{c_0d}$ is parallel to $\overleftrightarrow{ab}$, $c_0d = ab$, and b and c_0 lie on the same side of $\overleftrightarrow{ad}$. We have only to show that c_0 equals the point c that makes $\square abcd$ into a parallelogram. The two lines $\overleftrightarrow{cd}$ and $\overleftrightarrow{c_0d}$ contain the point d and are parallel to $\overleftrightarrow{ab}$. Thus, by Axiom PP, the *Parallel Postulate*, the two lines $\overleftrightarrow{cd}$ and $\overleftrightarrow{c_0d}$ must be equal. Moreover, since c_0 and b lie on the same side of $\overleftrightarrow{ad}$ and c and b also lie on the same side of this line, c and c_0 must lie on the same side of $\overleftrightarrow{ad}$. In particular, the point d cannot lie between c and c_0, so the rays $\overrightarrow{dc}$ and $\overrightarrow{dc_0}$ must be identical. However, since $dc = ab = dc_0$, *Segment Construction* (Proposition 4.10) gives that c and c_0 are indeed the same point. This finishes the proof. $\square$

Theorem 10.18. ***Desargues' Little Theorem***.
Suppose $\square ABy_1x_1$ *and* $\square ABy_2x_2$ *are parallelograms with a common side* $\overline{AB}$ *and the parallel sides* $\overline{x_1y_1}$ *and* $\overline{x_2y_2}$ *forming distinct lines. Then* $\square x_1x_2y_2y_1$ *is also a parallelogram.*

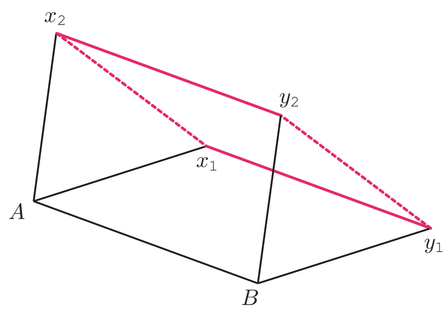


Figure 10.19. Parallelograms with a common side.

Desargues' Little Theorem, as illustrated in Figure 10.19, is an intuitively appealing result that is surprisingly difficult to prove! We leave the proof to Exercises 10.13 and 10.14, where all the details are carefully outlined. One key step in the proof is *Parallelogram Construction* (Theorem 10.16).

Desargues' Little Theorem is the basis for many of our fundamental results concerning translation and directed angle measure. As such it provides the underpinning for much of this book.

The Independence of the Parallel Postulate. For many centuries mathematicians were convinced that the *Parallel Postulate* had to be a consequence of the more basic axioms of standard geometry. And indeed, many extraordinary scholars (and many not-so-extraordinary would-be scholars) attempted to prove the *Parallel Postulate* — the history of this effort is fascinating. However, their intuition was wrong: in the first half of the nineteenth century it was shown that the *Parallel Postulate* was indeed **independent** of all the other standard axioms for plane geometry: there are geometric models in which the *Parallel Postulate* is true, and there are models in which the *Parallel Postulate* is false.

Euclidean geometry in the plane, as modeled by the *real Cartesian plane* $\mathbb{R}^2$ (Example 2.2), is a system for which the *Parallel Postulate* is valid. However, the *Poincaré disk* $\mathbb{P}^2$ (Example 2.4) is a system for which the *Parallel Postulate* is false, and in an extreme fashion: for any point p not on a line ℓ there are an *infinite* number of lines containing p that are parallel to ℓ. You can likely convince yourself of the truth of this statement by studying Figure 2.5. In that picture the point p_1 is not on the line ℓ_2, but both the lines ℓ_3 and ℓ_4 contain p_1 yet do not intersect ℓ_2. Hence both ℓ_3 and ℓ_4 are parallel to ℓ_2.

However, it is not yet obvious that all of our earlier axioms are valid for $\mathbb{P}^2$. In particular, we have not specified the coordinate systems we use on lines (hence the distance function has not yet been specified), nor have we produced an angle measure function. We will not go into these matters at this time, but will simply state that definitions for all of these quantities can indeed be made so that the *Poincaré disk* will satisfy all of our previous axioms. The *Poincaré disk* is a model for what is known as **hyperbolic geometry**, an example of a **non-Euclidean geometry** since a negation of the *Parallel Postulate* is valid for the system. We will study such geometries in Volume II of this text.

Exercises I.10

Exercise 10.1.

Let m_1 and m_2 be two distinct lines cut by a transversal ℓ. If $\angle x$ and $\angle y$ are alternate interior angles and $\angle y$ and $\angle z$ are vertical angles, then $\angle x$ and $\angle z$ are **corresponding angles**. Prove m_1 and m_2 are parallel if and only if a pair of corresponding angles are congruent.

Exercise 10.2.

Suppose m_1 and m_2 are parallel lines and ℓ a third line which intersects m_1 in a single point. Prove ℓ also intersects m_2 in a single point and hence must be a transversal for m_1 and m_2 if these two lines are distinct.

Exercise 10.3.

(a) Prove the forward direction of Proposition 10.7:

If two lines m_1 and m_2 are parallel then any line ℓ perpendicular to m_1 is also perpendicular to m_2.

Hints: Assume m_1 and m_2 are parallel and take ℓ to be any line perpendicular to m_1 at a point $p \in m_1$. You must show ℓ is perpendicular to m_2. Since the case $m_1 = m_2$ is trivial you may assume m_1 and m_2 are disjoint. To show $\ell \perp m_2$ you must first show that ℓ actually intersects m_2. This is Exercise 10.2.

(b) Prove the reverse direction of Proposition 10.7:

The two lines m_1 and m_2 will be parallel if whenever a line ℓ is perpendicular to m_1, ℓ is also perpendicular to m_2.

Hint: First show there does exist a line ℓ perpendicular to m_1. Then, by our current assumption, ℓ is also perpendicular to m_2.

(c) Give a second proof of the transitivity of parallelism (Theorem 10.6) based on Proposition 10.7.

Exercise 10.4.

Suppose ℓ and ℓ' are distinct parallel lines and a is a point on ℓ.

(a) Prove there exists a point $a' \in \ell'$ such that $\overline{aa'}$ is perpendicular to ℓ. Further prove a' is unique and $\overline{aa'}$ is also perpendicular to ℓ'.

(b) Prove that the number aa' is independent of the choice of the point a on ℓ, i.e., if b is any other point on ℓ and b' is the corresponding point on ℓ' such that $\overline{bb'}$ is perpendicular to ℓ, then $aa' = bb'$. This number is called the **distance** between the parallel lines ℓ and ℓ'. Does this result depend on the *Parallel Postulate?* Explain.

Exercise 10.5.

(a) Prove Proposition 10.8: Every Saccheri quadrilateral is a rectangle when the *Parallel Postulate* is assumed. Be sure to carefully point out how your proof depends on the *Parallel Postulate.*
Hint: In the Saccheri quadrilateral $\square abcd$ shown in Figure 9.12 you need to show that $\angle abc$ and $\angle bcd$ are both right angles. Showing the first of these angles is a right angle can be accomplished by verifying that $\triangle abc \cong \triangle cda$.

(b) For this exercise let the *Rectangle Hypothesis* be the statement that every Saccheri quadrilateral is a rectangle. Without using the *Parallel Postulate* or any result derived from the *Parallel Postulate*, prove that the *Rectangle Hypothesis* implies the *Triangle Sum Hypothesis*, i.e., that the sum of the measures of the interior angles of any triangle equals $180°$.
Hint: Modify the arguments of Exercise 9.3. In particular, in part (a) use Exercise 9.1 in place of Exercise 9.2; parts (b) and (c) require little alteration.

(c) Show that the *Parallel Postulate* and the *Rectangle Hypothesis* are equivalent as described in Exercise 9.4. *Hint:* You will need the result of Exercise 9.4.

Exercise 10.6.

(a) For any triangle $\triangle ABC$ prove that

> $\overline{AB} \cong \overline{BC}$ *if and only if the angle bisecting ray* $\overrightarrow{r}$ *for* $\angle B$ *is parallel to the perpendicular bisector* ℓ *of* $\overline{AC}$.

If $\overrightarrow{r}$ and ℓ are indeed parallel, further show that $\overrightarrow{r} \subset \ell$.

(b) Prove the following for any triangle $\triangle ABC$:

> *If* $\triangle ABC$ *is a triangle for which* $AB \neq BC$, *then the angle bisecting ray* $\overrightarrow{r}$ *for* $\angle B$ *intersects the perpendicular bisector* ℓ *for* $\overline{AC}$ *at a single point* O.

Hints: This is a challenging exercise. Let m be the line containing the angle bisecting ray $\overrightarrow{r}$. From (a) ℓ and m intersect in a unique point O. You must prove $\overrightarrow{BO}$ is the angle bisecting ray $\overrightarrow{r}$ for $\angle B$ (or equivalently, that the point O is interior to the angle $\angle B$).

Exercise 10.7.

(a) For any triangle, prove that the measure of an exterior angle equals the sum of the measures of the two remote interior angles.

Recall that **circles** were defined in Exercise 7.5.

(b) *The Theorem of Thales*. Suppose $\overline{AB}$ is a diameter of a circle $\mathcal{C}$ and C is a point on $\mathcal{C}$ other than A or B. Prove that $\angle ACB$ is a right angle. *Hint:* Suppose p is the center of the circle $\mathcal{C}$. Analyze the two triangles $\triangle ApC$ and $\triangle BpC$.

(c) Prove the converse of the *Theorem of Thales*: Suppose $\angle ApB$ is a right angle. Then p is on the circle $\mathcal{C}$ with diameter $\overline{AB}$. *Hint*: Let C be the midpoint of $\overline{AB}$. Then analyze the angles of the two triangles $\triangle ApC$ and $\triangle BpC$.

(d) *Challenge Exercise*. Prove that the *Theorem of Thales* is equivalent to the *Parallel Postulate* as described in Exercises 9.4 and 10.5.

Exercise 10.8. ***The Inscribed Angle Theorem***.

Let A, B, C be distinct points on a circle $\mathcal{C}$ with center P (Exercise 7.5).

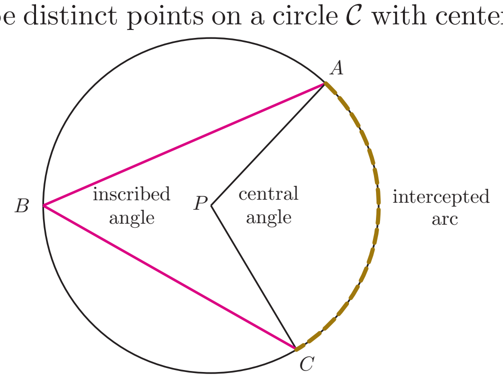


Figure 10.20. Inscribed angle, central angle, and intercepted arc.

The following objects are illustrated in Figure 10.20.

- $\angle ABC$ is an **inscribed angle** of the circle $\mathcal{C}$.
- The portion of $\mathcal{C}$ which is interior to $\angle ABC$, along with the points A and C, is the **intercepted arc** $\widehat{AC}$ for $\angle ABC$.
- The intercepted arc for $\angle ABC$ defines a **central angle** $\angle_c APC$, the measure of which varies between 0 and 360°. (The values between 180° and 360° occur when the intercepted arc comprises more than half of the circle.) The measure of this central angle is taken to be the **measure of the intercepted arc**.

In this exercise you will prove that *the measure of an inscribed angle $\angle ABC$ is half the measure of its intercepted arc.* The proof is done in three steps, covering all possible locations for A, B, and C.

(a) First suppose $\overline{BC}$ is a diameter of the circle, as shown in the first frame of Figure 10.21. You must show $m\angle 2 = 2\,m\angle 1$.

(b) Suppose $\overline{BD}$ is a diameter of the circle with A and C on opposite sides as shown in the second frame of Figure 10.21. You must show $m\angle 4 = 2\,m\angle 3$. *Hint:* Two applications of (a).

(c) Again suppose $\overline{BD}$ a diameter of the circle but this time with A and C on the same side as shown in the third frame of Figure 10.21. You must show $m\angle 6 = 2\,m\angle 5$.

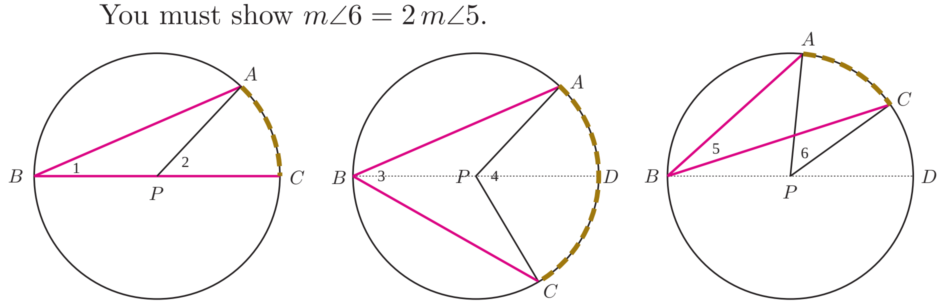
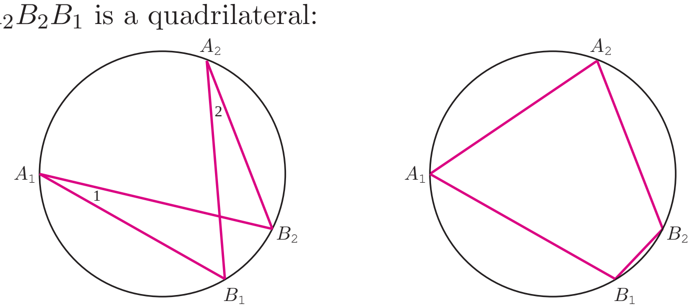


Figure 10.21. Cases for proving $m\angle ABC = \frac{1}{2}\,m\angle_c APC = \frac{1}{2}\,m(\widehat{AC})$.

Exercise 10.9.

Suppose A_1, A_2, B_1, B_2 are four points on a circle, ordered so that $\square A_1A_2B_2B_1$ is a quadrilateral:



(a) Prove that the angles $\angle 1 = \angle B_1A_1B_2$ and $\angle 2 = \angle B_1A_2B_2$ are congruent. This proves the following important property of circles:

Angles inscribed in the same arc of a circle are congruent.

(b) Show that the opposite interior angles in the inscribed quadrilateral $\square A_1A_2B_2B_1$ are supplementary.

Exercise 10.10.

(a) Prove that a parallelogram is a *convex* quadrilateral (Definition 5.10).

(b) Prove Proposition 10.10c: *The diagonals of a parallelogram bisect each other. Hint:* Recall Proposition 5.13.

(c) A **rhombus** is a parallelogram whose four sides are all congruent. Prove that the diagonals of any rhombus intersect at right angles.

(d) Prove a parallelogram vertex is in the interior of the opposite angle.

Exercise 10.11.

(a) Prove that a convex quadrilateral $\square abcd$ whose opposite sides are congruent (i.e., $\overline{ab} \cong \overline{cd}$ and $\overline{ad} \cong \overline{bc}$) is a parallelogram.

(b) Prove that every rectangle (Definition 9.11) is a parallelogram.

Exercise 10.12.

Suppose $\square ABy_1x_1$ and $\square ABy_2x_2$ are both parallelograms as shown in Figure 10.22. Prove $\overline{x_1x_2} \cong \overline{y_1y_2}$ *without the use of Desargues' Little Theorem!*[12] Structure your proof in the following way.

