A LOCAL APPROACH TO RINGS WITH FINITENESS CONDITIONS

By taking a local approach, we revisit in this note socle theory for semiprime rings, the structure theorem for prime rings with minimal one-sided ideals, and Martindale's theorem for prime rings satisfying a generalized polynomial identity. Inspired by Jordan methods, we also give a new proof of Kaplansky's structure theorem for prime rings with involution and minimal one-sided ideals.

For notation and terminology we follow basically our monograph [4]. Further and more complete information on ring structure theory can be found in [2, 3, 5].

1. Definitions and notation

From now on A will denote an associative algebra (non-necessarily with a unit element) over an arbitrary ring of scalars Φ, thus encompassing rings ($\Phi = \mathbb{Z}$ in this case) and classical associative algebras over a field $\mathbb{F}$. Since the role played by scalars is quite irrelevant in most cases, there is no problem in adopting the basic notions of ring theory to this lightly more general setting.

- A *right ideal* of A is a Φ-submodule I of A such that $IA \subset I$.
- For any subset $S \subset A$, the *right annihilator* of S is the set

$$\mathrm{Rann}(S) = \{a \in A : Sa = 0\}.$$

It is clear that $\mathrm{Rann}(S)$ is a right ideal of A. Left ideals and left annihilators, $\mathrm{Lann}(S)$, are defined in a similar way.

- Let $*$ be an involution (or a ring involution) of A. An element $a \in A$ is called *self-adjoint or symmetric* if $a = a^*$. We denote by $\mathrm{Sym}(A, *)$ the set of all symmetric elements of A. Similarly, $a \in A$ is called *skew-symmetric* if $a^* = -a$. We denote by $\mathrm{Skew}(A, *)$ (or simply by K when there is no risk of confusion) the set of all skew-symmetric elements of A.

- Right (left) A-modules are also required to be Φ-modules (this condition being by no means redundant since the existence of a unit element in A is not required in the definition of algebra. We adopt the convention of writing the maps of a left module on the right (thus composing them from left to right), and the maps of a right module on the left (thus composing them from right to left). This is notationally advantageous in dealing with pairs of dual vector spaces.

The following lemma collects some standard results on semiprime and prime associative algebras. The proof is left to the reader.

Lemma 1.1. *Let A be an associative algebra.*

(1) *A is semiprime (prime) if and only if $aAa = 0$ ($aAb = 0$) implies $a = 0$ ($a = 0$ or $b = 0$), for $a, b \in A$.*
(2) *Let A be semiprime and let I be an ideal of A. For any $a \in A$, we have*

$$a \in \mathrm{Ann}(I) \Leftrightarrow aI = 0 \Leftrightarrow Ia = 0 \Leftrightarrow aIa = 0.$$

(3) *Every ideal of a semiprime (prime) associative algebra inherits semiprimeness (primeness).*
(4) *Let A be semiprime and let $a, b \in A$. Then $aAb = 0 \Leftrightarrow bAa = 0$.*

2. Local algebras of an associative algebra

Attached to any element a of an associative algebra A, there is an algebra A_a called the local algebra of A at a. The relationship between the properties of an algebra and those of its local algebra is studied here. On the other hand, A_a keeps up relevant information about the element a; those elements whose associated local algebra satisfies some kind of finiteness condition, like being division, Artinian or PI, deserve special attention. This enables us to give a local characterization of the socle of a semiprime associative algebras, an intrinsic characterization of the rank of an element, and a new approach to Martindale's theorem on prime rings satisfying a generalized polynomial identity.

Definition 2.1. Let A be an associative algebra and let $a \in A$.

(i) The *a-homotope of A*, denoted by $A^{(a)}$, is the associative algebra defined by the same linear structure as A and the new product $x._a y = xay$ for all $x, y \in A$.
(ii) The set $\mathrm{Ker}(a) = \{x \in A : axa = 0\}$ is an ideal of $A^{(a)}$, and the quotient algebra $A^{(a)}/\mathrm{Ker}(a)$ is called the *the local algebra of A* at a and will be denoted by A_a.

We write $x \mapsto \overline{x}$ to denote the canonical projection of $A^{(a)}$ onto A_a, so the product of two elements $\overline{x}, \overline{y} \in A_a$ is given by $\overline{x} \cdot_a \overline{y} = \overline{xay}$.

Lemma 2.2. *Let A be an associative algebra with an involution $*$ and let $a \in \mathrm{Sym}(A, *)$. Then the map $\overline{x} \mapsto \overline{x}^*$ is well defined and an involution of the algebra A_a. Similarly, if $a \in \mathrm{Skew}(A, *)$, then the map $\overline{x} \mapsto -\overline{x}^*$ is an involution of A_a.*

Proof. Straightforward. □

The local algebra at an idempotent e is nothing else than the corner eAe. Indeed, something else can be said.

Proposition 2.3. *Let $a = aba$. Then $A_a \cong (aAa)^{(b)} \cong eAe$, where e is the idempotent ab. Hence A_a is unital with $\overline{b}$ being its unit. Conversely, if A is semiprime and A_a unital, then a is von Neumann regular.*

Proof. The map $\varphi : A_a \to (aAa)^{(b)}$, defined by $\varphi(\overline{x}) = axa$ for all $x \in A$, is an isomorphism of algebras: $\overline{x} = 0 \Leftrightarrow axa = 0$, and for all $x, y \in A$,

$$\varphi(\overline{xay}) = axaya = ax(aba)ya = \varphi(\overline{x})b\varphi(\overline{y}).$$

Similarly, the map $\psi : (aAa)^{(b)} \to eAe$ defined by $\psi(axa) = axab = axe$, for all $axa \in aAa$, is an algebra isomorphism:

(i) $axa = 0 \Rightarrow axe = axab = 0 \Rightarrow axa = axea = 0$,
(ii) $exe = abxab = \psi(abxa)$, and
(iii) $\psi((axa)b(aya)) = \psi(axaya) = axaye = axeaye = \psi(axa)\psi(aya)$.

Suppose now that A is semiprime and let $\bar{b}$ be the unit element of A_a. Then, for any $x \in A$, we have,

$$(a - aba)x(a - aba) = a(x - bax)(a - aba) = a(x - bax - xab + baxab)a = 0.$$

Now $(a - aba)A(a - aba) = 0$ implies that $a = aba$ by the semiprimeness of A. □

Proposition 2.4. *Let I be an ideal of A. Then the inclusion $j : I \to A$ induces a monomorphism $\overline{j}$ of I_a into A_a, which is onto if a is von Neumann regular.*

Proof. It is clear that $\overline{j} : I_a \to A_a$ is a monomorphism. Now if $a = aba$, then for any $x \in A$, $\overline{x} = \overline{y}$, with $y = baxab \in I$, which proves that $\overline{j}$ is onto. □

There is a global-to-local inheritance between niceness properties of A and those of its local algebras, which is a natural extension of the process of passing from A to the corner eAe and back.

Proposition 2.5. *The following properties pass from A to A_a, for any nonzero element a in A: semiprimeness, primeness and simplicity.*

Proof. Straightforward. □

Now we study how the centroid behaves when passing to local algebras. As usual, $\Gamma(A)$ stands for the centroid of the algebra A [4, Section 1.2].

Proposition 2.6. *Let A be semiprime and let $a \in A$ be nonzero. We have:*

(i) *The map $\Psi : \Gamma(A) \to \Gamma(A_a)$ defined by $\Psi(\gamma) = \overline{\gamma}$, where $\overline{\gamma}(\overline{x}) = \overline{\gamma(x)}$, $x \in A$, is a homomorphism of rings with* $\ker(\Psi) = \{\gamma \in \Gamma(A) : \gamma(a) = 0\}$.
(ii) *If A is prime, then Ψ is a monomorphism.*
(iii) *If A is simple, then $\Gamma(A_a)$ is a field extension of $\Gamma(A)$.*

Proof. (i) Let $\gamma \in \Gamma(A)$ and $x \in A$. If $\overline{x} = 0$ then $axa = 0$; hence $a\gamma(x)a = \gamma(axa) = 0$, which proves that Ψ is well defined. It is clear that Ψ is a ring homomorphism. Let's now compute $\ker(\Psi)$. Let $\gamma \in \Gamma(A)$ be such that $\gamma(a) = 0$. Then, for any $x \in A$, $a\gamma(x)a = \gamma(a)xa = 0$, which proves that $\gamma \in \ker(\Psi)$. Conversely, let $\gamma \in \ker(\Psi)$. Then $a\gamma(A)a = 0$ and hence $\gamma(a)A\gamma(a) = a\gamma^2(A)a \subset a\gamma(A)a = 0$, which implies $\gamma(a) = 0$ since A is semiprime.

