Supplementary Material for Chapter 1

Mesterton-Gibbons, STML 11 (September 17, 2009)

1.6. Remarks on the calculation of R_2

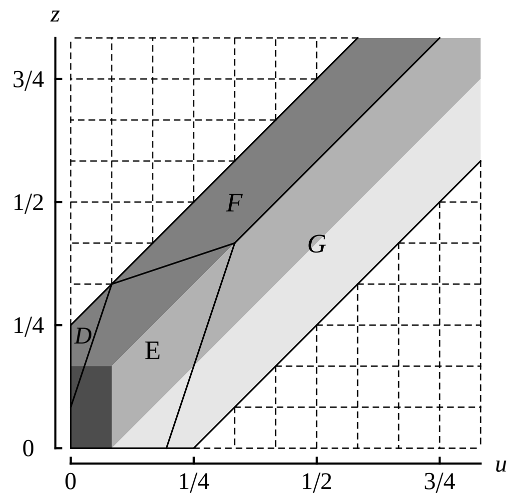
From the perspective of Van, $|u-v| \le \frac{1}{12}$, $|v-z| \le \frac{1}{6}$, $0 \le u, v, z \le \alpha$ translates to $u - \frac{1}{12} \le v \le u + \frac{1}{12}$, $z - \frac{1}{6} \le v \le z + \frac{1}{6}$, $0 \le v < \infty$ in the limit as $\alpha \to \infty$. That is,

$$\max\bigl(u - \tfrac{1}{12}, z - \tfrac{1}{6}, 0\bigr) \;\le\; v \;\le\; \min\bigl(u + \tfrac{1}{12}, z + \tfrac{1}{6}\bigr).$$

Because $z - \frac{1}{6}$ is bigger than $u - \frac{1}{12}$ above the line $z = u + \frac{1}{12}$ but $u - \frac{1}{12}$ is bigger below the line, whereas 0 exceeds both inside the rectangle $0 \le u \le \frac{1}{12}$, $0 \le z \le \frac{1}{6}$, and because $u + \frac{1}{12}$ is smaller than $z + \frac{1}{6}$ above the line $u = z + \frac{1}{12}$ but $z + \frac{1}{6}$ is smaller below the line, we find that there are four regions, say, I, II, III and IV, in which the above inequality reduces as follows:

$$\begin{aligned}
&\text{I (darkest region):} && 0 \;\le\; v \;\le\; u + \tfrac{1}{12}\\
&\text{II (second darkest region):} && z - \tfrac{1}{6} \;\le\; v \;\le\; u + \tfrac{1}{12}\\
&\text{III (third darkest region):} && u - \tfrac{1}{12} \;\le\; v \;\le\; u + \tfrac{1}{12}\\
&\text{IV (lightest region):} && u - \tfrac{1}{12} \;\le\; v \;\le\; z + \tfrac{1}{6}
\end{aligned}$$

We analyze each region in turn.



For I, it is clear that

$$\hat{v} \;=\; \tfrac{1}{4}\left(\tfrac{1}{4} + u + z\right)$$

always satisfies $\hat{v} \geq 0$. So the maximum on $0 \leq v \leq u + \frac{1}{12}$ is either at $v = \hat{v}$ or at $v = u + \frac{1}{12}$, according to whether $\hat{v} \leq u + \frac{1}{12}$ or $\hat{v} \geq u + \frac{1}{12}$, i.e., according to whether $z \leq 3u + \frac{1}{12}$ or $z \geq 3u + \frac{1}{12}$. The line $z = 3u + \frac{1}{12}$ is drawn on the diagram: it is the line that joins $(0, \frac{1}{12})$ to $(\frac{1}{12}, \frac{1}{3})$. Within this rectangle, the maximum is at $v = \hat{v}$ below or to the right of the line, but at $v = u + \frac{1}{12}$ above or to the left of the line. If the part of the rectangle above the line is denoted DI (for intersection of D with I) but the part below the rectangle is denoted EI, then what we have just shown is that strategy combinations of the following form belong to R_2:

u	v	z
u^{DI}	$u^{DI} + \frac{1}{12}$	z^{DI}
u^{EI}	$\frac{1}{4}\left(\frac{1}{4} + u^{EI} + z^{EI}\right)$	z^{EI}

Similarly, for II, the maximum on $z - \frac{1}{6} \leq v \leq u + \frac{1}{12}$ is at $z - \frac{1}{6}$ if $\hat{v} \leq z - \frac{1}{6}$, at $u + \frac{1}{12}$ if $\hat{v} \geq u + \frac{1}{12}$, and otherwise at $v = \hat{v}$. But $\hat{v} \leq z - \frac{1}{6}$ translates to $z \geq \frac{1}{3}u + \frac{11}{36}$ or above the line joining $(\frac{1}{12}, \frac{1}{3})$ to $(\frac{1}{3}, \frac{5}{12})$, and the other inequality is as before. Extending our notation in the obvious way, what we have just shown is that strategy combinations of the following form belong to R_2:

u	v	z
u^{DII}	$u^{DII}+\frac{1}{12}$	z^{DII}
u^{EII}	$\frac{1}{4}\left(\frac{1}{4}+u^{EII}+z^{EII}\right)$	z^{EII}
u^{FII}	$z^{FII}-\frac{1}{6}$	z^{FII}

Again, for III, the maximum on $u-\frac{1}{12}\leq v\leq u+\frac{1}{12}$ is at $u-\frac{1}{12}$ if $\hat{v}\leq u-\frac{1}{12}$, at $u+\frac{1}{12}$ if $\hat{v}\geq u+\frac{1}{12}$, and otherwise at $v=\hat{v}$. But $\hat{v}\leq u-\frac{1}{12}$ translates to $z+\frac{7}{12}\leq 3u$ or to the right (on the side not containing the origin) of the line joining $(\frac{1}{3},\frac{5}{12})$ to $(\frac{1}{4},0)$, the other inequality being as before; however, it is no longer possible to have $\hat{v}>u+\frac{1}{12}$, because the line joining $(\frac{1}{12},\frac{1}{3})$ to $(\frac{1}{3},\frac{5}{12})$ does not intersect III. What we have just shown is that strategy combinations of the following form belong to R_2:

u	v	z
u^{EIII}	$\frac{1}{4}\left(\frac{1}{4}+u^{EIII}+z^{EIII}\right)$	z^{EIII}
u^{GIII}	$u^{GIII}-\frac{1}{12}$	z^{GIII}

The analysis for IV is very similar: we need only note that $\hat{v}>z+\frac{1}{6}$ is impossible, because that translates to below (or to the right of) the line $u=3z+\frac{1}{12}$, which lies entirely outside the shaded regions. In other words, we add only these strategy combinations to R_2:

u	v	z
u^{EIV}	$\frac{1}{4}\left(\frac{1}{4}+u^{EIV}+z^{EIII}\right)$	z^{EIV}
u^{GIV}	$u^{GIV}-\frac{1}{12}$	z^{GIV}

The above four tables condense to yield Table 1.12.