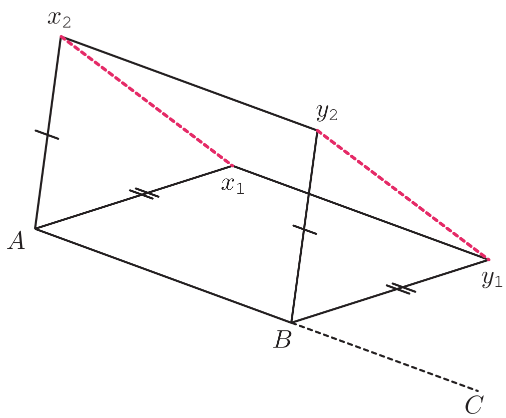


Figure 10.22. Given two parallelograms, prove $\overline{x_1x_2} \cong \overline{y_1y_2}$.

(a) First consider the case where A, x_1, and x_2 are collinear.

The remaining steps consider when A, x_1, and x_2 are *non*-collinear.

(b) Show that x_1 and y_1 both lie on the same side of the line $\overleftrightarrow{AB}$, and similarly for x_2 and y_2.

(c) Assume x_1, y_1, x_2, y_2 all lie on the same side of $\overleftrightarrow{AB}$, and let C be a point on the line $\overleftrightarrow{AB}$ such that B is between A and C. Then apply Exercise 5.5a to $\angle x_1AC$ and $\angle x_2AC$, and use Exercise 10.1 to conclude $\angle x_1Ax_2 \cong \angle y_1By_2$. Show how this implies $\overline{x_1x_2} \cong \overline{y_1y_2}$, the desired result.

(d) Finish the proof by showing $\overline{x_1x_2} \cong \overline{y_1y_2}$ when x_1 and y_1 lie on the side of $\overleftrightarrow{AB}$ opposite to x_2 and y_2.

[12]It is also true that $\overline{x_1x_2}$ and $\overline{y_1y_2}$ are parallel, but this is surprisingly hard to prove. In particular, it requires *Desargues' Little Theorem* (Theorem 10.18).

Exercise 10.13. ***The Ray Separation Theorem.***

This result will be needed for the proof of *Desarges' Little Theorem*:

> *Suppose two parallel rays $\overrightarrow{a_1b_1}$ and $\overrightarrow{a_2b_2}$ both lie on one side of a line m which is not parallel to the rays. If ℓ is a line that separates the rays, then ℓ must be parallel to the rays.*

Let m_1 and m_2 denote lines $m_1 = \overleftrightarrow{a_1b_1}$ and $m_2 = \overleftrightarrow{a_2b_2}$, let m intersect m_1 and m_2 at the points d_1 and d_2, respectively, and let $n = \overleftrightarrow{a_1a_2}$.

(a) Show that the rays $\overrightarrow{a_1b_1}$ and $\overrightarrow{a_2b_2}$ are on the same side of n.
Hint: Suppose that the rays $\overrightarrow{a_1b_1}$ and $\overrightarrow{a_2b_2}$ are on *opposite* sides of n. Then show $\overline{d_1d_2}$ intersects n at some point e. However, show that e cannot equal a_1 or a_2 (easy), but neither can it lie between a_1 and a_2, nor can it lie on the ray opposite $\overrightarrow{a_1a_2}$ (consider sides of m_1) or the ray opposite $\overrightarrow{a_2a_1}$ (consider sides of m_2).

(b) Prove that ℓ intersects $\overline{a_1a_2}$ at some point e. (Easy!)

(c) Suppose k is any line through e but unequal to n and *not* parallel to m_1 and m_2. Prove that k must intersect m_1 and m_2 at points $c_1 \in m_1$ and $c_2 \in m_2$ where c_1 and c_2 are *on opposite sides of* n.
Hint: The line k intersects n at a point q. Show q must lie between c_1 and c_2 by proving that the other locations for q are impossible.

(d) Show ℓ must be parallel to m_1 and m_2, as desired. *Hint:* If ℓ is not parallel to m_1 and m_2, then use $k = \ell$ in (c) to contradict (a).

Exercise 10.14. ***Desargues' Little Theorem.***

Suppose $\square ABy_1x_1$ and $\square ABy_2x_2$ are both parallelograms with $\overleftrightarrow{x_1y_1}$ and $\overleftrightarrow{x_2y_2}$ distinct lines as shown in Figure 10.22. Prove that $\square x_1x_2y_2y_1$ is also a parallelogram. This is *Desargues' Little Theorem* (Theorem 10.18). Structure your proof as follows.

(a) First show that the desired result will follow if you prove y_1 and y_2 lie on the same side of $\overleftrightarrow{x_1x_2}$. *Hint: Parallelogram Construction* (Theorem 10.16).

In view of (a) we will now assume y_1 and y_2 do *not* lie on the same side of $\overleftrightarrow{x_1x_2}$ and work towards a contradiction.

(b) Show $\overrightarrow{x_1y_1}$ and $\overrightarrow{x_2y_2}$ are on opposite sides of the line $\ell = \overleftrightarrow{x_1x_2}$.

(c) Consider the case where x_1 and x_2 are both on the same side of $\overleftrightarrow{AB}$. By interchanging the labels of x_1 and x_2 if necessary, you can assume $\angle BAx_2 \geq \angle BAx_1$. Prove the parallel rays $\overrightarrow{x_1y_1}$ and $\overrightarrow{x_2y_2}$ are then on the same side of $m = \overleftrightarrow{Ax_2}$. *Hint:* Show both $\overrightarrow{x_1y_1}$ and $\overrightarrow{x_2y_2}$ must lie on the same side of m as $\overrightarrow{AB}$.

(d) Consider the case where x_1 and x_2 are on opposite sides of $\overleftrightarrow{AB}$. As in (c), show that (by relabeling if necessary) you can arrange to have the parallel rays $\overrightarrow{x_1y_1}$ and $\overrightarrow{x_2y_2}$ on the same side of $m = \overleftrightarrow{Ax_2}$.

(e) In all cases you now have (by relabeling if necessary) the parallel rays $\overrightarrow{x_1y_1}$ and $\overrightarrow{x_2y_2}$ separated by $\ell = \overleftrightarrow{x_1x_2}$ but on the same side of $m = \overleftrightarrow{Ax_2}$. Conclude that this is impossible, and hence y_1 and y_2 must lie on the same side of $\overline{x_1x_2}$, as desired. *Hint:* Exercise 10.13, the *Ray Separation Theorem*.

§I.11 Directed Angle Measure and Ray Translation

Global Rotational Direction. In §6 we defined the concept of *directed angle measure* at each point in the plane. We further observed that at each point we had a choice of exactly two different directed angle measures, each one the negative of the other. Essentially, one measure gives a positive sign to "clockwise" rotation while the other gives a positive sign to "counterclockwise" rotation. However, we also noted a difficulty: there is no good definition for clockwise versus counterclockwise. They are simply the "opposites" of each other.

And herein lies a seriously thorny problem. We wish to consistently choose a directed angle measure at *all* points of the plane, say *counterclockwise* directed angle measure at every point. But though we are quite sure what this means intuitively, as just observed, there is no simple, rigorous way to define "counterclockwise" rotation at a point.

The intuitive meaning of "counterclockwise" suggests a solution. "Counterclockwise" means the opposite direction to a clock. In other words, rotations are classified by comparison against a clock. So pick some object with a fixed rotational direction (like a clock), arbitrarily designate its rotational direction as "clockwise," and then classify all other rotations as *clockwise* or *counterclockwise* via comparison with the fixed, "standard" clock.

However, there is still a difficulty. The comparison process assumes we can take the "standard" clock and move it to any point p in the plane and compare its rotation with any other rotation at p. However, how do we know this process will be consistent, i.e., might the process of "moving the clock around the plane" actually *change the rotational direction of the clock?*

There are geometric spaces in which this actually happens! Consider the **Möbius band**. To construct a Möbius band take a rectangular strip of paper which is much longer than it is wide. If you wrap the ends of the strip around and glue them together in the ordinary fashion, you obtain the side of a cylinder, as shown in the left half of Figure 11.1. However, if before gluing the ends together you give the strip a half-twist, i.e., a rotation of 180°, you obtain a Möbius band. This is shown in the right half of Figure 11.1.

A Möbius band has the curious property of being one-sided. To see this, start with a dot anywhere on the band and move the dot along the band as though you were going around a cylinder. When you go all the way around, you will find your dot on the "other side" of the surface! In this way the Möbius band has only one side.

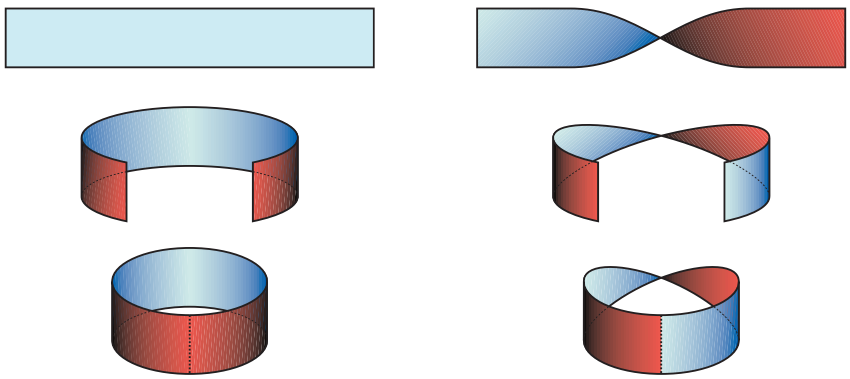


Figure 11.1. A cylinder compared with a Möbius band.

However, suppose you repeat the movement of the previous paragraph, but this time not with a simple dot but with a "clockwise angle" of, say, 30°. As with the dot, when you move the angle around the band, you will end up on the "other side" of the surface. However, an idealized surface has no thickness, so that any collection of points on the surface really exists on both sides of the surface simultaneously! So our angle can be considered as an object on both sides of the band, including *the side of the band on which the angle started.* However, as can be seen in Figure 11.2, *the rotation direction of the angle has changed from counterclockwise to clockwise!* So translating a rotation on a Möbius band will interchange clockwise and counterclockwise.

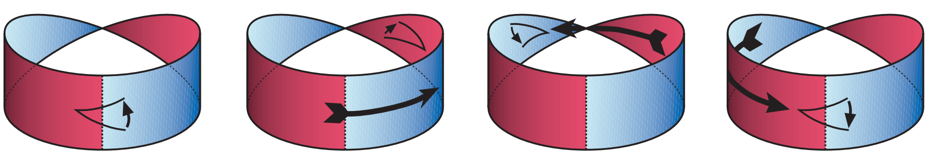


Figure 11.2. Going around a Möbius band reverses rotational direction.

We should not expect that moving an angle around in the Euclidean plane $\mathcal{E}$ could produce such a reversal of rotational direction. However, this belief needs to be verified from our axioms for the Euclidean plane.

The way we handle all the problems in this section is to show that directed angle measure assignments can be made at each point in the plane that are *translationally invariant*, meaning that "parallel translation" of a directed angle (i.e., moving it "without rotation") from one point to another will not change its directed angle measure. Hence the Euclidean plane does *not* behave like a Möbius band. No new axioms need be added to our collection, i.e., translationally invariant notions of clockwise and counterclockwise are already inherent in the Euclidean plane.

Translational Invariance. Choose a point a_0 in the plane. According to Proposition 6.16, two directed angle measures exist at a_0. Choose one of these measures, denoted by m, and interpret positive values of m to indicate *counterclockwise rotation* and negative values to indicate *clockwise rotation*.

We wish to choose a "counterclockwise" rotational direction at each point of the plane which is "consistent" with the choice made at a_0. Following our analogy of moving a clock around from point to point in the plane, the consistency we desire is **translational invariance**. To develop this concept, we first consider the translation of rays, then of directed angles.

Ray Translation. Though the concept of *ray translation* is intuitively simple — "move" a ray in the plane without rotating it or changing its length by expansion or contraction — capturing this concept in a formal definition and rigorously establishing its properties is far deeper than you would imagine. The key foundational principle is *Desargues' Little Theorem*. However, a careful development of ray translation yields a rich harvest: the fundamental properties of *directed angle measure*, *rotations*, and *translations*. We would have no book without these concepts.

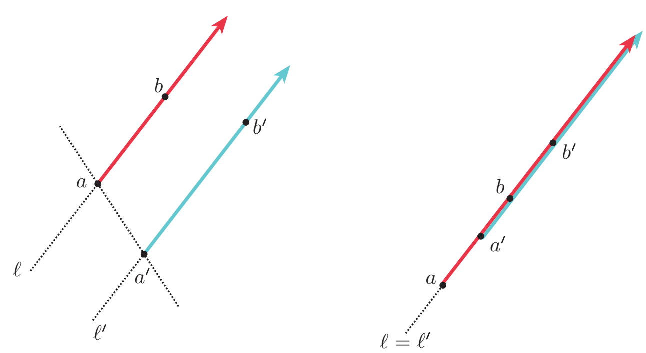


Figure 11.3. In each picture $\overrightarrow{ab}$ and $\overrightarrow{a'b'}$ are translates of each other.

Definition 11.4.

Two rays $\overrightarrow{ab}$ and $\overrightarrow{a'b'}$ are **translates** of each other if either

(1) the lines $\ell = \overleftrightarrow{ab}$ and $\ell' = \overleftrightarrow{a'b'}$ are distinct parallel lines such that b and b' lie on the same side of $\overleftrightarrow{aa'}$ or

(2) one of the rays $\overrightarrow{ab}$ and $\overrightarrow{a'b'}$ is a subset of the other.

The two types of ray translates are shown in Figure 11.3.

The properties we need of ray translation are intuitive and unsurprising: given a ray $\overrightarrow{a_1b_1}$, (1) there exists a *unique* translate of this ray to any other point a_2 in the plane and (2) if you translate $\overrightarrow{a_1b_1}$ to $\overrightarrow{a_2b_2}$ and then translate this new ray to $\overrightarrow{a_3b_3}$, the result is the same as translating $\overrightarrow{a_1b_1}$ to $\overrightarrow{a_3b_3}$. These properties are given below in Proposition 11.5 and Theorem 11.6.

Proposition 11.5.

Given a ray $\overrightarrow{ab}$ and any point a', there exists exactly one ray $\overrightarrow{a'b'}$ with initial point a' that is a translate of $\overrightarrow{ab}$.

Proof. Case (1). Suppose a' is not on the line $\ell = \overleftrightarrow{ab}$. To prove the existence of a translate ray $\overrightarrow{a'b'}$, note that by *Parallelogram Existence* (Theorem 10.14) there exists a unique point b' such that $\square abb'a'$ is a parallelogram. The ray $\overrightarrow{a'b'}$ is easily seen to be a translate of $\overrightarrow{ab}$.

To prove $\overrightarrow{a'b'}$ is the unique translate of $\overrightarrow{ab}$ starting at a', suppose $\overrightarrow{a'b''}$ is another such ray. By Axiom PP, the *Parallel Postulate*, there exists only one line ℓ' through a' which is parallel to ℓ. Thus the two rays $\overrightarrow{a'b'}$ and $\overrightarrow{a'b''}$ both lie on this line, and thus $\overrightarrow{a'b''}$ equals $\overrightarrow{a'b'}$ or its opposite. However, the ray opposite to $\overrightarrow{a'b'}$ is not a translate of $\overrightarrow{ab}$ and thus $\overrightarrow{a'b''}$ must equal $\overrightarrow{a'b'}$, as desired. This proves the desired uniqueness in case (1).