(ii) Suppose that A is prime and let $\gamma \in \ker(\Psi)$, equivalently, by (i), $\gamma(a) = 0$. Since $\gamma(A)$ is an ideal of A, $a\gamma(A) = \gamma(a)A = 0$ implies $a \in \mathrm{Ann}(\Gamma(A))$ and hence that $\gamma = 0$ by primeness of A.

(iii) Let A be simple. Then A_a is also simple (Proposition 2.5) and $\Gamma(A_a)$ is a field extension of $\Gamma(A)$ by [4, Proposition 1.10] and (ii). □

• A semiprime ring A is said to be *centrally closed* if its centroid $\Gamma(A)$ coincides with its extended centroid $\mathcal{C}(A)$. For a prime $\mathbb{F}$-algebra A this means that $\mathcal{C}(A)$ is just the field $\mathbb{F}$. For information relative to extended centroid and *central closure* the reader is referred to Section 1.3 of [4].

Lemma 2.7. *Let A be a centrally closed prime associative algebra over a field $\mathbb{F}$ and let $a \in A$. If A_a has nontrivial center, then A_a is unital and $Z(A_a) \cong \mathbb{F}$.*

Proof. Let $\overline{z} \in A_a$ be a nonzero central element. Then $(aza)xa = ax(aza)$ for every $x \in A$, Since A is centrally closed over $\mathbb{F}$, it follows from [2, Theorem 2.3.4] that $a = \lambda aza = a(\lambda z)a$ for some $\lambda \in \mathbb{F}$. Thus a is von Neumann regular and hence A_a is unital, with $\overline{b} := \overline{\lambda z} = \lambda\overline{z}$ as unit element (Proposition 2.3). Then $\overline{z} = \lambda^{-1}\overline{b}$, which proves that $Z(A_a) = \mathbb{F}\overline{b} \cong \mathbb{F}$, as required. □

Corollary 2.8. *Let A be a simple associative algebra and let $0 \neq a \in A$ be von Neumann regular. Then $\Gamma(A) \cong \Gamma(A_a) \cong Z(A_a)$.*

Proof. As A is simple, it is centrally closed, and since $a = aba$ is von Neumann regular, A_a is unital with $\overline{b}$ as unit element. In particular, the center of A_a is nontrivial. Then, by Lemma 2.7, $\Gamma(A_a) \cong Z(A_a) = \Gamma(A)\overline{b} \cong \Gamma(A)$. □

Exercises

Ex 2.9. Show that the localization at elements is transitive: $(A_a)_{\overline{b}} \cong A_{aba}$ for all $a, b \in A$.

Ex 2.10. Let $\mathrm{Rad}(A)$ denote the Jacobson radical and of an associative algebra A and let $\varphi : A_a \to A$ be the linear map $\overline{x} \mapsto axa,\ x \in A$. Show that $\varphi(\mathrm{Rad}(A_a)) \subset \mathrm{Rad}(A)$.

Ex 2.11. Show that if A is semiprimitive, then A_a is semiprimitive.

Ex 2.12. Show that if A is right primitive, then A_a is right primitive.

Ex 2.13. Show that if A is prime and A_a is right primitive for some nonzero element $a \in A$, then A is right primitive.

Ex 2.14. Let A be semiprime. Show that for $a, x \in A$ the following conditions are equivalent:

(i) $\mathrm{Lann}(a) = \mathrm{Lann}(ax)$ whenever $ax \neq 0$.
(ii) A_a is a domain, i.e. $\overline{x} \cdot_a \overline{y} \neq 0$ whenever $\overline{x} \neq 0$ and $\overline{y} \neq 0$.
(iii) $\mathrm{Rann}(a) = \mathrm{Rann}(xa)$ whenever $xa \neq 0$.

3. Socle of a semiprime associative algebra

Following [5, IV.3], we recall in this section the structure theory of semiprime algebras with minimal one-sided (in particular, the notion of socle), and prove a *local* characterization of the elements of the socle of a semiprime associative algebra. Since the definition of semiprime algebra is left-right symmetric, every statement on right ideals does have its corresponding version for left ideals.

Proposition 3.1. [5, III.9. Proposition 1] *Let I be a minimal right ideal of A. Then either $I^2 = 0$ or $I = eA$ where $e \in I$ is an idempotent.*

Proof. Suppose that $xI \neq 0$ for some $x \in I$. Then $xI = I$ by minimality of I. Let $e \in I$ be such that $xe = x$. We claim that e is an idempotent. Since $\mathrm{Rann}(x)$ is a right ideal, we have by minimality of I that either $I \cap \mathrm{Rann}(x) = I$ or $I \cap \mathrm{Rann}(x) = 0$, with the first possibility leading to a contradiction since $xe = x \neq 0$. Thus $I \cap \mathrm{Rann}(x) = 0$. Then $xe^2 = xe = x$ implies $e^2 - e \in I \cap \mathrm{Rann}(x) = 0$, so $e^2 = e$, as claimed. Since $e \in eA \subset I$, $eA = I$ by minimality of I. □

Corollary 3.2. *Let A be semiprime and let $a \in A$ be such that aA is a minimal right ideal. Then $a \in aA$.*

Proof. By Proposition 3.1, $aA = eA$ for an idempotent e. Hence $(a - ea)A = 0$, which implies, by semiprimeness of A, that $a = ea \in eA = aA$. □

• Let e_1, e_2 be nonzero idempotents of a semiprime associative algebra A over the ring of scalars Φ. Regarding Ae_1 and Ae_2 as left A-modules, it is not difficult to check that the map $b \mapsto \rho_b$, where $(x)\rho_b = xb$ for all $x \in Ae_1$ defines a Φ-module isomorphism of e_1Ae_2 onto $\mathrm{Hom}_A(Ae_1, Ae_2)$. When $e_1 = e_2$, this module isomorphism is actually an isomorphism of associative algebras.

Proposition 3.3. [5, IV.3. Proposition 1 and Corollary] *Let A be semiprime and let e be an idempotent of A. Then Ae is a minimal left ideal if and only if eAe is a division algebra if and only if eA is a minimal right ideal.*

Proof. For any idempotent $e \in A$, eAe is isomorphic to the ring $\mathrm{End}_A(Ae)$. Hence, by Schur's lemma, if Ae is a minimal left ideal, then eAe is a division algebra. Suppose conversely that eAe is a division algebra. If $ea \in eA$ is nonzero, we have by semiprimeness of A that $eaAea \neq 0$, so $eabe \neq 0$ for some $b \in A$. Since eAe is a division algebra, $eabece = e$ for an element $ece \in eAe$. This proves that eA is a minimal right ideal. By symmetry, we also have that Ae is a minimal right ideal if and only if eAe is a division algebra. □

Definition 3.4. Let A be semiprime. Every idempotent $e \in A$ satisfying the equivalent conditions of the proposition above is called a *division idempotent*.

Lemma 3.5. *Every division idempotent e of a semiprime algebra A generates an ideal which is simple as an algebra. Moreover, if two division idempotents e_1, e_2 generate the same ideal, then $e_1Ae_1 \cong e_2Ae_2$.*

Proof. Denote by I the ideal of A generate by e. By Lemma 1.1(3), I is semiprime, and since $eAe = e^2Ae^2 = eIe$, e remains a division idempotent in I. Let J be an ideal of I. Then eJe is an ideal of eIe, and since the latter is a division algebra, either $eJe = eIe$ or $eJe = 0$. If the first, $e \in J$ implies $J = I$; if the second, $e \in \mathrm{Ann}_I(J)$ (Lemma 1.1(2)) and hence, $J \subset I = \mathrm{Ann}_I(J)$, which implies $J = 0$ by semiprimeness of I. This proves that I is simple.

