Errata to "An Introduction to Ramsey Theory"

Last updated: June 11, 2019

If you think you found an error that is not listed below, please contact Jan Reimann (`jan.reimann@psu.edu`).

page 54, proof of Theorem 2.12

The proof as written does not work, as it is not guaranteed that the n_i will go to infinity. Thanks to ***Shamil Asgarli*** *for pointing this out, and for suggesting the following proof.*

We simultaneously define, inductively, an infinite path $\vec{t} \in [T]$ and a subsequence $(\vec{s}_{n_k})$ such that $\vec{s}_{n_k} \to \vec{t}$ for $k \to \infty$. We put $t^0 = r$ and $\vec{s}_{n_0} = \vec{s}_0$. Note that the set

$$S_0 = \{m : s_m^0 = t^0\}$$

is infinite (every sequence passes through the root node). Now assume we have defined $t^0 < \ldots < t^k$, where each t^i is in T and an immediate predecessor of t^{i+1}, and a sequence $\vec{s}_{n_0}, \ldots, \vec{s}_{n_k}$ such that the set

$$S_k = \left\{m : s_m^0 = t^0, \ldots, s_m^k = t^k\right\}$$

is infinite. Note that all sequences in S_k have distance at most 2^{-k} from each other in the path metric, since they all have the same first k elements. S_k is infinite and T is finitely branching, hence by the pigeonhole principle we can find an immediate successor $t^{k+1} \in T$ of t^k such that

$$S_{k+1} = \left\{m : s_m^0 = t^0, \ldots, s_m^k = t^k, s_m^{k+1} = t^{k+1}\right\}$$

is infinite. Let n_{k+1} be the smallest number $m \in S_{k+1}$ that is greater than n_k.

Since all t^k are on T, they define an infinite path

$$\vec{t} = t^0\, t^1\, t^2 \, \ldots \in [T].$$

By definition of t^k, $d(\vec{t}, \vec{s}_{n_k}) \le 2^{-k}$, and thus $(\vec{s}_{n_k})$ converges to $\vec{t}$ in the path metric.

page 75, line 20

The sentence starting with "Pick the $<$-least element $\ldots$" *should read:* "Pick the $<$-least element x_{α_1} of Z_1 (which must exist since $<$ is a well-ordering) and observe that $\{y \in Z_1 : c(x_{\{}\alpha_1\}, y) = \text{red}\}$ is again uncountable."

[*Thanks to* ***Shamil Asgarli*** *for catching this.*]

page 158, big formula (formal statement of Ramsey's theorem)

The last line of the formula is incorrect – we need to check ***each*** *entry in the argument of the function whether it is an element of the set coded by z. The line should read as follows:*

$$\forall m \le l \;(\; \forall s \le p \exists i \le k (\text{decode}(\text{arg}(f,m), s) = \text{decode}(z,i)) \;\Rightarrow\; \text{val}(f,m) = j \;)$$

[*Thanks to* ***Vineet Gupta*** *and* ***Adnan Aziz*** *for pointing this out.*]

page 180, proof of Proposition 4.46

The transition from the formula

$$\mathcal{N} \models \exists x_1 \forall x_2 \ldots \exists x_n\, \psi(a, \vec{c}, \vec{x})$$

to the Δ_0 formula using "meta"-quantifiers needs further justification. In particular, it does not *follow inductively by simply applying logical equivalences. Instead, the property of indiscernibles has to be* *invoked at this step already. An improved argument is given below. Thanks to* ***Michael Weiss*** *for bringing up this important issue. Michael has a blog –* ***https://diagonalargument.com*** *– that I recommend. Among other entries, there is a series of "conversations" with John Baez about non-standard models of PA. Check it out! Michael has also provided a proof of the above transition that does not use indiscernibility. It is also given below.*

Improved argument for proof of Proposition 4.46, starting at top of page 180:
By contracting quantifiers and possibly adding "dummy" variables and expressions like $x_i = x_i$, we can assume that a given formula φ is of the form

$$\exists x_1 \, \forall x_2 \, \ldots \, Qx_r \; \psi(\vec{y}, x_1, \ldots, x_r) \tag{0.1}$$

or

$$\forall x_1 \, \exists x_2 \, \ldots \, Qx_r \; \psi(\vec{y}, x_1, \ldots, x_r), \tag{0.2}$$

where Q is either $\exists$ or $\forall$, and ψ is quantifier-free. In the following, we focus on the form given in (0.1). The argument for the other form is similar.