Case (2). Suppose a' is on the line $\ell = \overleftrightarrow{ab}$. Using the *Ruler Placement Theorem* (Proposition 3.6), choose a coordinate system $\chi : \ell \to \mathbb{R}$ such that $\chi(a) = 0$ and $\chi(b) > 0$. In addition, choose a point b' on ℓ such that $\chi(b') > \chi(a')$. Then, as in Exercise 4.4, we obtain

$$\overrightarrow{ab} = \{c \in \ell \mid 0 \leq \chi(c)\},$$
$$\overrightarrow{a'b'} = \{c \in \ell \mid \chi(a') \leq \chi(c)\}.$$

Thus, if $\chi(a') \geq 0$, then $\overrightarrow{a'b'} \subset \overrightarrow{ab}$, but if $\chi(a') < 0$, then $\overrightarrow{ab} \subset \overrightarrow{a'b'}$. In either case, $\overrightarrow{a'b'}$ is a translate of $\overrightarrow{ab}$, as desired. Uniqueness of $\overrightarrow{a'b'}$ is verified as follows. Since a' is on the line $\ell = \overleftrightarrow{ab}$, any translate of $\overrightarrow{ab}$ which starts at a' will lie on ℓ. But then there are only two possibilities for the translate: $\overrightarrow{a'b'}$ or its opposite. But the opposite ray, which consists of all the points c on the line ℓ with $\chi(c) \leq \chi(a')$, cannot contain or be contained in $\overrightarrow{ab}$. Thus the assumed translate ray *must* be $\overrightarrow{a'b'}$, proving the desired uniqueness. □

In view of Proposition 11.5 we see that any ray $\overrightarrow{ab}$ starting at the point a can be uniquely *translated* to any other point a', by which we mean the ray $\overrightarrow{ab}$ has a unique translate ray $\overrightarrow{a'b'}$ starting at a'. However, we need to know more. In particular, we need to know that translation of rays is *transitive*. This is a much deeper result than you would ever suspect!

Theorem 11.6. ***Transitivity of Ray Translation.***

Translation of rays is transitive. Thus if $\overrightarrow{a_1b_1}$ translates to $\overrightarrow{a_2b_2}$ and $\overrightarrow{a_2b_2}$ translates to $\overrightarrow{a_3b_3}$, then $\overrightarrow{a_1b_1}$ translates to $\overrightarrow{a_3b_3}$.

Proof. For convenience we assume the points b_1, b_2, b_3 have been picked so that the distances a_1b_1, a_2b_2, a_3b_3 are all equal. This can always be done by *Segment Construction* (Proposition 4.10). Since parallelism is transitive (Theorem 10.6) the lines $\ell_1 = \overleftrightarrow{a_1b_1}$ and $\ell_3 = \overleftrightarrow{a_3b_3}$ are parallel to each other since both are parallel to $\ell_2 = \overleftrightarrow{a_2b_2}$.

Now assume the generic case where all three lines ℓ_1, ℓ_2, ℓ_3 are distinct. To show $\overrightarrow{a_1b_1}$ translates to $\overrightarrow{a_3b_3}$, we have only to show that b_1 and b_3 lie on the same side of the line $m = \overleftrightarrow{a_1a_3}$. See Figure 11.7.

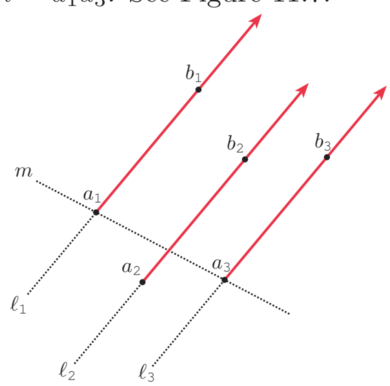


Figure 11.7. Need to verify b_1 and b_3 on the same side of $m = \overleftrightarrow{a_1a_3}$.

However, we know that the two line segments $\overline{a_1b_1}$ and $\overline{a_2b_2}$ are parallel, have the same length, and have the endpoints b_1 and b_2 on the same side of the line $m_3 = \overleftrightarrow{a_1a_2}$. But these are the exact conditions needed in *Parallelogram Construction* (Theorem 10.16) to conclude that $\square a_1b_1b_2a_2$ is a parallelogram. The same argument shows that $\square a_2b_2b_3a_3$ is a parallelogram. Thus, since the three lines ℓ_1, ℓ_2, and ℓ_3 are all distinct, *Desargues' Little Theorem* (Theorem 10.18) gives that $\square a_1b_1b_3a_3$ is also a parallelogram,[13] proving that the ray $\overrightarrow{a_3b_3}$ is indeed a translate of the ray $\overrightarrow{a_1b_1}$.

The various "degenerate" cases, where two or more of the lines ℓ_1, ℓ_2, ℓ_3 are the same, will be considered in Exercise 11.1. Most are not trivial. $\square$

The rest of this section may be skipped by readers needing only an intuitive treatment of clockwise/counterclockwise rotation, ray translation, and directed angle measure.

[13] Recall that the difficult part of the proof of *Desargues' Little Theorem* is that b_1 and b_3 are indeed on the same side of the line $\overleftrightarrow{a_1a_3}$. Simple statements are sometimes the most difficult!

Directed Angle Translation. We translate directed angles by translating each of the two rays of the angle. Ultimately this will allow us to choose a "counterclockwise" directed angle measure at one point a_0 and then translate this choice to all other points in the plane. However, before we can carry out this plan, we need some properties of directed angle translation.

Definition 11.8.

Two directed angles $\measuredangle abc$ and $\measuredangle a'b'c'$ are said to be **translates** of each other, denoted by $\measuredangle abc \equiv_\tau \measuredangle a'b'c'$, if

- $\overrightarrow{ba}$ and $\overrightarrow{b'a'}$ are translates of each other and
- $\overrightarrow{bc}$ and $\overrightarrow{b'c'}$ are translates of each other.

Proposition 11.9.

Given a directed angle $\measuredangle abc$ and any point b', there exists exactly one directed angle $\measuredangle a'b'c'$ with vertex b' that is a translate of $\measuredangle abc$.

Proof. This is a quick corollary of Proposition 11.5: simply apply that result to each ray $\overrightarrow{ba}$ and $\overrightarrow{bc}$. □

Proposition 11.10.

Translation of directed angles is an equivalence relation, i.e., the relationship $\equiv_\tau$ is reflexive, symmetry, and transitive.

Proof. To prove that $\equiv_\tau$ is reflexive means verifying any directed angle is a translate of itself. This is trivial from the definition of translation.

To prove that $\equiv_\tau$ is symmetric means showing $\measuredangle A' \equiv_\tau \measuredangle A$ whenever we have $\measuredangle A \equiv_\tau \measuredangle A'$. This is easy to verify from the definition of directed angle translation since if $\overrightarrow{ab}$ is a translate of $\overrightarrow{a'b'}$, then $\overrightarrow{a'b'}$ is a translate of $\overrightarrow{ab}$.

Transitivity for directed angle translation is a corollary of the transitivity of ray translation as given in Theorem 11.6. For suppose $\measuredangle abc \equiv_\tau \measuredangle a'b'c'$ and $\measuredangle a'b'c' \equiv_\tau \measuredangle a''b''c''$. Then rays $\overrightarrow{ba}$ and $\overrightarrow{b'a'}$ are translates of each other and rays $\overrightarrow{b'a'}$ and $\overrightarrow{b''a''}$ are translates of each other. Thus Theorem 11.6 gives that $\overrightarrow{ba}$ and $\overrightarrow{b''a''}$ are translates of each other. Similarly rays $\overrightarrow{bc}$ and $\overrightarrow{b'c''}$ are translates of each other, proving $\measuredangle abc \equiv_\tau \measuredangle a''b''c''$, as desired. □

Recall that a **trivial** directed angle is a directed angle made from two identical rays, so that any directed angle measure will assign 0 (mod 360) to this angle. A **straight** directed angle is a directed angle made from two opposite rays, so that any directed angle measure will assign 180° (mod 360) to such a directed angle. It is quite natural to expect directed angle translation to preserve such angles; this is the content of the next proposition. Its proof is quite easy and is left to Exercise 11.3.

Proposition 11.11.

A translate of a trivial directed angle is another trivial directed angle.
A translate of a straight directed angle is another straight directed angle.

The next proposition handles translates of non-trivial and non-straight directed angles. It too is a highly intuitive result: translation does not change (ordinary) angle measure, so that directed angles which are translates of each other are **congruent** as (ordinary) angles.

Proposition 11.12.

Non-trivial, non-straight directed angles that are translates of each other are congruent, i.e., they have the same (ordinary) angle measure.

Proof. Let $\measuredangle abc$ and $\measuredangle a'b'c'$ be translates of each other where, for convenience, a, c, a', and c' are chosen so that $ba = b'a'$ and $bc = b'c'$. We first consider the generic situation: $\overleftrightarrow{ba}$ and $\overleftrightarrow{b'a'}$ are distinct parallel lines and $\overleftrightarrow{bc}$ and $\overleftrightarrow{b'c'}$ are distinct parallel lines. If necessary, adjust the length $ba = b'a'$ so that lines $\overleftrightarrow{aa'}$ and $\overleftrightarrow{cc'}$ are distinct parallel lines. See Figure 11.13.

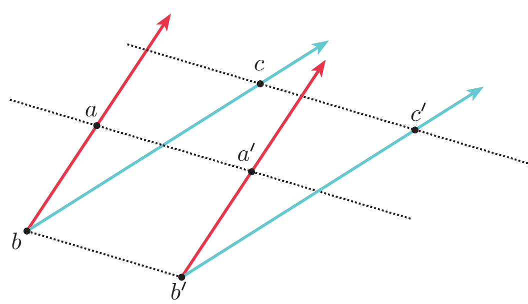


Figure 11.13. $\measuredangle abc$ and $\measuredangle a'b'c'$ are translates of each other.

With these choices the definition of ray translation along with *Parallelogram Construction* (Theorem 10.16) show $\square b'baa'$ and $\square b'bcc'$ to be parallelograms. Then *Desargues' Little Theorem* (Theorem 10.18) shows $\square aa'c'c$ to be a parallelogram. In particular, $ac = a'c'$. Therefore triangles $\triangle abc$ and $\triangle a'b'c'$ are congruent by *SSS* (Theorem 7.11). Thus $\angle abc \cong \angle a'b'c'$.

The degenerate cases, $\overleftrightarrow{ba} = \overleftrightarrow{b'a'}$ and/or $\overleftrightarrow{bc} = \overleftrightarrow{b'c'}$, are in Exercise 11.2. $\square$

Translationally Invariant Directed Angle Measure. As shown in Propositions 11.11 and 11.12, translation of a directed angle does not change its *ordinary* angle measure. However, we would also like its *directed* angle measure not to change. For this to be true, we must make consistent choices of directed angle measure at each point in the plane. So fix a point a_0, pick one of the two choices of directed angle measure at that point, and then "translate" that choice to all other points by translation of directed angles. The details follow....

Definition 11.14.

A directed angle measure m defined at every point is **translationally invariant** if the measure is unchanged under directed angle translation:

if $\measuredangle A$ and $\measuredangle A'$ are translates of each other, then $m\measuredangle A = m\measuredangle A'$.

We finally come to the big result: there exist exactly two translationally invariant directed angle measures in the plane. These two measures justify our intuitive notions of *clockwise* and *counterclockwise*. We interpret one of the directed angle measures — denote it as m_+ — as measuring directed angles using a *counterclockwise orientation* and the other — denote it as m_- — as using a *clockwise orientation*. Thus, under m_+, a positive directed angle measure indicates a *counterclockwise* turn from the initial ray to the terminal ray, while a negative value indicates a *clockwise* turn (also called a *negative counterclockwise* turn). For m_- the roles of clockwise and counterclockwise are reversed.

Theorem 11.15.

Exactly two translationally invariant directed angle measures exist in the Euclidean plane $\mathcal{E}$. Each measure is the negative of the other.

Proof. We first show that there can be *at most two* translationally invariant directed angle measures on the plane. We already know from Proposition 6.16 that at a given point a_0 there are precisely two directed angle measures. However, the assumed translation invariance of the angle measure shows that when the directed angle measure values are established at one point a_0, then they are immediately determined at every other point a' in the plane. To see this, take any directed angle $\measuredangle A'$ with vertex at a' and let $\measuredangle A$ be the translate of $\measuredangle A'$ with vertex a_0. By our construction the directed angle measure of $\measuredangle A'$ must equal the directed angle measure of $\measuredangle A$. Hence establishing the values of the direct angle measure at a_0 also determines the values at a'. Since there are only two choices for the directed angle measure at a_0, this proves that there can be *at most two* translationally invariant directed angle measures in the plane.

To show the existence of the two translationally invariant directed angle measures, pick a point a_0 in the plane. We know from Proposition 6.16 that there exist precisely two directed angle measures at a_0. Pick one of these directed angle measures at a_0, denoting it by m_0.

Let a' be any point in the plane and $\measuredangle A'$ any directed angle with vertex a'. Then, letting $\measuredangle A_0$ denote the translate of $\measuredangle A'$ to the vertex a_0 as shown in Figure 11.17, define the directed angle measure of $\measuredangle A'$ to be

$$m\,\measuredangle A' = m_0\,\measuredangle A_0. \tag{11.16}$$

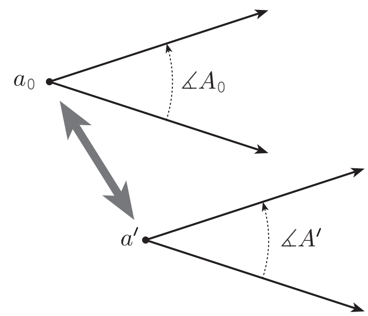


Figure 11.17. $m\measuredangle A'$ is defined to be $m_0\measuredangle A_0$.

To finish the proof requires showing that (11.16) defines m to be a translation invariant directed angle measure. Verifying that at each point a' we obtain a directed angle measure is straightforward. However, as we will see, the fact that m is translationally invariant is a deeper result.

We first verify that m, as defined by (11.16), is a directed angle measure at a'. We do so by checking the four conditions of Definition 6.15:

(1) Suppose $\measuredangle A'$ is trivial, i.e., it is one ray starting at a'. Then $\measuredangle A_0$, the translate of $\measuredangle A'$ to vertex a_0, is also trivial by Proposition 11.11. Hence

$$m\measuredangle A' = m_0\measuredangle A_0 = 0 \mod 360.$$

(2) Suppose $\measuredangle A'$ is a straight angle, i.e., it is composed of two opposite rays starting at a'. Then $\measuredangle A_0$, the translate of $\measuredangle A'$ to vertex a_0, is also a straight angle by Proposition 11.11. Hence

$$m\measuredangle A' = m_0\measuredangle A_0 = 180^\circ \mod 360.$$

(3) Suppose $\measuredangle A'$ is a directed angle at a' that is neither trivial nor straight. If θ' is that value of $m\measuredangle A'$ such that $-180^\circ < \theta' < 180^\circ$, then we wish to show $|\theta'| = m\angle A'$, i.e., that $|\theta'|$ is the measure of the (ordinary) angle $\angle A'$. By Proposition 11.12, $\measuredangle A'$ is the translate of a directed angle $\measuredangle A_0$ with vertex at a_0 that is neither trivial nor straight, and the (ordinary) angles $\angle A'$ and $\angle A_0$ are congruent. Moreover, by definition of m, $m\measuredangle A' = m_0\measuredangle A_0$. Hence θ' is that value of $m_0\measuredangle A_0$ such that $-180^\circ < \theta' < 180^\circ$. Thus

$$\begin{aligned} |\theta'| &= m\angle A_0 \text{ from (3) of Definition 6.15, } m_0 \text{ a directed angle measure,} \\ &= m\angle A' \text{ since } \angle A_0 \cong \angle A'. \text{ This is the desired result.} \end{aligned}$$

(4) Suppose b', c', d' are any three points other than a', and let $\overrightarrow{a_0b}$, $\overrightarrow{a_0c}$, $\overrightarrow{a_0d}$ be the translates to a_0 of the rays $\overrightarrow{a'b'}$, $\overrightarrow{a'c'}$, $\overrightarrow{a'd'}$, respectively. Then

$$\begin{aligned} m\measuredangle b'a'd' &= m_0\measuredangle ba_0d \text{ by definition of } m \text{ at } a', \\ &= m_0\measuredangle ba_0c + m_0\measuredangle ca_0d \text{ by condition (4) for } m_0 \text{ at } a_0, \\ &= m\measuredangle b'a'c' + m\measuredangle c'a'd' \text{ by definition of } m \text{ at } a'. \end{aligned}$$

Finally, we must verify that m is translationally invariant. So consider a directed angle $\measuredangle A'$ at a point a' and its translate $\measuredangle A''$ to a point a'' as shown in Figure 11.18. Thus $\measuredangle A' \equiv_\tau \measuredangle A''$. We must show

$$m\measuredangle A' = m\measuredangle A''.$$

To do so, recall that $m\measuredangle A'$ is defined by

$$m\,\measuredangle A' = m_0\,\measuredangle A_0,$$

where $\measuredangle A_0$ is the translate of $\measuredangle A'$ to the point a_0 as shown to the right. Thus $\measuredangle A' \equiv_\tau \measuredangle A_0$. However, since directed angle translation is transitive by Proposition 11.10, $\measuredangle A'' \equiv_\tau \measuredangle A_0$. Thus

$$m\measuredangle A'' = m_0\measuredangle A_0 = m\measuredangle A',$$

proving $m\measuredangle A' = m\measuredangle A''$, as desired. This finishes the proof that m, as constructed above, is a translationally invariant directed angle measure. □

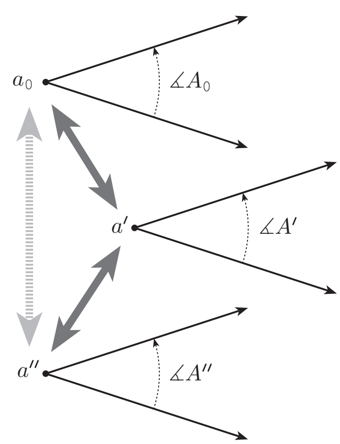


Figure 11.18. Proving that m is translationally invariant.