Let $e_1, e_2 \in A$ be division idempotents generating the same ideal. By semiprimeness of A, this implies that $\mathrm{Hom}_A(Ae_1, Ae_2) \cong e_1Ae_2 \neq 0$. Since both Ae_1 and A_2 are irreducible A-modules, it follows by Shur's lemma that the left A-modules Ae_1 and Ae_2 are isomorphic. Hence, $e_1Ae_2 \cong \mathrm{End}_A(Ae_1) \cong \mathrm{End}_A(Ae_2) \cong e_2Ae_2$. □

Teorema 3.6. [5, IV.3. Theorem 1] *For a semiprime algebra A, the sum of its minimal left ideals coincides the sum of its minimal right ideals. This ideal, called the* socle *of A and denoted by* $\mathrm{Soc}(A)$*, is a direct sum of ideals each which is a simple algebra.*

Proof. It follows from Propositions 3.1 and 3.3, and Lemma 3.5. □

Proposition 3.7. *Let A be semiprime and let I be a right ideal of A lying in a finite sum of minimal right ideals. Then every orthogonal subset of division idempotents of A contained in I is finite, and if $\{e_1, e_2, \ldots, e_n\}$ is a maximal such subset, then $e = \sum e_i \in \mathrm{Soc}(A)$ and $I = eA$.*

Proof. By module theory, we may assume that $I \subset I_1 \oplus I_2 \oplus \ldots \oplus I_r$, where the I_j are minimal right ideals. Hence any chain of right ideals contained in I has length $\leq r$, and any orthogonal subset of idempotents of A contained in I has at most r elements. Let $\{e_1, e_2, \ldots, e_n\}$ be a maximal set of mutually orthogonal division idempotents contained in I and set $e = \sum e_i$. Then $I = eA \oplus ((1-e)A \cap I)$, where $(1-e)A$ makes sense even if A has no a unit element. We claim that $(1-e)A \cap I = 0$ and hence that $I = eA$. Otherwise $(1-e)A \cap I$ would contain a division idempotent, say f. Taking $u := f - fe$ we would obtain a division idempotent which is orthogonal to e and lies in I, a contradiction. Thus $I = eA$, as claimed. □

Corollary 3.8. *Let A be semiprime. Then every element in the socle of A is von Neumann regular.*

Proof. Let $a \in \mathrm{Soc}(A)$. By Proposition 3.7, $aA = eA$ for some idempotent $e \in \mathrm{Soc}(A)$. Then $a = ea = axa$ for some $x \in A$, as required. □

Recall that an associative algebra is called *right* (*left*) *Artinian* if it satisfies the descending chain condition on right (left) ideals.

Corollary 3.9. *Let A be semiprime. The following conditions are equivalent:*

(i) *A is right Artinian.*
(ii) *A is unital and coincides with its socle.*
(iii) *$\mathbf{1} \in A$ and $1 = e_1 + \cdots + e_n$, where the e_i are division idempotents mutually orthogonal.*

In this case, the number n is uniquely determined and it is called the capacity *of A.*

Proof. (i) $\Rightarrow$ (ii). If A is right Artinian, then every nonzero right ideal of A contains a minimal right ideals. Hence A contains a maximal set $\{e_1, \ldots, e_n\}$ of mutually orthogonal division idempotents. Set $e = e_1 + \cdots + e_n$. Then $(1-e)A = 0$; since otherwise it would contain a nonzero division idempotent g and taking $f = g - ge = g(1-e)$ we would have a division idempotent orthogonal to e, which contradicts the maximality of $\{e_1, \ldots, e_n\}$. Thus $A = eA \oplus (1-e)A = eA$. We claim that A is unital with e as unit element. It is clear that $ea = a$ for every $a \in A$. Consider the left ideal $A(1-e)$. Then $A(1-e)A = A(1-e)eA = 0$ implies $A(1-e) = 0$ by semiprimeness of A, that is, $a = ae$ for all $a \in A$. This proves that A is unital and coincides with its socle.

(ii) $\Rightarrow$ (iii). If A has a unit element and coincides with its socle, then it follows from Proposition 3.7 that A contains a (finite) maximal set of mutually orthogonal division idempotents whose sum is $\mathbf{1}$.

(iii) $\Rightarrow$ (i). Let $\mathbf{1} = e_1 + \cdots + e_n$ be a sum of mutually orthogonal division idempotents. Taking $I_r := (\sum_{i=1}^{r} e_i)A$, for $1 \le r \le n$, we obtain a composition series $0 \subset I_1 \subset I_2 \subset \ldots \subset I_n = A$. Hence A is right Artinian and n is uniquely determined by A. □

Notation 3.10. Since the statements (i) and (ii) of the theorem above are left-right symmetric, a semiprime algebra is right Artinian if and only if it is left Artinian. Thus, in what follows, by a *semiprime Artinian* algebra we will mean a semiprime algebras which is left (equivalently, right) Artinian.

4. An elemental characterization of the socle

In Section 2, we associated an associative algebra A_a to any element a of an associative algebra A. In the present section we prove that when A is semiprime, a lies in the socle of A if and only if A_a is Artinian. We also show that the role played by an idempotent in classical socle theory of semiprime algebras is not exclusive for this kind of elements but it is also played by any element of the socle.

Definition 4.1. Let A be an associative algebra over a ring of scalars Φ. A J-*inner ideal* of A is a Φ-submodule B of A such that $bAb \subset B$ for any $b \in B$. A *minimal* J-*inner ideal* is a nonzero J-inner ideal of A which does not contain other J-inner ideals different from 0 and itself.

It is clear that the intersection of J-inner ideals of A is a J-inner ideal, that one-sided inner ideals are J-inner ideals, and that for any $a \in A$, aA, bA and aAa are J-inner ideals, the latter called the *principal* J-*inner ideal* determined by a.

Lemma 4.2. *Given $a \in A$, the map $\varphi : A_a \to A$, $\overline{x} \mapsto axa$, is a Φ-module isomorphism from A_a onto aAa satisfying:*

(i) *$\varphi(\overline{x} \cdot_a \overline{y} \cdot_a \overline{z}) = \varphi(\overline{x}) y \varphi(\overline{z})$, for all $x, y, z \in A$.*
(ii) *$\overline{I} \mapsto \varphi(\overline{I})$ is an isomorphism from the lattice of the* J-*inner ideals of A_a onto the lattice of the* J-*inner ideals of A contained in aAa.*

Proof. Straightforward. □

Proposition 4.3. *For an element a in a semiprime algebra A, the following conditions are equivalent:*

(i) *A_a is a division algebra.*

(ii) *aAa is a minimal inner ideal.*
(iii) *aA is a minimal right ideal.*

Proof. (i) $\Leftrightarrow$ (ii). A semigroup G is a group if and only if for every $a \in G$, $aGa = G$. Hence it is clear that a semiprime associative algebra is a division algebra if and only if it has no nontrivial J-inner ideals. Now use Lemma 4.2(ii) to get that A_a is a division algebra if and only if the J-inner ideal aAa is minimal.

(ii) $\Rightarrow$ (iii). Suppose that aAa is a minimal J-inner ideal (equivalently, A_a is a division algebra). Then A_a is unital and hence a is von Neumann regular by Proposition 2.3, i.e. $a \in aAa$. Let $0 \neq ab \in aA$. Since A is semiprime, $abxa \neq 0$ for some $x \in A$. Hence, by minimality of the J-inner ideal aAa, there exists $y \in A$ such that $(abxa)y(abxa) = a$. Then $a \in abA$ and hence $abA = aA$, which proves that the right ideal aA is minimal.

(iii) $\Rightarrow$ (i). Let aA be a minimal right ideal. By Proposition 3.1 and Corollary 3.1, $aA = eA$ for a division idempotent e and $a = ea = aba$ for some $b \in A$. Then, by Proposition 2.3, $A_a \cong eAe$ is a division algebra (3.3). □

Definition 4.4. A nonzero element a in a semiprime algebra A will be called a *division* element if it satisfies the equivalent conditions of Proposition 4.3.

Lemma 4.5. *Let a be a nonzero element of a semiprime associative algebra A and let $\overline{b}$ a division element of A_a. Then aba is a division element of A.*

Proof. It follows from the isomorphism $(A_a)_{\overline{b}} \cong A_{aba}$ (see Ex. 2.9). □

Corollary 4.6. *Let A be a semiprime associative algebra. We have:*

(i) *The socle of A is the linear span of its division elements, equivalently, the sum of its minimal inner ideals.*
(ii) *Every division element $a \in A$ generates an ideal which is simple as an algebra.*
(iii) *If two division elements $u, v \in A$ generate the same ideal, equivalently, $uAv \neq 0$, then its respective local algebras A_u and A_v are isomorphic.*

Proof. Only the statements (ii) and (iii) require some commentary. (ii) Let $a \in A$ a division element, equivalently, aA is a minimal right ideal. By Proposition 3.1, $aA = eA$ for a division idempotent $e \in A$, and since a is von Neumann regular, a and e generate the same ideal. Now Lemma 3.5 applies.