With any $\varphi(\vec{y})$ in prenex normal form (0.1) we associate a Δ_0 formula $\varphi^*(\vec{y}, z_1, \ldots, z_r)$ given as

$$\exists x_1 < z_1 \, \forall x_2 < z_2 \, \ldots \, Qx_r < z_r \; \psi(\vec{y}, x_1, \ldots, x_r).$$

Claim: For any formula φ in prenex normal form, for any $\vec{a} \in N$, and any $i_0 < i_1 < i_2 < \ldots < i_r$ with $\vec{a} < b_{i_0}$,

$$\mathcal{N} \vDash \varphi[\vec{a}] \;\Leftrightarrow\; \mathcal{M} \vDash \varphi[\vec{a}, b_{i_1}, \ldots, b_{i_r}]. \tag{0.3}$$

The claim is proved by induction on the formula length (see also Lemma 4.47, where this technique was first described). If φ has no quantifiers at all, the claim is clear. So assume now $\varphi(\vec{y})$ is as in (0.1) with $r \geq 1$. Then the claim is that $\varphi[\vec{a}]$ holds in $\mathcal{N}$ if and only if

$$\exists x_1 < b_{i_1} \, \forall x_2 < b_{i_2} \, \ldots \, Qx_r < b_{i_r} \; \psi(\vec{a}, x_1, \ldots, x_r)$$

holds in $\mathcal{M}$.[1]

The formula $\varphi^*(\vec{y}, z_1, \ldots, z_r)$ is

$$\exists x_1 < z_1 \, \forall x_2 < z_2 \, \ldots \, \exists x_r < z_r \; \psi(\vec{y}, x_1, \ldots, x_r, z_1, z_2, \ldots, z_r).$$

Let $\theta(\vec{y}, x_1)$ be

$$\forall x_2 \, \ldots \, Qx_r \; \psi(\vec{y}, x_1, \ldots, x_r),$$

so $\varphi(\vec{y}) = \exists x_1 \theta(\vec{y}, x_1)$. As θ is a shorter formula, by inductive hypothesis the claim has already been verified for θ.

Let $\vec{a} \in N$ and assume $i_0 < i_1 < \ldots < i_r$ are such that $\vec{a} < b_{i_0}$. $\varphi[\vec{a}]$ holds in $\mathcal{N}$ iff there exists a $c \in N$ such that $\theta[\vec{a}, c]$ holds in $\mathcal{N}$. Pick $j_1 < j_2 < \ldots < j_r$ such that $i_0 < j_1$ and $c < b_{j_1}$. By inductive hypothesis,

$$\mathcal{N} \vDash \theta[\vec{a}, c] \quad \text{iff} \quad \mathcal{M} \vDash \theta^*[\vec{a}, c, b_{j_2}, \ldots, b_{j_r}].$$

If we write it out, the expression on the right is

$$\mathcal{M} \vDash \forall x_2 < b_{j_2} \, \ldots \, Qx_r < b_{j_r} \; \psi(\vec{a}, c, x_2, \ldots, x_r).$$

By choice of b_1, this is equivalent to

$$\mathcal{M} \vDash \exists x_1 < b_{j_1} \, \forall x_2 < b_{j_2} \, \ldots \, Qx_r < b_{j_r} \; \psi(\vec{a}, x_1, \ldots, x_r),$$

[1] The notation in the preceding formula is, of course, a little sloppy, as the b_i and $\vec{a}$ are not variables but elements of the structure over which we interpret. But we feel this notation improves readability.

in other words, it is equivalent to

$$\mathcal{M} \vDash \varphi^*[\vec{a}, b_{j_1}, \dots, b_{j_r}].$$

As $i_0 < j_1$ and the (b_i) are diagonal indiscernibles for all Δ_0 formulas in $\mathcal{M}$, the last expression is equivalent to

$$\mathcal{M} \vDash \varphi^*[\vec{a}, b_{i_1}, \dots, b_{i_r}],$$

which proofs the claim.