Exercises I.11

Exercise 11.1.

In this exercise you complete the proof Theorem 11.6, the transitivity of ray translation, by verifying the various degenerate cases. Assume $\overrightarrow{a_1b_1}$ translates to $\overrightarrow{a_2b_2}$, $\overrightarrow{a_2b_2}$ translates to $\overrightarrow{a_3b_3}$, and (for convenience) b_1, b_2, and b_3 are chosen so that $a_1b_1 = a_2b_2 = a_3b_3$. You need to prove $\overrightarrow{a_1b_1}$ translates to $\overrightarrow{a_3b_3}$. As in the theorem define lines $\ell_1 = \overleftrightarrow{a_1b_1}$, $\ell_2 = \overleftrightarrow{a_2b_2}$, and $\ell_3 = \overleftrightarrow{a_3b_3}$. The proof for the generic case, when the three lines are all distinct, was given in the text. Now handle the other cases:

(a) Assume ℓ_1 and ℓ_2 are distinct but a_3 lies on ℓ_2. Then $\ell_2 = \ell_3$.
Proof sketch: If $a_2 = a_3$, then $b_2 = b_3$ and the desired result follows. If $a_2 \neq a_3$, then either $\overrightarrow{a_2b_2}$ contains $\overrightarrow{a_3b_3}$ or $\overrightarrow{a_3b_3}$ contains $\overrightarrow{a_2b_2}$. Assume the second case; the first case is similar. Then a_2 is between

a_3 and b_2, and the three points a_2, b_2, b_3 all lie on the same side of $\overleftrightarrow{a_1a_3}$ (why?). You must show b_1 lies on this same side of $\overleftrightarrow{a_1a_3}$. Suppose not. Then show the line segment $\overline{a_1a_3}$ must intersect the line $m = \overleftrightarrow{b_1b_2}$, showing a_1 and a_3 to be on opposite sides of m. But a_2 and a_3 must lie on the same side of m (why?), which means a_1 and a_2 must lie on opposite sides of m. But this is not possible (why?). Use this to finish the proof.

(b) Assume ℓ_1 and ℓ_2 are distinct but a_3 lies on ℓ_1. Then $\ell_1 = \ell_3$.
Hints: There are two non-trivial cases to consider: b_1, a_3 on the same side or on opposite sides of a_1. Consider the same side case: you need to show $\overrightarrow{a_3b_3} \subseteq \overrightarrow{a_1b_1}$, which will be true if a_3 is between a_1 and b_3. This is equivalent to showing a_1 and b_3 are on opposite sides of $m_3 = \overleftrightarrow{a_2a_3}$. To prove this, first show b_1, b_2, a_3 all lie on the same side of $m_1 = \overleftrightarrow{a_1a_2}$. However, $\square a_1a_2b_2a_3$ is a convex quadrilateral (Definition 5.10) and so the diagonal line segments intersect by Proposition 5.13. Use this to show a_1 and b_3 are on opposite sides of m_3, as desired. Then handle the case where b_1 and a_3 are on opposite sides of a_1 $\ldots$.

(c) Assume $\ell_1 = \ell_2$ but a_3 is not on this line. *Hint:* A variant of (a).

(d) Assume $\ell_1 = \ell_2$ and a_3 is on this line. Then $\ell_1 = \ell_2 = \ell_3$.
Hint: Use a coordinate system on the line as done in Exercise 4.4.

Exercise 11.2.
Proposition 11.12 states that if $\measuredangle abc$ and $\measuredangle a'b'c'$ are translates of each other, then $\angle abc \cong \angle a'b'c'$. The proof given in the text covered the generic case, i.e., when $\overleftrightarrow{ba}$ and $\overleftrightarrow{b'a'}$ are distinct parallel lines and $\overleftrightarrow{bc}$ and $\overleftrightarrow{b'c'}$ are distinct parallel lines. Complete the proof by handling the degenerate cases when $\overleftrightarrow{ba} = \overleftrightarrow{b'a'}$ and/or $\overleftrightarrow{bc} = \overleftrightarrow{b'c'}$.

Exercise 11.3.
Prove Proposition 11.11: *A translate of a trivial directed angle is another trivial directed angle. A translate of a straight directed angle is another straight directed angle.*

§I.12 Similarity

Recall the definition for *similar triangles*:

Definition 12.1. ***Triangle Similarity.***

(a) A **similarity** between two triangles, written $\triangle abc \sim \triangle ABC$, means that the ratios of corresponding side lengths are equal and corresponding angles are congruent, i.e.,

$$\frac{AB}{ab} = \frac{BC}{bc} = \frac{AC}{ac} \quad \text{and} \quad \angle a \cong \angle A,\ \angle b \cong \angle B,\ \angle c \cong \angle C.$$

(b) Two triangles are **similar** if there exists a similarity between them.

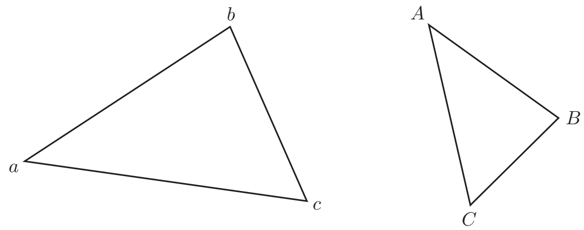


Figure 12.2. $\triangle abc$ and $\triangle ABC$ are similar via $abc \leftrightarrow ABC$.

It is a well-known theorem of geometry (*Angle-Angle-Angle*, or AAA) that two triangles are similar if and only if the three angles of the first triangle are congruent to the three angles of the second, as shown in Figure 12.3. In other words, the angle conditions for similarity are enough to imply the truth of the ratio conditions for the sides.

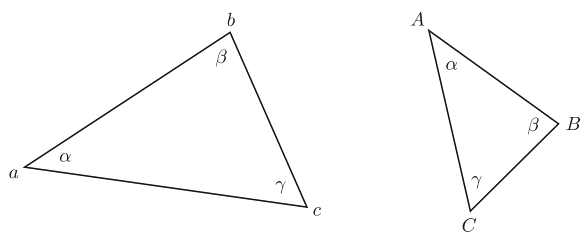


Figure 12.3. Two similar triangles showing the equal angles.

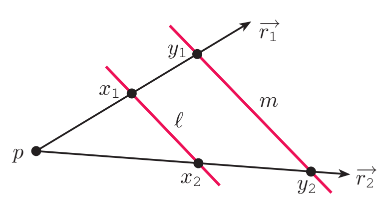
However, deriving this theorem from our previously established results is more difficult than initially expected: the proof requires a non-trivial passage from the rational numbers to the real numbers. We isolate this difficult passage in the *Similarity Theorem* and then use this result to prove AAA. Although AAA is the result of most immediate interest, the *Similarity Theorem* will play a major role in Chapter IV, *Similarities in the Plane*.

Theorem 12.4. ***The Similarity Theorem.***
Let ℓ, m be lines not containing the point p. Suppose $\overrightarrow{r_1}$, $\overrightarrow{r_2}$ are distinct rays from p such that $\overrightarrow{r_1}$ intersects ℓ and m at x_1, y_1, respectively, and $\overrightarrow{r_2}$ intersects ℓ and m at x_2, y_2, respectively. Then the ratios

$$\frac{py_1}{px_1} \text{ and } \frac{py_2}{px_2}$$

are equal if and only if the lines ℓ and m are parallel, in which case both of the ratios equal

$$\frac{y_1y_2}{x_1x_2}.$$



Proof. To save time, you may decide to skip the proof—it is not trivial. Hence we place it as optional reading at the end of the section. □

Here is the most common method for verifying triangle similarity.

Theorem 12.5. ***AAA (Angle-Angle-Angle) Similarity.***
Two triangles are similar if and only if the three angles of the first triangle are congruent to the three angles of the second.

Proof. Suppose $\triangle ABC$ and $\triangle abc$ have congruent corresponding angles, i.e.,

$$\begin{aligned} m\angle BAC &= m\angle bac = \alpha, \\ m\angle CBA &= m\angle cba = \beta, \\ m\angle ACB &= m\angle acb = \gamma. \end{aligned}$$

Using Proposition 4.10, *Segment Construction*, let b' be that point on the ray $\overrightarrow{ab}$ such that $\overline{ab'} \cong \overline{AB}$, and let c' be that point on the ray $\overrightarrow{ac}$ such that $\overline{ac'} \cong \overline{AC}$ (see Figure 12.6).

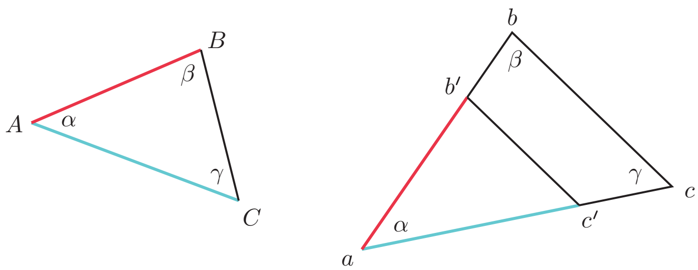


Figure 12.6. $\triangle ABC$ and $\triangle abc$ have congruent corresponding angles.

Then, since $m\angle b'ac' = \alpha = m\angle BAC$, *SAS* proves $\triangle ABC$ and $\triangle ab'c'$ are congruent. Thus $m\angle c'b'a = m\angle CBA = \beta$. This shows that the lines $\overleftrightarrow{b'c'}$ and $\overleftrightarrow{bc}$ are parallel since they have a congruent pair of corresponding angles (Exercise 10.1). Hence, from the *Similarity Theorem* we conclude

$$\frac{ab}{ab'} = \frac{ac}{ac'} = \frac{bc}{b'c'}.$$

Since $\triangle ABC$ and $\triangle ab'c'$ are congruent, our equalities can be rewritten as

$$\frac{ab}{AB} = \frac{ac}{AC} = \frac{bc}{BC},$$

verifying $\triangle ABC$ and $\triangle abc$ are similar. □

Actually, we have only to check *two* pairs of angles to verify similarity:

Corollary 12.7. ***AA (Angle-Angle) Similarity.***
Two triangles are similar if and only if two of the three angles of the first triangle are congruent to two of the three angles of the second.

Proof. *AAA* (Theorem 12.5) combined with the fact that the measures of the angles of any triangle add to 180° (Theorem 10.3). □

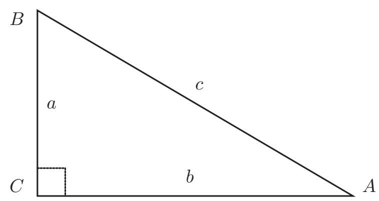
We can now state and prove what is perhaps the most famous theorem in all of Euclidean geometry: the *Pythagorean Theorem*. The proof we give is based on similarity of triangles via the use of AAA. We will give other proofs, based on area, in Chapter IX.

Recall that a **right triangle** is any triangle with a right angle. The **hypotenuse** is the side of the triangle opposite the right angle.

Theorem 12.8. ***The Pythagorean Theorem.***

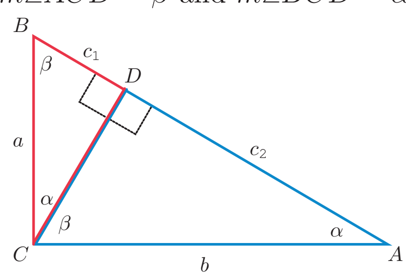
In any right triangle, the square of the length c of the hypotenuse is equal to the sum of the squares of the lengths a and b of the other two sides. Thus

$$c^2 = a^2 + b^2.$$

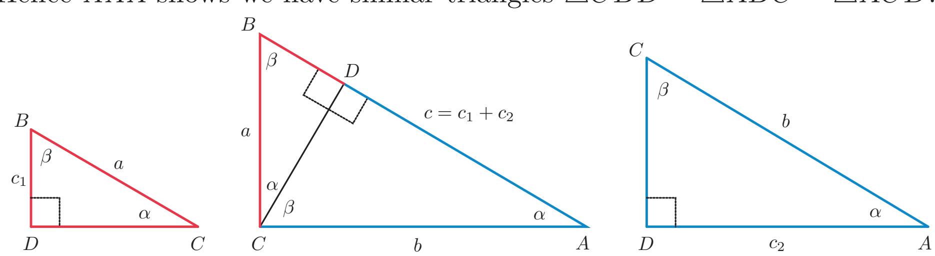


Proof. Let D be the point on the line $\overleftrightarrow{AB}$ such that $\overline{CD}$ is perpendicular to $\overleftrightarrow{AB}$. Since $m\angle ADC = 90°$, $m\angle ACD$ is less than $90°$ since the measures of all the angles in $\triangle ACD$ sum to $180°$ by Theorem 10.3. Hence D is in the interior of angle $\angle ACB$ by Exercise 6.4, therefore falling on the line segment $\overline{AB}$ between A and B.

Let $\alpha = m\angle A$ and $\beta = m\angle B$. Since the sum of the measures of the angles of the right triangle $\triangle ABC$ equals $180°$, we obtain $\alpha + \beta = 90°$. However, this equality implies $m\angle ACD = \beta$ and $m\angle BCD = \alpha$:



Hence AAA shows we have similar triangles $\triangle CBD \sim \triangle ABC \sim \triangle ACD$:



The first similarity implies

$$\frac{c_1}{a} = \frac{a}{c}, \text{ which gives } c_1 = \frac{a^2}{c}.$$

The second similarity implies

$$\frac{c_2}{b} = \frac{b}{c}, \text{ which gives } c_2 = \frac{b^2}{c}.$$

Hence $c = c_1 + c_2 = \dfrac{a^2}{c} + \dfrac{b^2}{c}$, which yields $c^2 = a^2 + b^2$. $\square$

Diagonals of a Parallelogram. There is a useful generalization of the Pythagorean Theorem to parallelograms. Its proof is left to Exercise 12.5.

Proposition 12.9.