(iii) Let $u, v \in A$ be division elements generating the same ideal. By Proposition 2.3, $A_u \cong eAe$ and $A_v \cong fAf$ where e, f are division idempotents with $\mathrm{id}(e) = \mathrm{id}(u) = \mathrm{id}(v) = \mathrm{id}(f)$. Then we have by Lemma 3.5 that $A_u \cong eAe \cong fAf \cong A_v$. □

Definition 4.7. Let A be a prime associative algebra with nonzero socle. The division algebra A_u, where u is a division element in A, uniquely determined by A (Corollary 4.6(iii)), is called the *division algebra of A*.

The structure of the minimal J-inner ideals in a semiprime associative algebra A is now determined.

Proposition 4.8. *Let A be semiprime and let M be a minimal* J*-inner ideal of A. Then either $M^2 = 0$ or $M = eAe$ for a division idempotent $e \in A$.*

Proof. Assume $M^2 \neq 0$. Since A is semiprime and M is minimal, for any nonzero $x \in M$ we have $M = xAx$ and hence $x^2 \neq 0$. Then $x^2Ax^2 = xMx = M$. Now for any $0 \neq xax \in M$, we have $M = (xax)A(xax) = xa(xAx)ax = xaMxa = (xax)M(xax)$ which implies (by the group characterization used in the proof of Proposition 4.3) that M is a division algebra. Then $M = eAe$, where $e = e^2$ is the unit element of M. □

The following useful result appears in [2, Lemma 4.61] for prime algebras.

Corollary 4.9. *Let A be prime and let u, v be division elements of A. Then uAv is a minimal* J*-inner ideal, and if $vu \neq 0$, then $uAv = eAe$ for a division idempotent $e \in A$.*

Proof. For any nonzero element $uav \in uAv$, we have by minimality of the right ideal uA (Proposition 4.3), $uavA = uA$ and hence $(uav)A(uav) = uA(uav) = uAv$, by minimality of the left ideal Av, which proves that uAv is minimal. Set $M = uAv$. If $vu \neq 0$, then u, v lie in the same simple component of the socle, so we my assume that A is simple and hence $M^2 \neq 0$ (in fact $M^2 = (uAv)(uAv) = u(AvuA)v = uAv = M$). Then, by Proposition 4.8, $uAv = eAe$ for a division idempotent $e \in A$. □

The following theorem is a useful characterization of the elements of the socle of a semiprime associative algebra.

Teorema 4.10. *Let A be semiprime. Then $a \in A$ lies in the socle if and only if A_a is Artinian.*

Proof. Since A is semiprime, A_a is semiprime by Proposition 2.5. Suppose that A_a is Artinian. Then we have by Corollary 3.9 that A_a is unital and coincides with its socle. Let $\bar{b}$ be the unit element of A_a. By Corollary 4.6, $\bar{b} = \bar{x}_1 + \cdots + \bar{x}_n$, where $\bar{x}_i \in \bar{I}_i$ for each $1 \leq i \leq n$, with $\bar{I}_i$ being a minimal inner ideal of A_a. Hence, by (Proposition 2.3, $a = aba = ax_1a + \cdots + ax_na \in \varphi(\bar{I}_1) + \cdots + \varphi(\bar{I}_n)$, where the $\varphi(\bar{I}_i)$ are minimal J-inner ideals of A (4.2(ii)). Then $a \in \mathrm{Soc}(A)$ by Corollary 4.6 again.

Conversely, let $a \in \mathrm{Soc}(A)$. By Corollary 3.8, $a = aba$ is von Neumann regular, with $b \in \mathrm{Soc}(A)$ since this is an ideal of A. Then write $a = aba = ax_1a + \cdots + ax_na$, where $x_i \in I_i$ for some minimal inner ideal I_i of A. As before, we have $\bar{b} = \bar{x}_1 + \cdots + \bar{x}_n \in \mathrm{Soc}(A_a)$, with $\bar{b}$ being the unit element of A_a. Therefore, A_a is unital and coincides with its socle, equivalently, A_a is Artinian. □

Corollary 4.11. *Let A be a semiprime associative $\mathbb{F}$-algebra and let $a \in A$. If aAa is finite-dimensional, then $a \in \mathrm{Soc}(A)$. Moreover, if A is prime, then its associated division algebra is finite-dimensional over $\mathbb{F}$.*

Proof. Since $A_a \cong aAa$ as vector spaces (Lemma 4.2), A_a is finite-dimensional and therefore Artinian. Then $a \in \mathrm{Soc}(A)$ by Theorem 4.10. Suppose now that A is prime, and let $\bar{b}$ be a division idempotent of A_a. By Lemma 4.5, aba is a division element of A and $A_{aba} \cong (A_a)_{\bar{b}}$ is finite-dimensional. □

Theorem 4.10 provides an intrinsic characterization of the rank of an element of the socle of a semiprime associative algebra.

Definition 4.12. Let A be semiprime. For an element $a \in \mathrm{Soc}(A)$, the *rank* of a, $\mathrm{rank}(a)$, is defined to be the capacity of A_a.

We can also compute the extended centroid of a prime associative algebra in terms of its local algebras.

Proposition 4.13. *Let A be a prime associative algebra with nonzero socle. For any nonzero $a \in \mathrm{Soc}(A)$, $\mathcal{C}(A) \cong Z(A_a)$. Hence $\mathcal{C}(A) \cong Z(\Delta)$, where Δ is the division algebra of A.*

Proof. Set $M = \mathrm{Soc}(A)$ and let $0 \neq a \in M$. Since M is a minimal ideal (Theorem 3.6), we have by [4, Lemma 1.23] that $\mathcal{C}(A) \cong \Gamma(M)$, and since, $a \in M$ is von Neumann (Corollary 3.8), it follows from Corollary 2.8 and Proposition 2.4 that $\Gamma(M) \cong \Gamma(M_a) \cong Z(M_a) \cong ZA_a)$. The second part follows taking a to be a division element of A. □

5. Pairs of dual vector spaces

A *pair of dual spaces* $\mathcal{P} = (X, Y, \langle\, ,\rangle)$ consists of a left vector space X, a right vector space Y (both over the same division algebra Δ), and a nondegenerate bilinear form $\langle\, ,\rangle : X \times Y \to \Delta$:

(i) $\langle x_1 + x_2, y_1\rangle = \langle x_1, y_1\rangle + \langle x_2, y_1\rangle$,
(ii) $\langle \alpha x_1, y_1\rangle = \alpha\langle x_1, y_1\rangle$,
(iii) $\langle x_1, y_1 + y_2\rangle = \langle x_1, y_1\rangle + \langle x_1, y_2\rangle$,
(iv) $\langle x_1, y_1\alpha\rangle = \langle x_1, y_1\rangle\alpha$,
(iv) $\langle x_1, Y\rangle = 0 \Rightarrow x_1 = 0$ and $\langle X, y_1\rangle = 0 \Rightarrow y_1 = 0$,

for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $\alpha \in \Delta$.

Example 5.1. Any left vector space X over Δ yields the *canonical pair* $(X, X^*, \langle\, ,\rangle)$, where X^* is the *dual* of X and $\langle x, \varphi\rangle) = x\varphi$, for all $x \in X$, $\varphi \in X^*$.

Example 5.2. Let e be a division idempotent of a prime associative algebra A. Then $(eA, Ae, \langle\, ,\rangle)$, with $\langle x, y\rangle = xy$ for all $x \in eA$, $y \in Ae$, is a pair of dual vector spaces over the division algebra eAe. Note that primeness is required in orden to prove that the bilinear form $\langle\, ,\rangle$ is nondegenerate.

Definition 5.3. Given a pair $\mathcal{P} = (X, Y, \langle\, ,\rangle)$ of dual vector spaces over a division algebra Δ, we define the *opposite* of $\mathcal{P}$ as the pair of dual vector spaces $\mathcal{P}^{op} = (Y^{op}, X^{op}, \langle\, ,\rangle^{op})$ over Δ^{op} ($\alpha \cdot \beta := \beta\alpha$ for all $\alpha, \beta \in \Delta$), where Y^{op} (resp. X^{op}) denotes the left (resp. right) vector space over Δ^{op} defined on the additive group of Y (respectively of X) by putting $\alpha \cdot y = y\alpha$ (resp. $x \cdot \alpha = \alpha x$) for all $y \in Y$, $x \in X$ and $\alpha \in \Delta$, and where $\langle y, x\rangle^{op} := \langle x, y\rangle$.

