The Great Prime Number Race

Update as of June 11, 2021

TYPOS

p.7, line 6, replace the 2nd and 3rd inequalities by equalities; line 8, replace the 2nd inequality by an equality

p.29, line -5, replace I_{n-1} by nI_{n-1}

p.43, line -4, insert $\zeta(s)$

My book has been written as an introduction to the explicit formulas of analytic number theory. There is a more sophisticated version of these formulas due to Weil [We]. We follow the exposition of Bombieri [Bo].

Consider the class $\mathcal{W}$ of complex-valued functions $f(x)$ on the positive half-line $\mathbb{R}_+$, continuous and continuously differentiable except for finitely many points at which both $f(x)$ and $f'(x)$ have at most a discontinuity of the first kind, and at which the value of $f(x)$ and $f'(x)$ is defined as the average of the right and left limits there. Suppose also that there is $\delta > 0$ such that $f(x) = O(x^{\delta})$ as $x \to 0_+$ and $f(x) = O(x^{-1-\delta})$ as $x \to +\infty$.

Let $\widetilde{f}(s)$ be the Mellin transform

$$\widetilde{f}(s) = \int_0^\infty f(x) x^s \frac{dx}{x}$$

which is an analytic function of s for $\delta < \Re(s) < 1+\delta$. We denoted $\widetilde{f}(s)$ by $(\mathcal{M}f)(s)$ in Definition 4.3.

For the Riemann zeta function, Weil's formula can be stated as follows. Let $\Lambda(n) = \log p$ if $n = p^a$ is a power of a prime p, and 0 otherwise. This is the familiar von Mangoldt function in Definition 2.5.

EXPLICIT FORMULA. For $f \in \mathcal{W}$ we have

$$\begin{aligned} \widetilde{f}(0) - \sum_\rho \widetilde{f}(\rho) + \widetilde{f}(1) &= \sum_{n=1}^\infty \Lambda(n) \Big\{ f(n) + \frac{1}{n} f(\frac{1}{n}) \Big\} + (\log 4\pi + \gamma) f(1) \\ &+ \int_0^\infty \Big\{ f(x) + \frac{1}{x} f(\frac{1}{x}) - \frac{2}{x} f(1) \Big\} \frac{dx}{x - x^{-1}} \end{aligned}$$

Note that this formula exhibits a familiar pattern: the prime powers enter into one side of the formula, and the non-trivial zeta zeros enter into the other side.

If we think of the elements in $\mathcal{W}$ as test-functions, then this explicit formula is an *identity of distributions*.

For a very recent discussion of the Weil explicit formula, see §2.1.1 in [CC].

Supplemental Exercises.

1. *The value of the gamma function on the critical line.* Prove that, for all $t \in \mathbb{R}$, we have

$$|\Gamma(1/2 + it)| = \sqrt{\frac{\pi}{\cosh \pi t}}$$

Hint. Use the reflection principle for the gamma function, see p.43.

2. We have the von Mangoldt formula, the Riemann formula, and the Weil explicit formula. How are these formulas related?

Hint. The Wikipedia article *Explicit formulae for L-functions* is the most helpful source which I can find.

References

[Bo] E. Bombieri, *The Riemann hypothesis.* The millennium prize problems, 107–124, Clay Math. Inst., Cambridge, MA, 2006.

[CC] A. Connes, C. Consani, *Spectral Triples and ζ-Cycles*, arXiv:2106.01715.

[WD] Walter Dittrich, *Reassessing Riemann's Paper*, SpringerBriefs in History of Science and Technology, 2nd Edition 2021. See, especially, chapter 11 on the Zeta Function in Quantum Electrodynamics (QED).

[We] A. Weil, *Sur les "formules explicites" de la théorie des nombres premiers*, Oeuvres Scientifiques, volume II, 48–61.