Suppose $\square ABCD$ is a parallelogram with side lengths a and b and diagonal lengths d_1 and d_2. Then

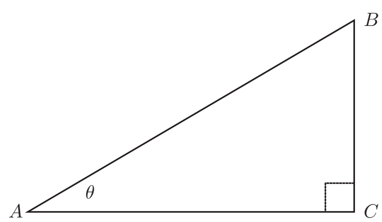
$$2(a^2 + b^2) = d_1^2 + d_2^2.$$

The Trigonometric Functions. Our results on triangle similarity are needed to justify the usual geometric definitions of the trigonometric functions. We illustrate this procedure with the *sine* function.

Given a right triangle $\triangle ABC$ with right angle at $\angle C$, the **sine** of angle $\angle A$ is defined to be BC/AB, i.e., the length of the "opposite" side divided by the length of the hypotenuse.

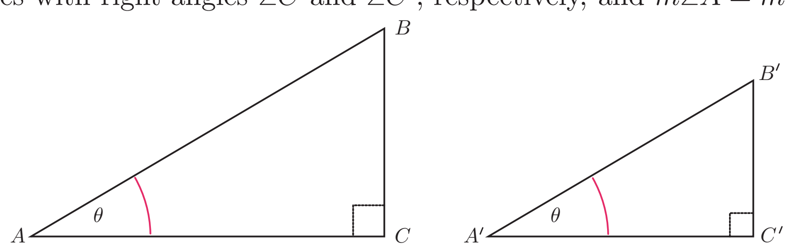
A major property of the sine of an angle $\angle A$ is that it should depend *only on the angular measure of* $\angle A$, i.e., only on the value

$$\theta = m\angle A.$$



This property justifies the customary notation of $\sin\theta$ for the sine of any angle whose measure is θ.

To prove our claimed property, suppose $\triangle ABC$ and $\triangle A'B'C'$ are two right triangles with right angles $\angle C$ and $\angle C'$, respectively, and $m\angle A = m\angle A'$:



We wish to show that the sine of $\angle A$ equals the sine of $\angle A'$, i.e.,

$$BC/AB = B'C'/A'B'.$$

However, $m\angle A = m\angle A'$ by assumption, and $m\angle C = m\angle C'$ since all right angles have measure 90°. Thus AA (Corollary 12.7) shows that the two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar. Hence

$$BC/B'C' = AB/A'B'.$$

This proves $BC/AB = B'C'/A'B'$, so the sine of $\angle A$ does equal the sine of $\angle A'$, as desired. Hence the sine of an angle depends only on the measure of the angle.

Justifications for the usual geometric definitions of the other trigonometric functions are identical to the one just given for the sine — see Exercises 12.8 and 12.9.

Proof of the *Similarity Theorem* (*Optional*). The proof of the *Similarity Theorem*, based on our previously established axioms and theorems, is rarely given in elementary geometry courses as it involves an intricate, technically complicated argument proceeding from integer ratios to rational ratios and finally to real number ratios. To save time, you may simply decide to accept the truth of the *Similarity Theorem* and thus skip the following proof. However, the proof utilizes several beautiful arguments and illustrates the central importance of the real number system in our development of geometry.

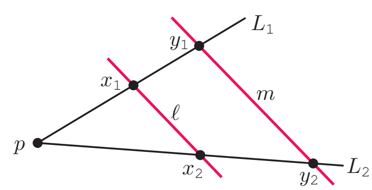
Our proof of the *Similarity Theorem* begins with establishing a slightly reduced version of the result: the *Basic Similarity Principle*. The full theorem can then be deduced from this principle.

Theorem 12.10. ***The Basic Similarity Principle.***

Let ℓ and m be parallel lines with ℓ not containing the point p. Suppose L_1 and L_2 are two lines containing p such that L_1 intersects ℓ and m at x_1, y_1, respectively, and L_2 intersects ℓ and m at x_2, y_2, respectively. Then the ratios

$$\frac{py_1}{px_1} \text{ and } \frac{py_2}{px_2}$$

are equal. In particular, this ratio δ depends only on the choices of p, ℓ, and m, not on the choice of the line L through p.



Proof. If the parallel lines ℓ and m are equal, then the claims in the theorem are trivially true (the ratios are all 1). We thus assume that ℓ and m are distinct parallel lines.

There are various cases to consider, depending on the placement of the point p relative to the lines ℓ and m. The most trivial case is when p lies on the line m for then $y_1 = y_2 = p$ and the ratios in the theorem are equal (since they are both zero), as desired. Now consider the case when the point p is *not* between the two lines m and ℓ (the case where p is between the two lines is treated similarly and is left to the reader). By interchanging the labels of m and ℓ if necessary, we may assume that the point p and the line m are on *opposite sides* of the line ℓ, as shown in the diagram with Theorem 12.10.

The difficulty in the proof arises from the possible irrationality of the ratio δ. The technique of the following proof is to verify the theorem first for $\delta = 2$, then for δ any integer ≥ 1, then for δ any rational number ≥ 1, then finally for δ any irrational number ≥ 1.

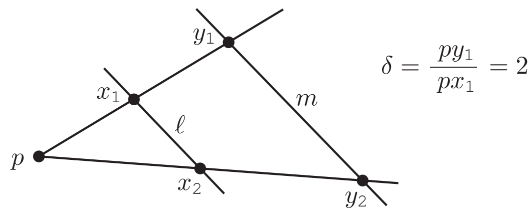


Figure 12.11. Setup for the *Basic Similarity Principle* when $\delta = 2$.

So suppose p, ℓ, m, x_1, y_1, x_2, and y_2 are as in the hypothesis of the theorem. We first assume the ratio δ of py_1 to px_1 equals 2 as shown in Figure 12.11. This is equivalent to x_1 being the midpoint of $\overline{py_1}$. We need to show that the ratio of py_2 to px_2 also equals 2. This means showing that x_2 is between p and y_2 and that $px_2 = x_2y_2$.

The point x_2 lies between p and y_2 since ℓ and m are parallel lines.[14] Hence we need only show that $px_2 = x_2y_2$. To do so, let y_0 be that point on m between y_1 and y_2 such that $\overline{x_1y_0}$ is parallel to $\overline{py_2}$. We will then show two facts (see Figure 12.12):

(1) triangles $\triangle px_1x_2$ and $\triangle x_1y_1y_0$ are congruent, so $px_2 = x_1y_0$,
(2) the quadrilateral $\square x_1y_0y_2x_2$ is a parallelogram; hence $x_1y_0 = x_2y_2$ from Proposition 10.10b.

These two facts give $px_2 = x_2y_2$, as needed for the case $\delta = 2$.

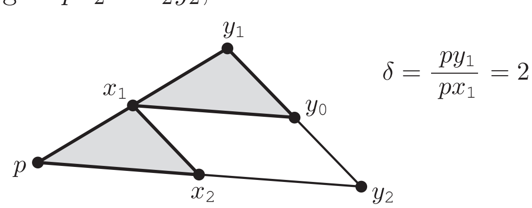


Figure 12.12. Two congruent triangles and a parallelogram.

[14]Since p lies on the side of ℓ *not* containing m, p and y_1 are on opposite sides of ℓ, and hence x_1 must lie between p and y_1. However, since m is parallel to ℓ, the segment $\overline{y_1y_2}$ remains on the same side of ℓ. Therefore p and y_2 will be on opposite sides of ℓ, proving that x_2 is between p and y_2, as desired.

To prove (1), that $\triangle px_1x_2 \cong \triangle x_1y_1y_0$, we employ *ASA* (Theorem 7.8):

Angle: $\overline{px_2}$ is parallel to $\overline{x_1y_0}$, so $\angle x_1px_2 \cong \angle y_1x_1y_0$ by Exercise 10.1.

Side: $px_1 = x_1y_1$ by the assumption that $\delta = 2$.

Angle: $\overline{x_1x_2}$ is parallel to $\overline{y_1y_0}$, so $\angle px_1x_2 \cong \angle x_1y_1y_0$ by Exercise 10.1.

To prove (2), that $\Box x_1y_0y_2x_2$ is a parallelogram, we have only to observe that each pair of opposite sides of the quadrilateral are comprised of parallel lines. This proves the *Basic Similarity Principle* when the ratio δ equals 2.

Now suppose the ratio δ of py_1 to px_1 equals n, a positive integer. Then select those $n-1$ points which divide the line segment $\overline{py_1}$ into n equal pieces (such points exist as a consequence of the coordinate system on $L_1 = \overleftrightarrow{py_1}$). The first of these segments will be $\overline{px_1}$ (see Figure 12.13).

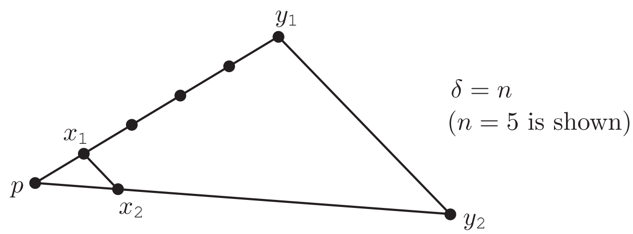


Figure 12.13. Analyzing the case $\delta = n$, an integer.

From each of the points just selected in $\overline{py_1}$ construct two lines: one parallel to $\overline{py_2}$ and the other parallel to $\overline{y_1y_2}$. In this way the line segment $\overline{py_2}$ is itself divided up into n pieces, the first of which is $\overline{px_2}$ (see Figure 12.14). We must show that all these new line segments in $\overline{py_2}$ are the same length. When this is done, we will have that $py_2 = n\,px_2$, the desired relationship between these two distances.

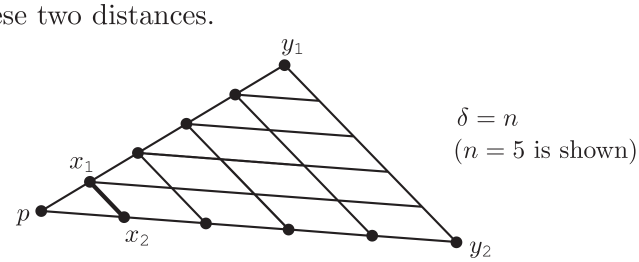


Figure 12.14. Subdivisions for the case $\delta = n$, an integer.

Along the line segment $\overline{py_1}$ we now have $n-1$ small overlapping triangles like those considered in the previous case where $\delta = 2$. One such triangle is shown in yellow in Figure 12.15.

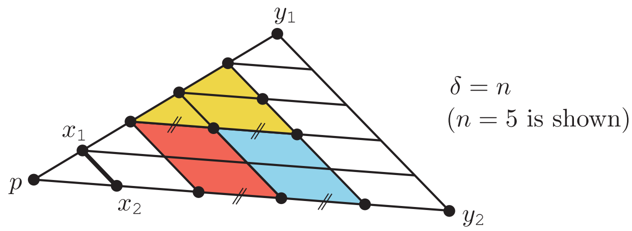


Figure 12.15. Any two adjacent line segments in $\overline{py_2}$ are of equal length.

From the previous case of $\delta = 2$ we see that the two line segments comprising the base of the small yellow triangle in Figure 12.15 (parallel to $\overline{py_2}$) are of equal length. Moreover, each of these line segments is the top side of a parallelogram (shaded in blue and red) whose opposite side is one of the line segments of $\overline{py_2}$ constructed above. Hence the two line segments in $\overline{py_2}$ so picked out (the bottom sides of the blue and red parallelograms) must be of equal length.

Since this relationship holds for each of the $n-1$ *overlapping* triangles, *all* of the line segments constructed in $\overline{py_2}$ are of equal length, as desired. Thus the theorem is verified for the case $\delta = n$.

Now suppose the ratio δ of py_1 to px_1 equals n/m, a rational number ≥ 1. Select those $n-1$ points which divide the line segment $\overline{py_1}$ into n equal pieces. The first m of these segments will comprise $\overline{px_1}$ (see Figure 12.16).

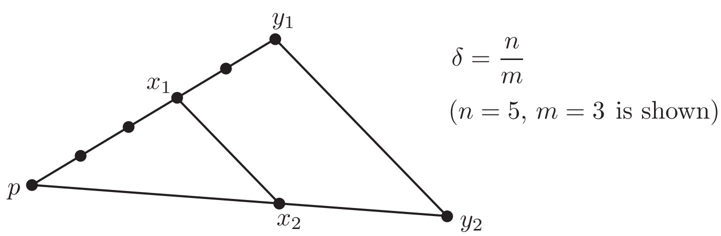


Figure 12.16. Analyzing the case $\delta = n/m$, a rational number.

From each of the points just selected in $\overline{py_1}$ construct two lines: one parallel to $\overline{py_2}$ and the other parallel to $\overline{y_1y_2}$. In this way the line segment $\overline{py_2}$ is itself divided up into n pieces, the first m segments of which comprise $\overline{px_2}$ (see Figure 12.17).

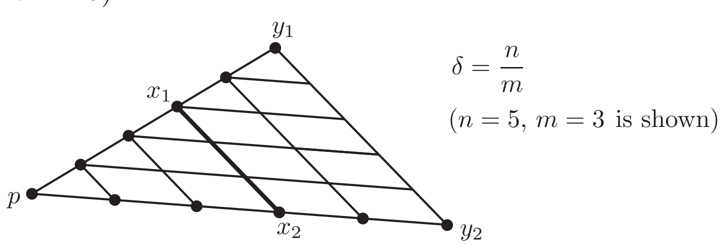


Figure 12.17. Subdivisions for the case $\delta = n/m$, a rational number.

However, from the case $\delta = n$ just considered, we see immediately that all the line segments in $\overline{py_2}$ are of equal length. Hence

$$py_2 = n\,(px_2/m) = \delta\ px_2,$$

proving the theorem for the case $\delta = n/m$.

Finally suppose the ratio $\delta = py_1/px_1$ equals a real number ≥ 1. We will use the following important fact about the relationship between the rational and real numbers. Suppose χ_1 and χ_2 are two real numbers such that

- if r is a rational number such that $r < \chi_1$, then $r < \chi_2$, and
- if r is a rational number such that $r > \chi_1$, then $r > \chi_2$.

From Exercise 1.2 we can conclude $\chi_1 = \chi_2$, i.e., χ_1 and χ_2 are the same real number. For the case at hand we apply this result to the two numbers

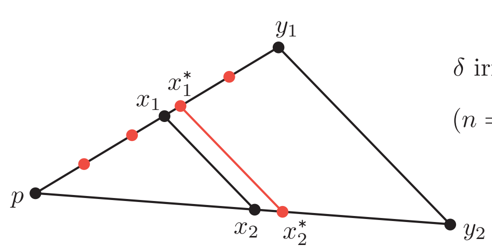
$$\chi_1 = \frac{py_1}{px_1} \quad \text{and} \quad \chi_2 = \frac{py_2}{px_2}.$$

So suppose $r = n/m$ is a rational number such that

$$\chi_1 = \frac{py_1}{px_1} > r.$$

Then divide $\overline{py_1}$ into n equal pieces, and take the first m of them, starting at p. The new line segment so formed will have an endpoint x_1^*, and this point x_1^* will be (slightly) further away from p than is x_1 (see Figure 12.18). This comes from the following computation:

$$px_1^* = m\,(\frac{py_1}{n}) = \frac{1}{r}\,py_1 > px_1.$$



δ irrational, $\quad r = \dfrac{n}{m}$

($n = 5$, $m = 3$ is shown)

Figure 12.18. The general case, when δ is a real number ≥ 1.