• Let $(X, Y, \langle\, ,\rangle)$ be a pair of dual vector spaces. By $V \leq X$ (resp. $W \leq Y$), we will mean that V is a subspace of X (resp. W is a subspace of Y). Given $S \subset X$ (resp. $T \subset Y$), we set $S^{\perp} = \{y \in Y : \langle S, y\rangle = 0\}$ (resp. $T^{\perp} = \{x \in X : \langle x, T\rangle = 0\}$. It is clear that $S^{\perp} \leq Y$ and $T^{\perp} \leq X$ and that the couple of maps $V \mapsto V^{\perp}$ and $W \mapsto W^{\perp}$ defines a Galois connection between the lattices of the subspaces of X and Y:

(i) $V_1 \leq V_2 \Rightarrow V_2^{\perp} \leq V_1^{\perp}$ and $W_1 \leq W_2 \Rightarrow W_2^{\perp} \leq W_1^{\perp}$, for all $V_1, V_2 \leq X$, $W_1, W_2 \leq Y$.

(ii) $V \leq V^{\perp\perp}$ and $W \leq W^{\perp\perp}$, for all $V \leq X$, $W \leq Y$.

Example 5.4. Let $(X, Y, \langle\, ,\rangle)$ be a pair of dual vector spaces and let $V \leq X$. Then $(X, Y/V^{\perp}, \langle\, ,\rangle)$ is a pair of dual vector spaces, with $\langle v, \overline{y}\rangle = \langle v, y\rangle$, for all $v \in V$, $y \in Y$, where $\overline{y}$ denotes the coset $y + V^{\perp}$.

• Let $\mathcal{P} = (X, Y, \langle\, ,\rangle)$ be a pair of dual vector spaces over Δ. The following statements can be easily proved.

(i) The map $x \mapsto \hat{x}$, $\hat{x}(y) = \langle x, y\rangle$ for all $y \in Y$, is a monomorphism of X into Y^*.

(ii) Similarly, $y \mapsto \hat{y}$, $(x)\hat{y} = \langle x, y\rangle$, $x \in X$, is a monomorphism of Y into X^*.

Definition 5.5. Let $\mathcal{P} = (X, Y, \langle\, ,\rangle)$ and $\mathcal{P}_0 = (X_0, Y_0, \langle\, ,\rangle_0)$ be two pairs of dual vector spaces over the same division algebra Δ. Then $\mathcal{P}_0$ is said to be a *subpair* of $\mathcal{P}$ if $X_0 \leq X$, $Y_0 \leq Y$ and $\langle\, ,\rangle_0$ is the restriction of $\langle\, ,\rangle$ to $X_0 \times Y_0$.

Example 5.6. Every pair $\mathcal{P} = (X, Y, \langle\, ,\rangle)$ of dual vector spaces can be regarded as a subpair of the canonical pair (X, X^*), with $\mathcal{P} = (X, X^*)$ if and only if $\mathcal{P}$ is finite-dimensional.

Definition 5.7. Let $\mathcal{P} = (X, Y, \langle\, ,\rangle)$ be a pair of dual vector spaces. The sequences $\{x_i\}_{i=1}^n \subset X$ and $\{y_i\}_{i=1}^n \subset Y$, where $n \in \mathbb{N} \cup \{\infty\}$, are called *biorthogonal* if $\langle x_i, y_j\rangle = 0$ whenever $i \neq j$ and $\langle x_i, y_i\rangle = 1$ for all i.

Lemma 5.8. *Every pair of biorthogonal sequences of pair $\mathcal{P}$ spans a subpair of $\mathcal{P}$.*

• If $(X, Y, \langle\, ,\rangle)$ is finite-dimensional and $\mathcal{B}$ is a basis of X, then there exists a basis $\mathcal{C}$ of Y such that $\mathcal{B}$ and $\mathcal{C}$ are biorthogonal. If X and Y are countably infinite-dimensional, then not every basis of X has a corresponding basis in Y such that these are biorthogonal. However we have the following result due to Mackey [5, IV.14, Proposition 1].

Lemma 5.9. *Let $(X, Y, \langle\, ,\rangle)$ be a pair of dual vector spaces over Δ such that X and Y are countably infinite-dimensional. Then there exist bases $\{x_n\}_{n=1}^{\infty} \subset X$ and $\{y_n\}_{n=1}^{\infty} \subset Y$ which are biorthogonal.*

Definition 5.10. A subpair $\mathcal{P}_0 = (X_0, Y_0)$ of a pair $\mathcal{P} = (X, Y, \langle\, ,\rangle)$ is said to be a *direct subpair* if $X = X_0 \oplus Y_0^{\perp}$ and $Y = Y_0 \oplus X_0^{\perp}$.

Proposition 5.11. *Let $\mathcal{P} = (X, Y, \langle\, ,\rangle)$ be a pair of dual vector spaces.*

(i) *If $V \leq X$ and $W \leq Y$ are finite-dimensional, then (V, W) can be embedded in a finite-dimensional subpair of $\mathcal{P}$.*

(ii) *Every finite-dimensional subpair of $\mathcal{P}$ is direct.*
(iii) *If $\mathcal{P}_0 = (X_0, Y_0)$ is a direct subpair of $\mathcal{P}$, then $\mathcal{P}_0^\perp := (Y_0^\perp, X_0^\perp)$ is also direct subpair of $\mathcal{P}$ and $\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_0^\perp$.*

Proof. (i) See [5, IV.15, Lemma]

(ii) Let $\mathcal{P}_0 = (X_0, Y_0)$ be a finite-dimensional subpair of $\mathcal{P}$, and let $\{x_1, \dots, x_n\} \subset X_0$ and $\{y_1, \dots, x_n\} \subset Y_0$ be biorthogonal bases. If $x = \sum \alpha_i x_i \in Y_0^\perp$, then for each $1 \leq j \leq n$ we have $0 = \langle x, y_j \rangle = \alpha_j$, which proves that $x = 0$. The other required conditions are proved similarly.

(iii) It suffices to show that $X_0 = X_0^{\perp\perp}$ (the proof of $Y_0 = Y_0^{\perp\perp}$ is similar). From $Y = Y_0 \oplus X_0^\perp$ we get

$$0 = Y^\perp = (Y_0 \oplus X_0^\perp)^\perp = Y_0^\perp \cap X_0^{\perp\perp},$$

and from $X = X_0 \oplus Y_0^\perp$, using the Modular Law (because $X_0 \subset X_0^{\perp\perp}$), we get

$$X_0^{\perp\perp} = X_0^{\perp\perp} \cap X = X_0^{\perp\perp} \cap (X_0 \oplus Y_0^\perp) = X_0 \oplus (Y_0^\perp \cap X_0^{\perp\perp}) = X_0.$$

□

6. Continuous linear maps

Let $\mathcal{P} = (X, Y, \langle\, , \rangle)$ be a pair of dual vector spaces over a division algebra Δ. Regarded Δ as a topological space with the discrete topology, we endow the vector space X with the *initial topology* defined by the family of linear forms $\{\hat{y} : y \in Y\} \subset X^*$. Similarly, Y is endowed with the initial topology defined by the family of linear forms $\{\hat{x} : x \in X\} \subset Y^*$

Proposition 6.1. *Let $\mathcal{P} = (X, Y, \langle\, , \rangle)$ and $\mathcal{P}' = (X', Y', \langle\, , \rangle')$ be two pair of dual vector spaces over the same division algebra Δ. Then a linear map $a : X \to X'$ is continuous* (relative to the topologies defined in X and Y by the respective pairs) *if and only if there exists a* unique *linear map $a^\# : Y' \to Y$ such that $\langle xa, y \rangle = \langle x, a^\# y \rangle$ for all $x \in X$, $y \in Y$.*

Proof. By definition of initial topology, $a \in \mathrm{Hom}_\Delta(X, X')$ is continuous if and only for every $y' \in Y'$, there exists $y'_a \in Y$ (necessarily unique by the nondegeneracy of the bilinear form) such that, for all $x \in X$, we have

$$\langle xa, y' \rangle' = \langle x, y'_a \rangle$$

Define $a^\# : Y' \to Y$ by $a^\# y' = y'_a$. It is easy to check that $a^\#$ is linear and continuous. □

- Note that any $a \in \mathrm{End}_\Delta(X)$ is continuous relative to the canonical pair.
- Denote by $\mathcal{L}_Y(X)$ the set of all continuous linear maps of X, and by $\mathcal{F}_Y(X)$ the subset of those continuous linear maps having finite rank. For $a, b \in \mathcal{L}_Y(X)$ and $\alpha \in Z(\Delta)$,

we have

$$(a+b)^{\#} = a^{\#} + b^{\#},\ (\alpha a)^{\#} = \alpha a^{\#},\ (ab)^{\#} = a^{\#}b^{\#},\ \mathrm{Id}_X^{\#} = \mathrm{Id}_Y.$$

Thus $\mathcal{L}_Y(X)$ is a unital associative algebra and $\mathcal{F}_Y(X)$ is an ideal of $\mathcal{L}_Y(X)$.