We can finally show that $\mathcal{N}$ satisfies induction. Recall that (Ind) is equivalent to the *least number principle* (LNP), as we saw in Section 4.1. Suppose $\mathcal{N} \vDash \varphi[a, \vec{c}]$, where $\varphi(v, \vec{w})$ is given in prenex normal form as

$$\exists x_1 \, \forall x_2 \, \dots \, Q x_n \, \psi(v, \vec{w}, \vec{x}), \quad \text{with } \psi \text{ quantifier free.}$$

As before, we choose i_0 such that $a, \vec{c} < b_{i_0}$. We can apply property (0.3) established in the Claim above and obtain the equivalence

$$\mathcal{N} \vDash \varphi[a, \vec{c}] \;\text{ iff }\; \mathcal{M} \vDash \exists x_1 < b_{i_0+1} \, \forall x_2 < b_{i_0+2} \, \dots \, Q x_n < b_{i_0+n} \, \psi(a, \vec{c}, \vec{x}).$$

Since induction (and hence LNP) holds in $\mathcal{M}$, there exists a *least* $\hat{a} < b_{i_0}$ such that

$$\mathcal{M} \vDash \exists x_1 < b_{i_0+1} \, \forall x_2 < b_{i_0+2} \, \dots \, Q x_n < b_{i_0+n} \, \psi(\hat{a}, \vec{c}, \vec{x}).$$

By the definition of $\mathcal{N}$, the existence of $\hat{a} \in N$, and the equivalence above, it follows that $\mathcal{N} \vDash \varphi[\hat{a}, \vec{c}]$. Finally, $\hat{a}$ has to be the smallest witness to φ in $\mathcal{N}$, because any smaller witness would also be a smaller witness in $\mathcal{M}$. This concludes the proof of Proposition 4.46.

Alternative proof of the equivalence (on page 180) of

$$\mathcal{N} \vDash \exists x_1 \forall x_2 \dots \exists x_n \, \psi(a, \vec{c}, \vec{x})$$

and

$$\exists i_1 > i_0 \, \forall i_2 > i_1 \, \dots \, \exists i_n > i_{n-1} \;\; \mathcal{N} \vDash \exists x_1 < b_{i_1} \, \forall x_2 < b_{i_2} \, \dots \, \exists x_n < b_{i_n} \, \psi(a, \vec{c}, \vec{x}).$$

by Michael Weiss (`diagonalargument.com`)

First do the equivalence

$$\begin{aligned}
&\mathcal{N} \vDash \forall x \, \exists y \, \psi(x, y, \vec{c}) \\
\Leftrightarrow\; &\forall p \, \exists q \, \mathcal{N} \vDash \psi(p, q, \vec{c}) \\
\Leftrightarrow\; &\forall i_1 {>} i_0 \;\; \forall p {<} b_{i_1} \;\; \exists i_2 {>} i_1 \;\; \exists q {<} b_{i_2} \;\; \mathcal{N} \vDash \psi(p, q, \vec{c}) \\
\Leftrightarrow\; &\forall i_1 {>} i_0 \;\; \forall p {<} b_{i_1} \;\; \exists i_2 {>} i_1 \;\; \exists q {<} b_{i_2} \;\; \mathcal{M} \vDash \psi(p, q, \vec{c})
\end{aligned}$$

Now that we are in $\mathcal{M}$, a model of PA, we can use the collection axioms. Define a function F by

$$F(x, \vec{u}) = \begin{cases} \mu y[\psi(x, y, \vec{u})] \text{ if } \exists y \, \psi(x, y, \vec{u}) \\ 0 \text{ otherwise} \end{cases}$$

The definition can be formalized in the language of PA (although it is not Σ_1). Using the collection axioms in $\mathcal{M}$, $\max_{0 \le z \le x} F(z, \vec{u})$ exists for all x and $\vec{u}$, and is attained at some z in the segment $[0, x]$. If x and $\vec{u}$ belong to N, then so does the maximum, since it is attained at some $z \in N$ and $\mathcal{N} \vDash \forall \vec{u} \, \forall x \, \exists y \, \mathcal{N} \vDash \psi(x, y, \vec{u})$. Using the cofinality of the b's in N, we conclude that F is bounded on the segment $[0, p]$ by some b_i, and we are entitled to switch the quantifiers.

page 182, Definition 4.48

The definition should read:

Let $X \subseteq \mathbb{N}$, $n \ge 1$, and suppose $f : [X]^n \to \mathbb{N}$. A set $M \subseteq X$ with $|M| > n$ is **min-homogeneous** if for every $s, t \in [M]^n$,

$$\min s = \min t \;\Rightarrow\; f(s) = f(t).$$

[Thanks to **Vineet Gupta** and **Adnan Aziz**]

page 186/187, proof of Lemma 4.52

At several places $[W]^{n+1}$ should be $[Y]^{n+1}$: page 186, lines 3, 6, last line of the second last paragraph, and page 187, line 2.

[Thanks to **Vineet Gupta** and **Adnan Aziz**]