Let x_2^* be that point on $\overline{py_2}$ such that $\overline{x_1^* x_2^*}$ is parallel to $\overline{x_1 x_2}$ (again, see Figure 12.18). According to the previous case, where δ was a rational number, we have that

$$\frac{py_2}{px_2^*} = \frac{py_1}{px_1^*} = \frac{n}{m} = r.$$

But since x_1 is between p and x_1^* and $\overline{x_1^* x_2^*}$ is parallel to $\overline{x_1 x_2}$, x_2 is between p and x_2^*. In particular, px_2 is less than px_2^*. Thus

$$\chi_2 = \frac{py_2}{px_2} > \frac{py_2}{px_2^*} = r.$$

Hence we have shown that if r is any rational such that $r < \chi_1$, then $r < \chi_2$. Demonstrating the corresponding result for $r > \chi_1$ is done in a similar fashion. This shows that $\chi_1 = \chi_2$, or

$$\frac{py_1}{px_1} = \frac{py_2}{px_2},$$

as was desired. Hence we have finally verified the *Basic Similarity Principle* in its full generality. (We told you it was complicated....) $\square$

The *Basic Similarity Principle* will now give the *Similarity Theorem*.

Proof of Theorem 12.4, the Similarity Theorem. Notice that $\overrightarrow{r_1}$ and $\overrightarrow{r_2}$ cannot be opposite rays. For if they were, then p, x_1, and x_2 would be collinear, contradicting the assumption that ℓ does not contain p. In particular, $x_1 \neq x_2$, and similarly $y_1 \neq y_2$. However, we allow the case $\ell = m$, which is equivalent to $x_1 = y_1$ and $x_2 = y_2$.

First consider the *reverse* direction of the theorem, i.e., assume the lines $\ell = \overleftrightarrow{x_1x_2}$ and $m = \overleftrightarrow{y_1y_2}$ are parallel. According to the *Basic Similarity Principle* the following two ratios are equal:

$$\frac{py_1}{px_1} \text{ and } \frac{py_2}{px_2}.$$

Hence we have only to show that the above two ratios are also equal to y_1y_2/x_1x_2 in order to finish the proof of this direction of the theorem.

If ℓ and m are the same line, then all the ratios are equal to 1 and we are done. So we now assume ℓ and m are distinct parallel lines. In particular, interchanging labels if necessary, we can assume that x_2 is between p and y_2, so that $px_2 < py_2$. We now show y_1y_2/x_1x_2 is equal to the two (equal) ratios py_1/px_1 and py_2/px_2.

To do this, we perform a little trick: we apply the *Basic Similarity Principle* about the point y_2. This is done by introducing the point y_0 on the line m between y_1 and y_2 such that the line $L = \overleftrightarrow{x_2y_0}$ is parallel to the line $M = \overleftrightarrow{py_1}$ (see Figure 12.19).[15]

[15] Such a point y_0 exists for the following reasons. By Proposition 9.3 there exists a line L through x_2 parallel to $\overleftrightarrow{py_1}$. However, L cannot be parallel to $m = \overleftrightarrow{y_1y_2}$ since m cannot be parallel to $\overleftrightarrow{py_1}$. Hence L must intersect m in a unique point, which we label y_0. Since by assumption x_2 is between p and y_2, then p and y_2 are on opposite sides of $L = \overleftrightarrow{x_2y_0}$. Moreover, $M = \overleftrightarrow{py_1}$ is parallel to L, and thus the points p and y_1 are on the same side of L. Hence y_1 and y_2 are on opposite sides of L, implying that the intersection point y_0 of L and $m = \overleftrightarrow{y_1y_2}$ must lie between y_1 and y_2. This establishes the existence and desired properties for y_0. Whew!

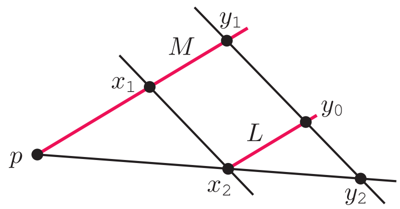


Figure 12.19. Preparing to use the *Basic Similarity Principle* about y_2.

The *Basic Similarity Principle* applied to the point y_2, the intersecting lines $L_1 = \overleftrightarrow{y_2p}$, $L_2 = m = \overleftrightarrow{y_2y_1}$, and the parallel lines $M = \overleftrightarrow{x_2y_0}$, $L = \overleftrightarrow{py_1}$ shows

$$\frac{y_2y_0}{y_1y_2} = \frac{y_2x_2}{py_2}. \tag{12.20}$$

We will now use this result to prove the desired equality

$$\frac{x_1x_2}{y_1y_2} = \frac{px_2}{py_2}. \tag{12.21}$$

First note that the quadrilateral $\square x_1y_1y_0x_2$ is constructed from two pairs of parallel lines and is therefore a parallelogram. Hence the lengths x_1x_2 and y_1y_0 are equal by Proposition 10.10b. A little algebra now gives (12.21):

$$\begin{aligned}
\frac{x_1x_2}{y_1y_2} &= \frac{y_1y_0}{y_1y_2} \quad \text{since } \square x_1y_1y_0x_2 \text{ is a parallelogram,} \\
&= \frac{y_1y_2 - y_2y_0}{y_1y_2} \quad \text{since } y_1y_0 + y_0y_2 = y_1y_2 \ (y_0 \text{ is between } y_1 \text{ and } y_2), \\
&= 1 - \frac{y_2y_0}{y_1y_2} \\
&= 1 - \frac{y_2x_2}{py_2} \quad \text{by (12.20)}, \\
&= \frac{py_2 - y_2x_2}{py_2} \\
&= \frac{px_2}{py_2} \quad \text{since } px_2 + x_2y_2 = py_2.
\end{aligned}$$

This establishes the desired (12.21).

We now must establish the *forward* direction of the *Similarity Theorem*, i.e., that the equality of the two ratios

$$\frac{py_1}{px_1} = \frac{py_2}{px_2} \tag{12.22}$$

guarantees that the two lines $\ell = \overleftrightarrow{x_1x_2}$ and $m = \overleftrightarrow{y_1y_2}$ are indeed parallel. There certainly is *some* point y_0 on $\overleftrightarrow{px_2}$ such that the line $m_0 = \overleftrightarrow{y_1y_0}$ is parallel to $\ell = \overleftrightarrow{x_1x_2}$. We wish to show $y_0 = y_2$ (see Figure 12.23).

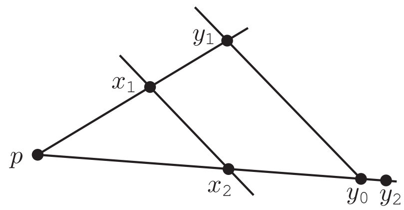


Figure 12.23. Prove $\ell = \overleftrightarrow{x_1x_2}$ and $m = \overleftrightarrow{y_1y_2}$ parallel by showing $y_0 = y_2$.

From the *Basic Similarity Principle* we have

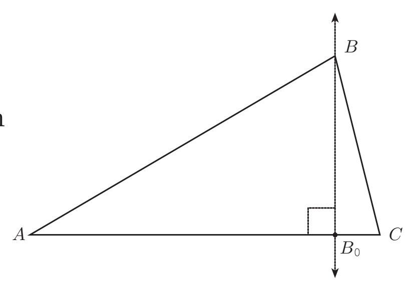
$$\frac{py_1}{px_1} = \frac{py_0}{px_2}.$$

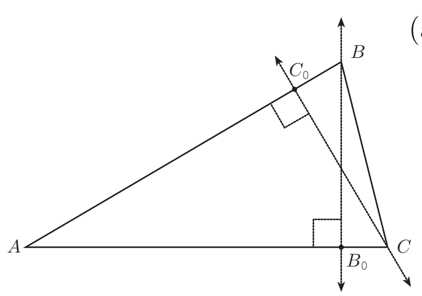
When combined with (12.22), this shows $py_0 = py_2$. Hence, since y_0 and y_2 must be on the same side of p, *Segment Construction* (Proposition 4.10) gives $y_0 = y_2$. This completes the proof of the *Similarity Theorem.* □

Exercises I.12

Exercise 12.1.

Given a triangle $\triangle ABC$, the line from vertex B which intersects side $\overleftrightarrow{AC}$ perpendicularly is called the **altitude** from vertex B. Let B_0 be the point of intersection of the altitude from B to the line $\overleftrightarrow{AC}$. The length of the line segment $\overline{BB_0}$ is the **height** of the triangle from vertex B, and the length AC is the **base** opposite B.





(a) Use similar triangles to prove that for $\triangle ABC$ the base times the height is the same no matter which vertex you choose. *Hint:* Let C_0 be the point of intersection of the altitude from C to the line $\overleftrightarrow{AB}$. You need to show $AC \cdot BB_0 = AB \cdot CC_0$. To do so, consider two triangles with A as a common vertex.

(There is a case to consider separately: when $m\angle BAC = 90°$.)

(b) We have not yet developed the concept of area — this will be done in Chapter IX. However, assuming the high school formulas for area, give an area-based proof for the result of part (a).

Exercise 12.2. ***SSS Similarity Theorem.***

Prove the *SSS Similarity Theorem:*

> *If the ratios of corresponding side lengths in triangles $\triangle ABC$ and $\triangle abc$ are all equal, then the triangles are similar.*

Hint: According to Definition 12.1 you need to show that the corresponding angles in the two triangles are congruent, i.e., $\angle A \cong \angle a$, $\angle B \cong \angle b$, and $\angle C \cong \angle c$. On the ray $\overrightarrow{ab}$ let b' be that point such that $ab' = AB$, and on the ray $\overrightarrow{ac}$ let c' be that point such that $ac' = AC$. Then compare triangles $\triangle abc$, $\triangle ab'c'$, and $\triangle ABC$.

Exercise 12.3. ***SAS Similarity Theorem.***

Prove the *SAS Similarity Theorem:*

> *Given a correspondence between two triangles, suppose two pair of corresponding side lengths are proportional and the included angles are congruent. Then the triangles are similar.*

Hint: Suppose the two triangles are $\triangle ABC$ and $\triangle abc$, with $\angle A \cong \angle a$ and $\dfrac{AB}{ab} = \dfrac{AC}{ac}$. Start by choosing b' on $\overrightarrow{ab}$ such that $ab' = AB$, and let c' be that point on $\overrightarrow{ac}$ such that $\overline{b'c'}$ is parallel to $\overline{bc}$.

Exercise 12.4.

Prove the converse of the Pythagorean Theorem:

> *Suppose a triangle Δ has side lengths a, b, c where $c^2 = a^2 + b^2$. Then Δ has a right angle opposite the side of length c.*

Exercise 12.5.

In this exercise you will prove Proposition 12.9:

> *Suppose $\square ABCD$ is a parallelogram with side lengths a and b and diagonal lengths d_1 and d_2. Then $2(a^2 + b^2) = d_1^2 + d_2^2$.*

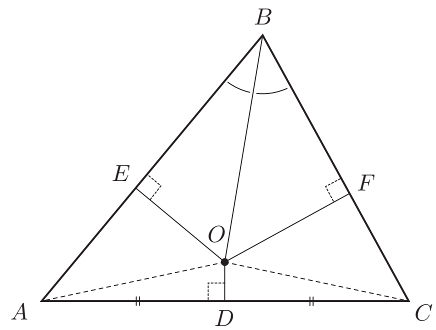
Hints: First verify the proposition if $\square ABCD$ is a rectangle. If not a rectangle, then label the parallelogram so that $a \le b$, $a = AD = BC$, $b = AB = CD$, $d_1 = AC$, and $d_2 = BD$. Assume $m\angle BAD < 90°$. (Show how the case $m\angle BAD > 90°$ can be reduced to $m\angle BAD < 90°$ via another relabeling of the parallelogram.) Drop a perpendicular from D to $\overleftrightarrow{AB}$, meeting $\overleftrightarrow{AB}$ at X. Prove that X must lie between A and B, and let $c = AX$, $h = DX$. Drop another perpendicular, this one from C to $\overleftrightarrow{AB}$, meeting $\overleftrightarrow{AB}$ at Y. Show that B is between A and Y and that $CY = h$. Then apply the Pythagorean Theorem.

Exercise 12.6. ***Varignon's Theorem***

Prove that the midpoints of the sides of any quadrilateral are the vertices of a parallelogram. *Hint*: Consider the diagonals of the quadrilateral.

Exercise 12.7. ***Deception!***

Consider a triangle $\triangle ABC$ where $AB \neq BC$. Let D be the midpoint of $\overline{AC}$ and let O be the point of intersection of the perpendicular bisector of $\overline{AC}$ and the angle bisector of $\angle ABC$ (as shown below). The point O must exist from Exercise 10.6b. From O drop lines perpendicular to $\overleftrightarrow{AB}$ and $\overleftrightarrow{CB}$, intersecting $\overleftrightarrow{AB}$ and $\overleftrightarrow{CB}$ at the points E and F, respectively:



(a) Prove that $\triangle BOE \cong \triangle BOF$.
Hint: Use a result from Exercises I.8.

(b) Prove that either $\triangle DOA \cong \triangle DOC$ or $O = D$.

(c) Use (a) and (b) to prove $\triangle EOA \cong \triangle FOC$.

(d) From (a) and (c) you obtain $BE = BF$ and $EA = FC$. It thus appears that $BA = BE + EA = BF + FC = BC$, contradicting the assumption $BA \neq BC$. Hence you have "proven" that any triangle $\triangle ABC$ is isosceles! *What is wrong with this "proof"?!*

Exercise 12.8.

(a) For a right triangle $\triangle ABC$ with right angle at $\angle C$ give the natural geometric definitions of the **cosine** and **tangent** of the angle $\angle A$. Then show these quantities depend *only on the angle measure θ of $\angle A$.* We can therefore employ the usual notations $\cos\theta$ and $\tan\theta$.

(b) Prove that $\sin^2\theta + \cos^2\theta = 1$ for any angle measure in the domain $0 < \theta < 90°$.

(c) Determine a formula for $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$.

(d) Define the **secant** for any $0 < \theta < 90°$ by $\sec\theta = 1/\cos\theta$. Determine a simple formula expressing $\sec\theta$ in terms of $\tan\theta$.

Exercise 12.9. ***Trigonometry with Directed Angles.***

The definitions given in §12 and Exercise 12.8 for $\sin\theta$, $\cos\theta$, $\tan\theta$, and $\sec\theta$ apply only to ordinary (non-directed) angle measures in the domain $0 < \theta < 90°$. In this exercise you will expand these definitions to directed angle measures of any value.

As discussed in §11 (in particular, Theorem 11.15), fix a directed angle measure function with *counterclockwise orientation* at each point in the plane. Fix a directed angle $\measuredangle XOY$ with $m\measuredangle XOY = 90^\circ \bmod 360$ where, for convenience, we pick X and Y so that $OX = OY = 1$ as shown in Figure 12.24. Furthermore, place coordinate systems χ and η on the lines $\ell_x = \overleftrightarrow{OX}$ and $\ell_y = \overleftrightarrow{OY}$, respectively, so that $\chi(O) = 0$, $\chi(X) = 1$, $\eta(O) = 0$, and $\eta(Y) = 1$. Finally, let $\mathcal{C}$ denote the circle with center O and radius 1. We will refer to $\mathcal{C}$ as the *unit circle*.

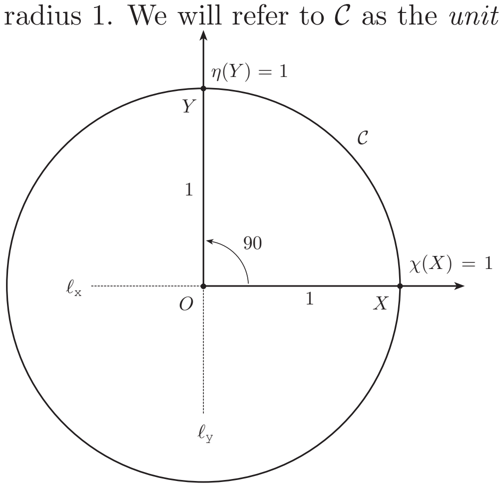


Figure 12.24. A unit circle given coordinate axes.