• We will simply write $\mathcal{L}(X)$ and $\mathcal{F}(X)$ instead of $\mathcal{L}_{X^*}(X)$ and $\mathcal{F}_{X^*}(X)$, with respect to the canonical pair (X, X^*).

Proposition 6.2. *Let $\mathcal{P} = (X, Y, \langle\,,\rangle)$ be a pair of dual vector spaces over Δ and let $V \leq X$ and $W \leq Y$. Then (V, W) is a directed subpair of $\mathcal{P}$ if and only if there exist an idempotent $e \in \mathcal{L}_Y(X)$ such that $V = Xe$ and $W = e^{\#}Y$.*

Proof. Let $e \in \mathcal{L}_Y(X)$ be an idempotent. To prove that $(Xe, e^{\#}Y)$ is a directed subpair of $\mathcal{P}$ we only need to show that $(Xe)^{\perp} = \ker(e^{\#})$ and that $(e^{\#}Y)^{\perp} = \ker(e)$. Since the argument is the same in both cases, we will only check the first equality:

$$y \in (Xe)^{\perp} \Leftrightarrow \langle Xe, y\rangle = 0 \Leftrightarrow \langle X, e^{\#}y\rangle = 0 \Leftrightarrow e^{\#}y = 0.$$

Conversely, suppose that (V, W) is a directed subpair of (X, Y), i.e $X = V \oplus W^{\perp}$ and $Y = W \oplus V^{\perp}$. Let $e : X \to X$ the projection on V given by the decomposition $X = V \oplus W^{\perp}$, i.e. $(v + y)e = v$, for all $v \in V$, $y \in W^{\perp}$. Then e is continuous with the adjoint $e^{\#}$ being the projection on W given by the decomposition $Y = W \oplus V^{\perp}$. □

• Let $(X, Y, \langle\,,\rangle)$ be a pair of dual spaces. For $x \in X$, $y \in Y$, write y^*x to denote the map of X defined by

$$(x')y^*x = \langle x', y\rangle x$$

for all $x' \in X$. Then $y^*x \in \mathcal{F}_Y(X)$, with adjoint $(y^*x)^{\#}y' = y\langle x, y'\rangle$ for all $y' \in Y$.

Given $V \leq X$ and $W \leq Y$ we denote by W^*V the subgroup of the abelian group $(\mathcal{F}_Y(X), +)$ generate by the set $\{w^*v : w \in W, v \in V\}$.

The following lemma, whose proof is left to the reader, shows how to handle these linear maps.

Lemma 6.3. *Let $(X, Y, \langle\,,\rangle)$ be a pair of dual spaces over a division algebra Δ.*

(i) *Every $a \in \mathcal{F}_Y(X)$ can be expressed as $a = \sum y_i^* x_i$, where both subsets of vectors $\{y_i\} \subset Y$ and $\{x_i\} \subset X$ are linearly independent, and $y^*\alpha x = (y\alpha)^*x$ for all $\alpha \in \Delta$, $x \in X$, $y \in Y$, which just mean that $\mathcal{F}_Y(X) = Y^*X$ is isomorphic as abelian group to the tensor product $Y \otimes_{\Delta} X$.*

(ii) *$a(y^*x)b = (a^{\#}y)^*xb$, for all $a \in \mathcal{L}_Y(X)$, $b \in \mathrm{End}_{\Delta}(X)$,.*

(iii) *$(y_1^*x_1)(y_2^*x_2) = y_1^*\langle x_1, y_2\rangle x_2$, for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$.*

Remark 6.4. *Let $\mathcal{P} = (X, Y, \langle\,,\rangle)$* be a pair of dual vector spaces over a division algebra Δ and let $\mathcal{P}^{op} = (Y^{op}, X^{op}, \langle\,,\rangle^{op})$ be its opposite pair (over Δ^{op}) (Example 5.3). Then the map $a \mapsto (a^{\#})^{op}$ (where, according our notational convention, $y(a^{\#})^{op} = a^{\#}y$, for all $y \in Y^{op}$, since Y^{op} is a left vector space over Δ^{op}) is an anti-isomorphism of $\mathcal{L}_Y(X)$ onto $\mathcal{L}_{X^{op}}(Y^{op})$.

Exercises

In all the exercises of this section, A will be the associative algebra $\mathcal{F}_Y(X)$ of the finite rank continuous linear maps with respect to a pair $(X, Y, \langle\, ,\rangle)$ of dual vector spaces over a division algebra Δ. Show:

Ex 6.5. A is simple.

Ex 6.6. $\mathcal{R}$ is a right ideal of A if and only if $\mathcal{R} = W^*X$ for some $W \leq Y$.

Ex 6.7. $\mathcal{L}$ is a left ideal of A if and only if $\mathcal{L} = Y^*V$ for some $V \leq X$.

Ex 6.8. $\mathcal{R} = W^*X$ is a principal right ideal of A if and only if W is finite-dimensional. Similarly, $\mathcal{L} = Y^*V$ is a principal left ideal of R if and only if V is finite-dimensional.

Ex 6.9. $0 \neq e \in A$ is an idempotent if and only if $e = \sum_{i=1}^n y_i^* x_i$, where $\{x_i\}_{i=1}^n \subset X$, $\{y_i\}_{i=1}^n \subset Y$ are biorthogonal.

Ex 6.10. $e \in A$ is an idempotent if and only if $e = y^*x$, with $\langle x, y\rangle = 1$.

Ex 6.11. Every nonzero idempotent $e \in A$ is a sum $e = \sum_{i=1}^n e_i$ of division idempotents pairwise orthogonal, with n being the rank of e.

Ex 6.12. Let $e = y^*x$ be a division idempotent of A. Then the map $\alpha \mapsto y^*\alpha x$ is an isomorphism of the division algebras Δ onto eAe.

Ex 6.13. The map $\gamma \mapsto \gamma(e)$ is an isomorphism of the centroid $\Gamma(A)$ onto the center $Z(eAe)$ of division algebra eAe, for any division idempotent e of A.

Ex 6.14. $\Gamma(A) \cong Z(\Delta)$.

7. Trace of a finite-rank linear map

We give in this section a characterization of the trace for finite rank linear maps $a \in \mathcal{F}_Y(X)$. This characterization is *intrinsic* (in the sense that it avoids imbeddings into finite matrices), *elementary* (since it can be easily computed in terms of any representation of a as a finite sum of rank-one operators), and *consistent* with the usual notion of trace of a square matrix over a commutative ring. However, unlike the commutative case (endomorphisms of free modules of finite rank over a commutative unital ring) the trace of a finite-rank linear map on a left vector space over a division algebra Δ is not an element of Δ, but a residue class module $[\Delta, \Delta]$.

Lemma 7.1. *Let $(X, Y, \langle\, ,\rangle)$ be a pair of dual vector spaces over Δ. For any $a \in \mathcal{F}_Y(X)$, the element $\sum_i \langle u_i, w_i\rangle \in \Delta$ is independent up to congruence modulo $[\Delta, \Delta]$ of the chosen representation $a = \sum w_i^* u_i$.*

Proof. Let $a = \sum z_j^* v_j$ $(1 \leq j \leq s)$ be another expression of the a as a sum of rank-one linear maps. If $V \leq X$ denotes the linear span of the set $\{u_i\} \cup \{v_j\}$, and $W \leq Y$ is the linear span of $\{w_i\} \cup \{z_j\}$, we have by Proposition 5.11 that (V, W) can be imbedded in a finite-dimensional subpair of dual vector spaces (X_0, Y_0) of $\mathcal{P}$. Let $\{x_k\} \subset X_0$, $\{y_k\} \subset Y_0$ $(1 \le k \le n)$ be dual bases. Then, for each $1 \le i \le r$,

$$u_i = \sum_k \alpha_{ik} x_k \text{ and } w_i = \sum_k y_k \beta_{ki} \tag{1}$$

where $\alpha_{ik}, \beta_{ki} \in \Delta$, $1 \le k \le n$. Similarly, for each $1 \le j \le s$, there exist $\lambda_{jk}, \mu_{kj} \in \Delta$ such that

$$v_j = \sum_k \lambda_{jk} x_k \text{ and } z_j = \sum_k y_k \mu_{kj}. \tag{2}$$

Fix $1 \le l \le n$ and compute $x_l a$ using (1). We have

$$\begin{aligned} x_l a &= x_l(\sum_i w_i^* u_i) = \sum_i g(x_l, w_i) u_i \\ &= \sum_i \langle x_l, \sum_k y_k \beta_{ki} \rangle (\sum_k \alpha_{ik} x_k) = \sum_k (\sum_i \beta_{li} \alpha_{ik}) x_k. \end{aligned}$$

In particular, the lth-coordinate of $x_l a$ with respect to the basis $\{x_k\}$ is given by $\sum_i \beta_{li}\alpha_{il}$. Taking the sum of all these, we get the element $\sum_l(\sum_i \beta_{li}\alpha_{il})$ of Δ which only depends on a and the chosen dual bases $\{x_k\}, \{y_k\}$ of (V, W). Therefore,

$$\sum_{i,l} \beta_{li}\alpha_{il} = \sum_{j,l} \mu_{lj}\lambda_{jl} \tag{3}$$

for $1 \le i \le r$, $1 \le l \le n$ and $1 \le j \le s$.

Using (1) we compute $\tau_1(a) := \sum_i \langle u_i, w_i \rangle$:

$$\tau_1(a) = \sum_i \langle u_i, w_i \rangle = \sum_i \langle \sum_k \alpha_{ik} x_k, \sum_l y_l \beta_{li} \rangle = \sum_{i,l} \alpha_{il}\beta_{li}. \tag{4}$$

Similarly, by using (2) instead of (1), we get

$$\tau_2(a) = \sum_j \langle v_j, z_j \rangle = \sum_{j,l} \lambda_{jl}\mu_{lj}. \tag{5}$$

Then it follows from (3), (4) and (5) that

$$\begin{aligned} \tau_1(a) - \tau_2(a) &= \sum_{i,l} \alpha_{il}\beta_{li} - \sum_{i,l} \beta_{li}\alpha_{il} + \sum_{j,l} \mu_{lj}\lambda_{jl} \\ &- \sum_{j,l} \lambda_{jl}\mu_{lj} = \sum_{i,l} [\alpha_{il}, \beta_{li}] + \sum_{j,l} [\mu_{lj}, \lambda_{jl}] \in [\Delta, \Delta]. \end{aligned}$$

□

- For $a = \sum_i y_i^* x_i \in \mathcal{F}_Y(X)$, set

$$\operatorname{tr}(a) := \overline{\sum_i \langle x_i, y_i \rangle} := \sum_i \langle x_i, y_i \rangle + [\Delta, \Delta]. \tag{6}$$

Note that

Proposition 7.2. *The map* $\mathrm{tr} : \mathcal{F}_Y(X) \to \frac{\Delta}{[\Delta,\Delta]}$ *is well defined, and it is clearly* $Z(\Delta)$*-linear and onto. Moreover, if* $a = \sum y_i{}^* \alpha_{ij} x_j$*, where* $\{x_i\}$*,* $\{y_i\}$ *are biorthogonal, we have;*

$$\mathrm{tr}(a) = \sum_i \alpha_{ii} + [\Delta, \Delta],$$

where $\sum_i \alpha_{ii}$ *coincides with the usual trace of the matrix* $(\alpha_{ij}) \in \mathrm{M}_n(\Delta)$*, with* n *being the rank of* a*.*

8. Structure theorems

Teorema 8.1. [5, IV.9, Structure theorem] *Up to isomorphism, an associative algebra* A *is prime with nonzero socle if and only if there exists a pair* $(X, Y, \langle\,,\rangle)$ *of dual vector spaces over a division algebra* Δ *such that*

$$\mathcal{F}_Y(X) \triangleleft A \leq \mathcal{L}_Y(X).$$

In this case, $\mathrm{Soc}(A) = \mathcal{F}_Y(X)$*,* A *is right and left primitive, and the division algebra* Δ *is uniquely determined by* A *up to isomorphism.*

Proof. Suppose that $\mathcal{F}_Y(X) \triangleleft A \leq \mathcal{L}_Y(X)$, where $(X, Y, \langle\,,\rangle)$ is a pair of dual vector spaces over a division algebra Δ. We claim that A is prime with minimal one-sided ideals. Let $a, b \in A$ be such that $aAb = 0$. Then

$$0 = a\mathcal{F}_Y(X)b = a(Y^*X)b = (a^\# Y)^*(Xb).$$

If $b \neq 0$, then $a^\# = 0$, equivalently, $a = 0$, which proves that A is prime. Let us now see that for any $0 \neq y \in Y$, y^*X is a minimal right ideal of A. Clearly y^*X is closed under addition, and for any $a \in A$ we have $(y^*X)a = y^*(Xa) \subset y^*X$, so y^*X is a right ideal. Moreover, if $x \neq 0$, then $(y^*x)A = y^*xA = y^*X$, since $\mathcal{F}_Y(X) \subset A$, which proves that y^*X is minimal.

Conversely, suppose that A is a prime associative algebra with nonzero socle. Then A contains a division idempotent e, so we can form the pair of dual vector spaces $(eA, Ae, \langle\,,\rangle)$, $\langle x, y\rangle = xy$, for all $x \in eA$, $y \in Ae$ (see Example 5.2). Define the map $\varphi : A \to \mathrm{End}_{eAe}(eA)$ by $x\varphi_a = xa$ for all $x \in eA$, $a \in A$. It is not difficult to see:

(i) for every $a \in A$, $\varphi_a \in \mathcal{L}_{Ae}(eA)$, with $\varphi_a^\# y = ay$ for all $y \in Ae$,
(ii) φ is a monomorphism: $\varphi_a = 0 \Rightarrow eAa = 0 \Rightarrow a = 0$, by primeness of A, and
(iii) for any $x \in eA$, $y \in Ae$, we have $y^*x = \varphi_{yx}$, which just means that

$$\mathcal{F}_{Ae}(eA) = (Ae)^*eA = \varphi_{AeA} = \varphi_{\mathrm{Soc}(A)},$$

as required. That every prime associative algebra A with nonzero socle is primitive can be proved in purely algebraic terms. Let e be a division idempotent of A. Then $(1-e)A$ (which makes sense even if A has no a unit element) is a maximal right ideal of A, and since any nonzero ideal of M contains $\mathrm{Soc}(A)$, $(1-e)A$ cannot contain nonzero ideals of A, which proves that A is right primitive. That A is left primitive follows by the symmetry of this argument.

Finally, as proved in [5, IV.11.Corollary 1], the pair of dual vector spaces $(X, Y, \langle\, ,\rangle)$ is uniquely determined up to equivalence (see [5, IV.11, page 79] for definition) by the algebra A. In particular, as also seen in Exercise 6.12, the division algebra Δ is isomorphic to eAe for every division idempotent e, and the corners generated by two division idempotents are isomorphic (Lemma 3.5). □

As a direct consequence of Theorem 8.1, we get

Corollary 8.2. *Up to isomorphism, an associative algebra A is simple and contains minimal one-sided ideals if and only if $R = \mathcal{F}_Y(X) = Y^*X$, for a pair $(X, Y, \langle\, ,\rangle)$ of dual vector spaces over a division algebra Δ.*

A natural example of simple associative algebra coinciding with its socle is the the algebra $\mathrm{M}_\infty(\Delta)$ of infinite matrices with a finite number of nonzero entries over a division algebra Δ. Note that $\mathrm{M}_\infty(\Delta) = \cup_{n=1}^{\infty}\mathrm{M}_n(\Delta)$ is clearly a simple associative algebra, and that it contains minimal one-sided ideals. Take, for instance, the principal left ideal generated by the entry matrix [11].

Proposition 8.3. [5, IV.15, Theorem 2] *Let $(X, Y, \langle\, ,\rangle)$ be a pair of dual vector spaces over Δ such that X and Y are countably infinite-dimensional. Then $\mathcal{F}_Y(X)$ is isomorphic to* $\mathrm{M}_\infty(\Delta)$.