(a) Show that any fixed ray $\overrightarrow{OQ}$ starting at O must intersect the unit circle $\mathcal{C}$ in exactly one point P as shown in Figure 12.25.

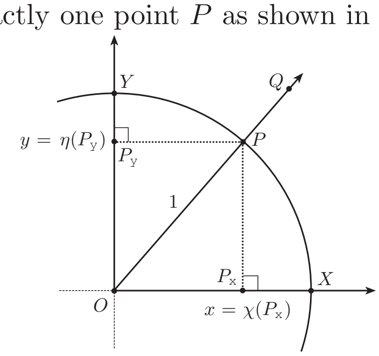


Figure 12.25. $\overrightarrow{OQ}$ intersects circle $\mathcal{C}$ in a unique point P.

(b) Show that any point P on the unit circle uniquely determines two points, $P_x \in \ell_x$ and $P_y \in \ell_y$, such that the line $\overleftrightarrow{PP_x}$ is perpendicular to ℓ_x and the line $\overleftrightarrow{PP_y}$ is perpendicular to ℓ_y.

(c) With P, P_x, P_y as in (b), the ordered pair of real numbers

$$(x, y) = (\chi(P_x), \eta(P_y))$$

is called the set of *xy-coordinates* for the point P. Determine the value of $x^2 + y^2$ and justify your answer.

For any $\theta \in \mathbb{R}$ let $\overrightarrow{OQ_\theta}$ be the unique ray such that the directed angle measure of $\measuredangle XOQ_\theta$ equals θ mod 360. From (a) this ray intersects the unit circle in a unique point P_θ and, from (b) and (c), P_θ has a unique set of xy-coordinates (x_θ, y_θ). Define the *sine* and *cosine* of θ by

$$(\cos\theta, \sin\theta) = (x_\theta, y_\theta) = \text{ the } xy\text{-coordinates of } P_\theta.$$

(d) Show that, for $0 < \theta_0 < 90°$, the definitions just given for $\sin\theta_0$ and $\cos\theta_0$ agree with the original definitions given following Proposition 12.9 and in Exercise 12.8. Then define the *tangent* and *cotangent* for any directed angle measure value θ. What difficulty arises with the definitions of $\tan\theta$ and $\cot\theta$?

(e) Compute the sine, cosine, tangent, and secant of $60°$, $90°$, $120°$, $180°$, $215°$, $495°$, $-60°$, $-90°$, and $-300°$. Interpret the results as best you can in terms of side lengths of triangles. Use both pictures and verbal descriptions.

(f) Do the various trigonometry identites of Exercise 12.8, developed for $0 < \theta < 90°$, remain valid for *all* values of θ? Prove your claims.

(g) Suppose a and b are real numbers such that $a^2 + b^2 = 1$. Prove there exists a unique (mod 360) directed angle measure θ such that

$$a = \cos\theta \text{ and } b = \sin\theta.$$

§I.13 Circles

Under what conditions do three given positive numbers a, b, c form the side lengths of a triangle? The answer to this is given in the *Triangle Theorem*, a simple and fundamental result that can, in some ways, be viewed as the converse of the *Triangle Inequality*.

Under what conditions do two circles in the plane intersect? This is an important question whose answer is given in the *Two Circle Theorem*. By considering the centers of the two circles and a potential point of intersection, the proof of the *Two Circle Theorem* will reduce to an application of the *Triangle Theorem*.

We will use the *Two Circle Theorem* in Chapter IV to give a geometric proof of the existence of a fixed point for a "strict similarity."

Theorem 13.1. ***The Triangle Theorem***

Suppose a, b, and c are positive numbers. If each of these numbers is less than the sum of the other two, then a triangle exists with side lengths a, b, and c.

Proof. We label the three side lengths so that $a \geq b \geq c > 0$. Then the condition that each of the numbers is less than or equal to the sum of the other two simply becomes $b + c > a$.

In order to guide our construction, first suppose $\triangle ABC$ is a triangle with the desired side lengths, as shown in Figure 13.2.

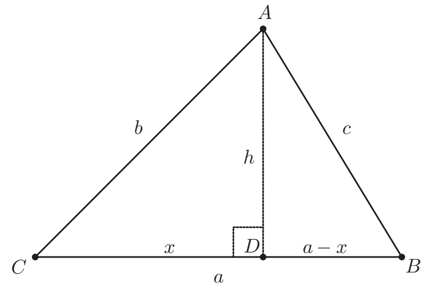


Figure 13.2. A triangle $\triangle ABC$ with the desired side lengths.

Let D be the foot of the altitude from A, with $CD = x$ and $AD = h$. Then the *Pythagorean Theorem* gives

$$\begin{aligned} x^2 + h^2 &= b^2, \\ (a-x)^2 + h^2 &= c^2. \end{aligned}$$

These equations are easily solved to give

$$\begin{aligned} x &= \frac{a^2 + b^2 - c^2}{2a}, \\ h &= \sqrt{b^2 - x^2}. \end{aligned}$$

Now return to the situation where no triangle is given and we need to construct $\triangle ABC$ with the desired side lengths a, b, and c. Begin by choosing two points B and C such that $BC = a$. Then, guided by our previous computations, define the number x by

$$x = \frac{a^2 + b^2 - c^2}{2a}.$$

Since $a \geq b \geq c > 0$, then $x > 0$. Moreover, simple algebra shows

$$a - x = \frac{a^2 - b^2 + c^2}{2a},$$

which must be positive since $a \geq b \geq c > 0$. Thus, since $0 < x < a = BC$, we know there exists a point D between B and C such that $CD = x$, as shown in Figure 13.2.

Now consider the quantity $b - x$. Simple algebra gives

$$b - x = \frac{c^2 - (a-b)^2}{2a}.$$

However, the condition $b + c > a \geq b$ becomes $c > a - b \geq 0$, showing $b > x$. Hence we can define the positive number h by $h = \sqrt{b^2 - x^2}$ and find a point A such that $\overline{AD}$ is perpendicular to $\overline{BC}$ and $AD = h$. This is shown in Figure 13.2. Applying the *Pythagorean Theorem* to each of the right triangles $\triangle ACD$ and $\triangle ABD$ gives

$$\begin{aligned} AC &= \sqrt{x^2 + h^2} = b, \\ AB &= \sqrt{(a - x)^2 + h^2} \\ &= \sqrt{(a^2 - 2ax + x^2) + (b^2 - x^2)} \\ &= \sqrt{a^2 - (a^2 + b^2 - c^2) + b^2} = c. \end{aligned}$$

Hence $\triangle ABC$ has the desired side lengths. □

We first defined circles in Exercise 7.5. We now give this definition a number and prove a fundamental result about such sets.

Definition 13.3.

A **circle** $\mathcal{C}$ in the plane is a collection of all points equidistance from one fixed point, i.e., there exists a point P and a real number $r > 0$ such that $\mathcal{C} = \{x \in \mathcal{E} \mid Px = r\}$. The point P is the **center** of the circle and r is the **radius**. A **chord** is any line segment joining two points on the circle. A **diameter** is any chord containing the center of the circle.

Theorem 13.4. ***The Two Circle Theorem***

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be circles with radii r_1 and r_2, respectively, with $c > 0$ the distance between their centers. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect if and only if each of the numbers r_1, r_2, c is less than or equal to the sum of the other two. In particular, we have the following.

(a) *$\mathcal{C}_1$ and $\mathcal{C}_2$ intersect in* **two distinct points** *p and $\bar{p}$ if and only if each of the numbers r_1, r_2, c is strictly less than the sum of the other two. In this case p and $\bar{p}$ are the reflections of each other in the line of centers for the two circles.*

(b) *$\mathcal{C}_1$ and $\mathcal{C}_2$ intersect in* **exactly one point** *p if and only if one of the numbers r_1, r_2, or c equals the sum of the other two. In this case p lies on the line of centers for the two circles.*

(c) *In all other cases $\mathcal{C}_1$ and $\mathcal{C}_2$* **do not intersect**.

Proof. We label the circles $\mathcal{C}_1$ and $\mathcal{C}_2$ so that $r_1 \geq r_2$. Let q_1 and q_2 be the centers of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.

Suppose one of the numbers r_1, r_2, c is greater than the sum of the other two. Then the circles $\mathcal{C}_1$ and $\mathcal{C}_2$ cannot intersect, as shown in Figure 13.5. For if there were a point p of intersection, then consider the three points p,

q_1, and q_2. Two of these points would be such that the distance between them would be strictly greater than the sum of the distances between the remaining two pair of points. This would contradict the *Triangle Inequality,* Corollary 8.11. This establishes (c).

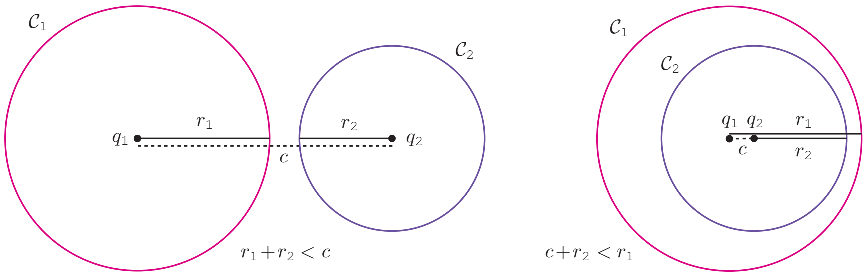


Figure 13.5. Case (c) of the *Two Circle Theorem* — no intersection.

Now suppose one of the numbers r_1, r_2, or c equals the sum of the other two. By our assumption $r_1 \geq r_2$ we have either $r_1+r_2 = c$ or $c+r_2 = r_1$. In either case there exists one point p_0 on the line $\overleftrightarrow{q_1q_2}$ that is on both circles, as seen in Figure 13.6. However, there cannot be another point of intersection, for if p were such a point, then the side lengths of the triangle $\triangle q_1pq_2$ would violate the *Strict Triangle Inequality,* Theorem 8.9. Hence the circles have exactly one point of intersection, verifying (b).

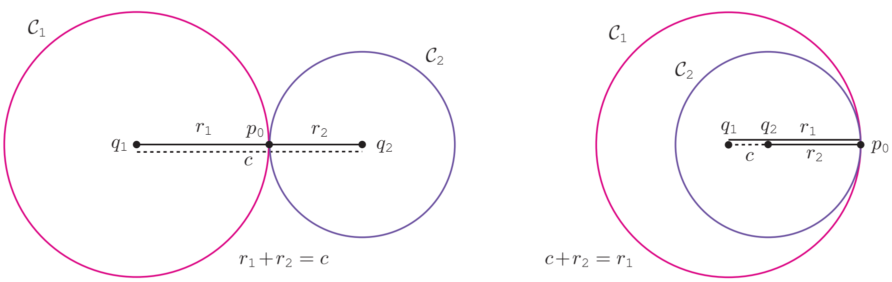


Figure 13.6. Case (b) of the *Two Circle Theorem* — a single intersection.

Finally, consider the case where each of the numbers r_1, r_2, c is strictly less than the sum of the other two. Then by the *Triangle Theorem*, Theorem 13.1, there exists a triangle $\triangle Q_1PQ_2$ such that $PQ_1 = r_1$, $PQ_2 = r_2$, and $Q_1Q_2 = c$. We can construct a copy of this triangle on the line segment $\overline{q_1q_2}$ in the following manner.

Let $\overrightarrow{q_1p}$ be the ray such that $\angle q_2q_1p \cong \angle Q_2Q_1P$ and $q_1p = Q_1P$. Since $q_1q_2 = c = Q_1Q_2$, *SAS* gives $\triangle q_1pq_2 \cong \triangle Q_1PQ_2$. In particular,

$$pq_1 = PQ_1 = r_1,$$
$$pq_2 = PQ_2 = r_2.$$

This proves that p is on both circles $\mathcal{C}_1$ and $\mathcal{C}_2$, as shown in Figure 13.6. Moreover, choose $\bar{p}$ to be the point on the side of $\overleftrightarrow{q_1q_2}$ not containing p such that $\angle q_2q_1\bar{p} \cong \angle q_2q_1p$ and $q_1\bar{p} = q_1p = r_1$. By *SAS*, $\triangle q_1\bar{p}q_2 \cong \triangle q_1pq_2$. In particular, $\bar{p}q_1 = pq_1 = r_1$ and $\bar{p}q_2 = pq_2 = r_2$, proving $\bar{p}$ is a second point of intersection of the two circles $\mathcal{C}_1$ and $\mathcal{C}_2$.

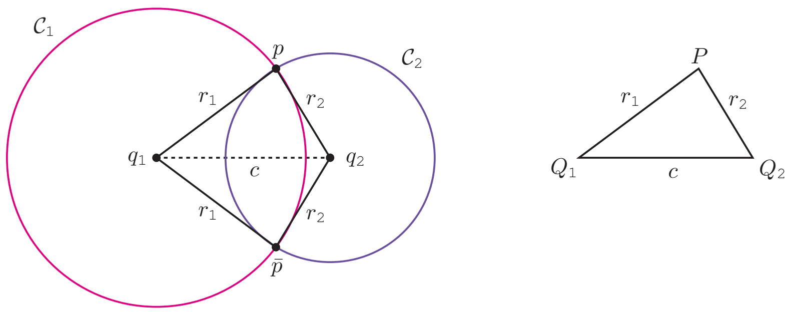


Figure 13.7. Case (a) of the *Two Circle Theorem* — a double intersection.

Suppose p' is a third point of intersection. Then $\triangle q_1p'q_2 \cong \triangle q_1pq_2$ by *SSS*. Hence $\angle q_2q_1p' \cong \angle q_2q_1p$, and $q_1p' = q_1p = r_1$. Thus the ray $\overrightarrow{q_1p'}$ must equal either $\overrightarrow{q_1p}$ or $\overrightarrow{q_1\bar{p}}$, and since $q_1p' = r_1$, the point p' must equal either p or $\bar{p}$. This proves that $\mathcal{C}_1$ and $\mathcal{C}_2$ have exactly two points of intersection, p and $\bar{p}$, where p and $\bar{p}$ are the reflections of each other across the line $\overleftrightarrow{q_1q_2}$. □

Corollary 13.8.
Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be circles of radius r_1 and r_2, with c the distance between their centers. Label the circles so that $r_1 \geq r_2$. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect if and only if

$$r_1 + r_2 \geq c \geq r_1 - r_2.$$

Proof. Exercise 13.2. Notice that in this result we allow $c = 0$.

Exercises I.13

Exercise 13.1. ***The Line-Circle Theorem***.
A line is said to be **tangent** to a circle if the line and the circle intersect in exactly one point. This point is called the **point of contact**.

(a) A **radius segment** for a circle is a line segment from the center of the circle to a point on the circle. If a line ℓ is perpendicular to a radius segment of a circle $\mathcal{C}$ at its endpoint on the circle, prove that ℓ is tangent to the circle.

(b) Suppose a line ℓ is tangent to a circle $\mathcal{C}$ with point of contact x. Prove ℓ is perpendicular to the radius segment with endpoint x.

(c) ***The Line-Circle Theorem***. Suppose a line ℓ intersects the interior of a circle $\mathcal{C}$, i.e., ℓ contains a point whose distance to the center of $\mathcal{C}$ is less than the radius of $\mathcal{C}$. Prove that ℓ intersects the circle in *exactly* two points.

Exercise 13.2.

Prove Corollary 13.8 from Theorem 13.4. *Hint:* First consider the case $c = 0$. Then consider the case $c > 0$ and apply the *Two Circle Theorem*.

Exercise 13.3.

(a) Prove that three circles in the plane whose centers A, B, and C are non-collinear can intersect in at most one point p — see the left panel of Figure 13.9.

(b) Explain the relevance of the two frames of Figure 13.9 to (a) and to Exercise 7.4c.

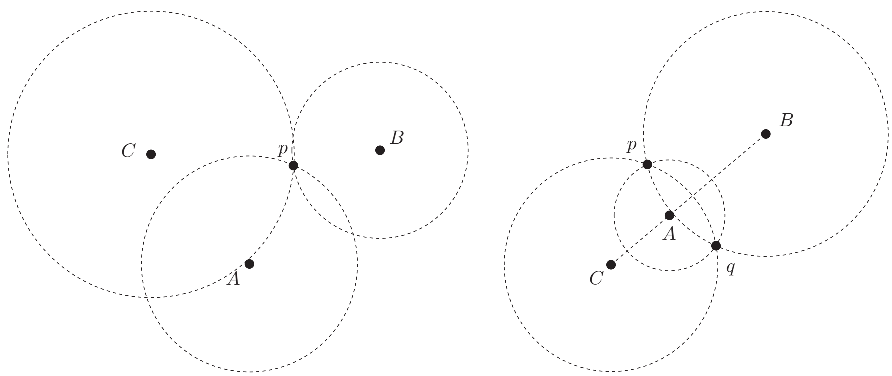


Figure 13.9. Intersections of three circles.

§I.14 Bolzano's Theorem

In §1 we stated *Bolzano's Theorem* (Theorem 1.4) for later use in the text. We now prove this result, showing it to be a direct consequence of the completeness of the real number system. The techniques of this proof will not be needed to understand the uses we make of *Bolzano's Theorem*. As a consequence, readers unfamiliar with the rigorous formulations of limits and convergence may find it best to skip this discussion.

A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is said to **converge to $\boldsymbol{x}$** if the terms in the sequence become arbitrarily close to x as the index n becomes large. Formulated in precise terms, given any $\epsilon > 0$, there exists an integer $N > 0$ such that $|x - x_n|$ is less than ϵ whenever n is greater than N.

A sequence $\{x_n\}_{n=1}^{\infty}$ is **bounded** if all the elements of the sequence are contained in a bounded interval, i.e., there exists a bounded interval $[a, b]$ such that $a \leq x_n \leq b$ for all n. A sequence can be bounded but not converge. However, the completeness of the real number system implies that *every* bounded sequence has a convergent *subsequence*. This is *Bolzano's Theorem*.

Theorem 1.4. ***Bolzano's Theorem.***

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof. Suppose $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$. Then the sequence is contained in a bounded interval $[a_0, b_0]$.

Let c_0 be the midpoint of this interval, i.e., $c_0 = (a_0 + b_0)/2$. Since there is an infinite number of terms in the original sequence, at least one of the two intervals $[a_0, c_0]$ and $[c_0, b_0]$ must also contain an infinite number of terms of the sequence. Choose one of these intervals that does indeed contain an infinite number of terms of the sequence and for convenience relabel this interval as $[a_1, b_1]$. Then choose x_{n_1} to be any element of the sequence that lies in $[a_1, b_1]$.

Let c_1 be the midpoint of the new interval $[a_1, b_1]$. Since there is an infinite number of terms of the original sequence in $[a_1, b_1]$, at least one of the intervals $[a_1, c_1]$ and $[c_1, b_1]$ also contains an infinite number of terms of the sequence. Choose one such interval and relabel it $[a_2, b_2]$. Then choose x_{n_2} to be any element of the sequence that lies in $[a_2, b_2]$ such that $n_1 < n_2$.

Continue in this way generating a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of the original sequence $\{x_n\}_{n=1}^{\infty}$ and a nested sequence of closed intervals

$$[a_0, b_0] \supseteq [a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots \supseteq [a_k, b_k] \supseteq \cdots$$

such that $x_{n_k} \in [a_k, b_k]$ for each integer $k \geq 1$. We claim the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ must converge, which will prove *Bolzano's Theorem*.

From the completeness of $\mathbb{R}$ (Theorem 1.2) there is at least one real number x contained in all the closed intervals $[a_0, b_0]$, $[a_1, b_1]$, $[a_2, b_2]$, ... We claim the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converges to x. To verify this claim, let $\epsilon > 0$ be any arbitrarily small number. We know the length $|b_k - a_k|$ of each interval $[a_k, b_k]$ is half that of its predecessor. Thus the lengths $|b_k - a_k|$ shrink closer and closer to zero (with zero as the limit) and, in particular, become less than the chosen $\epsilon > 0$ when k is sufficently large, say when $k > K$ for some integer K. Since the interval $[a_k, b_k]$ contains all the subsequence points x_{n_k} for $k > K$ as well as the point x, then the distances $|x - x_{n_k}|$ must all be less than ϵ when k is greater than K. This proves that the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converges to x, finishing our proof of *Bolzano's Theorem*. □

Later in the text certain desired results will require not only the completeness of $\mathbb{R}$ but also the completeness of the Euclidean plane $\mathcal{E}$. This result is a consequence of our set of axioms for $\mathcal{E}$, primarily the existence of a coordinate system for each line, which directly applies the completeness of $\mathbb{R}$ to every line in $\mathcal{E}$.

A sequence $\{p_n\}_{n=1}^{\infty}$ of points in the plane $\mathcal{E}$ is **bounded** if all the points of the sequence are contained in the interior of a circle. The sequence is said to **converge to the point *p*** if the points in the sequence become arbitrarily close to p ("within ϵ of p" for any small $\epsilon > 0$) as the index n becomes large ("whenever $n > N$" for some large N). If this is the case, then the sequence $\{p_n\}_{n=1}^{\infty}$ **converges** and p is the **limit point** of the sequence.

More generally, any subset C of the plane is **bounded** if it is contained in the interior of a circle. A subset C of the plane is said to be **closed** if it contains the limit point of every convergent sequence of points of C, i.e., if $\{p_n\}_{n=1}^{\infty}$ is a sequence of points of C which converges to a point p, then p would also have to be in C if C is a closed set. Intuitively a closed set C contains all its "boundary points."

A sequence of non-empty bounded closed sets in $\mathcal{E}$, $C_0, C_1, C_2, \ldots$ is **nested** if each set contains the next one as a subset, i.e.,

$$C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n \supseteq \cdots .$$

One way to characterize the *completeness of* $\mathcal{E}$ is to prove that every such nested sequence of non-empty bounded closed sets has at least one point p that belongs to *all* the sets. This is the same way we characterized the completeness of the real number system in §1.

Theorem 14.1. ***The Completeness of*** $\boldsymbol{\mathcal{E}}$**.**
For any nested sequence of non-empty bounded closed sets in $\mathcal{E}$ *there will always exist a point* p *that belongs to all the sets (i.e.,* p *is in the intersection of all the sets).*

Proof. Suppose $C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots$ is a nested sequence of non-empty bounded closed sets in $\mathcal{E}$. From each of these sets C_k select a point p_k. We will show that there is a convergent subsequence $\{p_{k_j}\}_{j=0}^{\infty}$ of the sequence $\{p_k\}_{k=0}^{\infty}$ which converges. The limit point p will have to be in each of the original sets C_k since these sets are closed. Here are the details.

Since C_0 is bounded, there exists a closed solid square S_0 which contains C_0 (and hence all the C_k) in its interior. Let L be the side length of S_0, and choose the first subsequence point to simply be the first point in the original sequence, i.e., choose p_{k_0} to equal p_0.

Divide S_0 into four closed solid subsquares, each with side length $L/2$. At least one of these subsquares contains an infinite number of the entries of the original sequence $\{p_k\}_{k=0}^{\infty}$. Let S_1 be such a subsquare, and choose the second subsequence point from the original sequence with index greater than $k_0 = 0$, i.e., choose p_{k_1} from $\{p_k\}_{k=0}^{\infty}$ to be a point in S_1 such that $k_1 > k_0$.

Continue in this way, generating a nested sequence of closed solid squares $S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$ and a corresponding sequence of points $p_{k_0} \in S_0$, $p_{k_1} \in S_1$, $p_{k_2} \in S_2$, $\ldots$, where $k_0 < k_1 < k_2 < \ldots$.

We first claim that there does indeed exist a point p in the intersection of all the closed solid squares $S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$. This will require careful use of coordinate systems. Choose a side of S_0 (call this the *base* of S_0). Then use the *Ruler Placement Theorem* (Proposition 3.6) to place a coordinate system χ on the line containing the base of S_0 such that the χ coordinates range from 0 to L on the base.

Each subsquare S_1, S_2, ... has a base, defined to be that side parallel to the base of S_0 which is the closest to the base of S_0. Then each subsquare has a "left side," defined to be the side which forms a directed angle with (counterclockwise) measure 90° from the base. Extend the left side of each square S_j to a line ℓ_j that intersects the base of S_0 perpendicularly at a χ coordinate x_j. Then each pair of numbers $\{x_{j-1}, x_j\}$ are the endpoints of a bounded closed interval I_j in $\mathbb{R}$ of length $L/2^j$. Moreover, these bounded closed intervals form a nested sequence in $\mathbb{R}$, so that the completeness of $\mathbb{R}$ (Theorem 1.2) implies that there exists a number x in all these intervals. In particular, x must be the χ coordinate of a point p_x in the base of S_0.

Now select a line ℓ through p_x which is perpendicular to the base of S_0. By the construction of p_x the line ℓ must intersect each of the closed solid squares S_j, $j = 0, 1, 2, \ldots$, in a closed bounded interval of length $L/2^j$. Moreover, these intervals are nested. Using the *Ruler Placement Theorem* (Proposition 3.6) to place a coordinate system η on ℓ and in that way identifying ℓ with the real number system $\mathbb{R}$, the completeness of $\mathbb{R}$ shows that there exists a point p in all these closed subintervals of ℓ. Hence p is indeed a point contained in all the subsquares S_0, S_1, S_2, ... , as we desired.

We claim p is the limit point of the subsequence $\{p_{k_j}\}_{j=0}^{\infty}$. This is easy to show since both p and p_{k_j} are in the closed solid square S_j and the distance between any two points in S_j is less than $2(L/2^j) = L/2^{(j-1)}$. Since this distance goes to zero as j increases toward infinity, this proves p is the limit of the subsequence $\{p_{k_j}\}_{j=0}^{\infty}$.

However, for each non-negative integer k this implies that p is the limit point of the subsequence $\{p_{k_j}\}_{j=k}^{\infty}$, i.e., the subsequence starting at the k-th point, and all the points of such a sequence lie in the closed set C_k. Hence p must also lie in C_k by the definition of closed set. Thus p lies in *all* the sets C_k, $k = 0, 1, 2, \ldots$, as desired. This finishes the proof of Theorem 14.1. □

If you examine the proof of Theorem 14.1, you will see that we actually proved a version of *Bolzano's Theorem* for the Euclidean plane $\mathcal{E}$. For suppose $\{p_k\}_{k=1}^{\infty}$ is a bounded sequence in $\mathcal{E}$. Then let S_0 be a closed solid square containing the sequence. The argument in our proof then goes through without change, showing that there exists a convergent subsequence $\{p_{k_j}\}_{j=0}^{\infty}$. This is the conclusion of *Bolzano's Theorem*.

Theorem 14.2. ***Bolzano's Theorem for $\mathcal{E}$.***
Every bounded sequence in the plane $\mathcal{E}$ has a convergent subsequence.

Exercises I.14

Exercise 14.1.

(a) Consider the sequence

$$\{\frac{1}{2}, -\frac{1}{2}, \frac{2}{3}, -\frac{2}{3}, \frac{3}{4}, -\frac{3}{4}, \frac{4}{5}, -\frac{4}{5}, \dots\}$$

from Exercise 1.5a. Starting with the interval $[a_0, b_0] = [-1, 1]$, illustrate the proof of *Bolzano's Theorem* by making explicit choices for $[a_1, b_1]$, $[a_2, b_2]$, $[a_3, b_3], \dots$ and x_{n_1}, x_{n_2}, x_{n_3}, ...

(b) Consider the sequence

$$\{1, -1, 1.4, -1.4, 1.41, -1.41, 1.414, -1.414, 1.4142, -1.4142, \dots\}$$

from Exercise 1.5b. Starting with the interval $[a_0, b_0] = [-2, 2]$, illustrate the proof of *Bolzano's Theorem* by making explicit choices for $[a_1, b_1]$, $[a_2, b_2]$, $[a_3, b_3], \dots$ and x_{n_1}, x_{n_2}, x_{n_3}, ...

Exercise 14.2.

Bolzano's Theorem is actually equivalent to the completeness of the real number system, i.e., Bolzano's Theorem is not only a consequence of the completeness of $\mathbb{R}$, it also *implies* the completeness of $\mathbb{R}$. Prove this result, i.e., show that if every bounded sequence in $\mathbb{R}$ has a convergent subsequence, then $\mathbb{R}$ must be complete in the sense of Theorem 1.2.
Hint: You must begin with a nested sequence of closed intervals

$$[a_0, b_0] \supseteq [a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots \supseteq [a_k, b_k] \supseteq \cdots$$

and show that there is at least one real number x which is an element of all the intervals. To do so, consider the sequence of left endpoints, i.e., $\{x_n\}_{n=0}^{\infty} = \{a_n\}_{n=0}^{\infty}$.

§I.15 Axioms for the Euclidean Plane

For convenience we list all the axioms we have developed for the Euclidean plane. Although we will not verify the details, this axiom system is indeed *consistant*, i.e., there are specific models in which all of the axioms can be shown to be true. The most important model used for this purpose is $\mathbb{R}^2$, the real Cartesian plane, as introduced in Example 2.2. However, adding all the necessary structures (such as angle measure) to this model involves some tedious technical details.[16]

[16] The interested reader is referred to Chapter 26 in Moise, *Elementary Geometry from an Advanced Standpoint*, for this development (though in fact even Moise does not develop the angle measure function).

Assume the existence of $\mathcal{E}$ (a set of **points**), a distinguished collection of subsets $\mathcal{L}$ (the collection of **lines**), a *coordinate system* on each line in $\mathcal{L}$ that defines a *distance function* $d : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ applicable to any pair of points $(p, q) \in \mathcal{E} \times \mathcal{E}$, and an *angle measure function* $m : \mathcal{A} \to \mathbb{R}$ applicable to any angle in $\mathcal{A}$ (the collection of angles) that satisfy the following axioms.

Incidence Axioms.

I-1. The plane $\mathcal{E}$ contains at least three non-collinear points.

I-2. Given two points $p \neq q$, there is *exactly one* line $\overleftrightarrow{pq}$ containing both.

The Plane Separation Axiom.

PS. If a line ℓ is removed from the plane $\mathcal{E}$, the result is a disjoint union of two non-empty convex sets $\mathcal{H}_\ell^1$ and $\mathcal{H}_\ell^2$ such that if $p \in \mathcal{H}_\ell^1$ and $q \in \mathcal{H}_\ell^2$, then the line segment $\overline{pq}$ intersects ℓ.

Angle Measure Axioms.

M-1. For every angle $\angle A$, $0 < m\angle A < 180°$.

M-2. *Angle Construction.* Suppose $\overrightarrow{ab}$ is a ray on line ℓ and $\mathcal{H}$ is one of the two half planes with edge ℓ. Then for every number $0 < r < 180°$ there is a unique ray $\overrightarrow{ac}$, with c in $\mathcal{H}$, such that $m\angle bac = r$.

M-3. *Angle Addition.* If c is in the interior of $\angle bad$, then
$$m\angle bad = m\angle bac + m\angle cad.$$

M-4. *Supplements.* If two angles $\angle bac$ and $\angle cad$ form a *linear pair*, then they are *supplementary*, i.e., $m\angle bac + m\angle cad = 180°$.

The SAS (Side-Angle-Side) Axiom.

SAS. Suppose a correspondence between two triangles is such that two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle. Then the correspondence is a congruence between the two triangles.

The Parallel Postulate.

PP. Suppose ℓ is a line and p is a point not on ℓ.
Then there exists a *unique* line ℓ' parallel to ℓ and containing p.