Proof. By Mackey Lemma 5.9, there exist bases $\{x_n\}_{n=1}^{\infty} \subset X$ and $\{y_n\}_{n=1}^{\infty} \subset Y$ which are biorthogonal. Then every $a \in \mathcal{F}_Y(X) = Y^*X$ has a unique expression of the form $\sum y_i^*\alpha_{ij}x_j$, with $\alpha_{ij} = 0$ up to a finite number of pairs $(i,j) \in \mathbb{N}\times\mathbb{N}$, and the rank-one linear maps $y_i^*x_j$ multiply as entry matrices: $(y_i^*x_j)(y_l^*x_k) = (y_i^*\langle x_j, y_l\rangle x_k = 0$ if $j \neq l$ and $(y_i^*x_j)(y_j^*x_k) = y_i^*x_k$. Hence the map $a = \sum y_i^*\alpha_{ij}x_j \mapsto (\alpha_{ij})$ is an algebra-isomorphism of $\mathcal{F}_Y(X)$ onto $\mathrm{M}_\infty(\Delta)$. □

Corollary 8.4. (Wedderburn-Artin) *For any nonzero associative algebra A the following conditions are equivalent:*

(1) *A is prime and Artinian.*
(2) *A is isomorphic to a full matrix algebra $\mathrm{M}_n(\Delta)$, where Δ is a division algebra and $n \geq 1$.*
(3) *A is simple, coincides with its socle, and has a unit element.*

Proof. Let A be prime and Artinian. Then A has nonzero socle. Hence, by Theorem 8.1, there exists a pair of dual vector spaces $(X, Y, \langle\, ,\rangle)$ over a division algebra Δ such that

$$\mathcal{F}_Y(X) \triangleleft A \leq \mathcal{L}_Y(X)$$

If X were infinite-dimensional, then X would contain a strictly descending chain

$$V_1 \supset V_2 \supset \cdots$$

of subspaces, which would yield a strictly descending chain

$$Y^*V_1 \supset Y^*V_2 \supset \cdots$$

of left ideals of A, a contradiction. Thus X has finite dimension, say n. Then $\mathcal{F}_Y(X) = \mathcal{L}_L(X) = \mathrm{End}_\Delta(X) \cong \mathrm{M}_n(\Delta)$, which proves that (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) is clear, and (3) $\Rightarrow$ (i) was proved in Corollary 3.9 □

Remark 8.5. If A is simple and coincides with its socle, then two elements $a, b \in A$ have equal rank if and only if $A_a \cong A_b$. In the particular case of an algebra $A = \mathrm{M}_n(\Delta)$ of $n\times n$ matrices with entries in a division algebra Δ, it is well known that two matrices a and b have the same rank if and only if they are *equivalent*, that is, there exist invertible matrices p and q in A such that $b = paq$. This can also be expressed by the isomorphism $A_a \to A_b$ given by $\overline{x} \mapsto \overline{q^{-1}xp^{-1}}$.

Teorema 8.6. (Litoff) *Let $A = \mathcal{F}_Y(X)$, where $(X, Y, \langle\,,\rangle)$ is a pair of dual vector spaces over Δ. We have:*

1. *For every idempotent $e \in A$, the unital algebra eAe is isomorphic to the matrix algebra $\mathrm{M}_r(\Delta)$, with $r = \mathrm{rank}(e)$.*
2. *For any finite subset $\{a_1, a_2, \ldots, a_n\}$ of elements of A, there exists and idempotent $e \in A$ such that $\{a_1, a_2, \ldots, a_n\} \subset eAe$.*

Proof. (1) By Proposition 6.2, $(Xe, e^{\#}Y)$ is a (finite-dimensional) direct subpair of $(X, Y, \langle\,,\rangle)$. Then $eAe = e(Y^*X)e = (e^{\#}Y)^*(Xe) = \mathcal{F}_{e^{\#}Y}(Xe) \cong \mathrm{M}_r(\Delta)$, with $r = \mathrm{rank}(e)$.

(2) Set $S = \cup_{i=1}^n Xa_i \subset X$ and $T = \cup_{i=1}^n a_i^{\#}Y \subset Y$. Then S and T generate respectively finite-dimensional subspaces $V \leq X$ and $W \leq Y$. By Proposition 5.11(i), (V, W) can be embedded in a finite-dimensional subpair (X_0, Y_0) of $(X, Y, \langle\,,\rangle)$. Since every finite-dimensional subpair is direct (5.11(ii)), $X_0 = Xe$ and $Y_0 = e^{\#}Y$ for an idempotent $e \in A$. Hence, by (1), $\{a_1, a_2, \ldots, a_n\} \subset (e^{\#}Y)^*(Xe) = eAe$. □

Remark 8.7. A ring R is isomorphic to the complete ring of linear maps of finite rank of a vector space over a division ring if and only if R is a simple ring containing minimal one-sided ideals and such that every left ideal is a left annihilator [5, IV.16.Theorem 3].

9. Martindale theorem's and PI-elements

Throughout this section, A will be a prime associative algebra (over a ring of scalars Φ, so by taking $\Phi = \mathbb{Z}$ we include the case that A is a ring), with Γ denoting its centroid, $\mathcal{C}$ its extended centroid, and $\tilde{A}$ its central closure. Notice that by [4, Propositions 1.11 and 1.19], both notions are independent of the underlying ring of scalars Φ. Furthermore, since $\tilde{A} = \mathcal{C}A$, the $\mathcal{C}$-algebra $\tilde{A}$ is PI if and only if so is the Γ-algebra A.

Definition 9.1. An element $a \in A$ is called a PI-*element* if A_a is PI, i.e. if A_a satisfies a polynomial identity. We denote by $\mathrm{PI}(A)$ the set of all PI-elements of A.

Proposition 9.2. *Let A be a centrally closed prime algebra over a field $\mathbb{F}$ and let $0 \neq a \in \mathrm{PI}(A)$. Then A_a is a finite-dimensional central simple $\mathbb{F}$-algebra. So A has nonzero socle and its associated division algebra is finite-dimensional over $\mathbb{F}$.*

Proof. Since A_a is PI, we have by Posner's theorem [3, Theorem 8.6.6.] that $Z(A_a) \neq 0$ and the central closure of A_a is a finite-dimensional central simple $\mathbb{F}$-algebra. But by Lemma 2.7, $Z(A_a)$ is itself the field $\mathbb{F}$, so A_a coincides with is central closure. Hence, by Corollary 4.11, A has nonzero socle and its associated division algebra is finite-dimensional over $\mathbb{F}$. □

Following [2, 6.1], let $Q_s(A)_{\mathcal{C}}\langle X\rangle$ be the coproduct of the Martindale symmetric algebra $Q_s(A)$ and the free associative $\mathcal{C}$-algebra $\mathcal{C}\langle X\rangle$, where X is an infinite set.

Definition 9.3. *Let U be an additive subgroup of A.* A nonzero element $f(x_1, \ldots, x_n) \in Q_s(A)_{\mathcal{C}}\langle X\rangle$ is said to be a *generalized polynomial* (in short, GPI) *on U* if $f(u_1, \ldots, u_n) = 0$ for all $u_1, \ldots, u_n \in U$.

• Anh and Márki gave in [1] internal characterizations of these rings eliminating the central closure. In fact, as will be seen next, a prime associative algebra A satisfies a GPI if and only if A contains a nonzero PI-element.

Lemma 9.4. *Every nonzero* PI-*element $a \in A$ gives rise to a* GPI*.*

Proof. Let $p(x_1, \ldots, x_n) = 0$ be a multilinear identity which is satisfied by A_a. Then $p(ax_1a, \ldots, ax_na)$ is a GPI on A. □

Conversely, by Martindale's theorem [2, Theorem 6.1.6], if a prime associative algebra A satisfies a GPI on a nonzero ideal I, then its central closure $\tilde{A}$ contains a division idempotent e such that $e\tilde{A}e$ is finite-dimensional over the extended centroid of A. As we will prove now, this implies the existence of a nonzero PI-element in A.

Definition 9.5. Let A be an associative algebra. An element $a \in A$ is called *square cancellable* if $a^2x = 0 \Rightarrow ax = 0$ and $xa^2 = 0 \Rightarrow xa = 0$.

Proposition 9.6. *Let $a \in A$ be square cancellable. Then A_{a^2} is isomorphic to the subalgebra aAa of A.*

Proof. Let $\varphi : A_{a^2} \to aAa$ given by $\varphi(\overline{x}) = axa$, for all $x \in A$. Since a is square cancellable, $\overline{x} = 0 \Leftrightarrow a^2xa^2 = 0 \Leftrightarrow axa = 0$, which proves that φ is well defined and one-to-one. Thus we only need to see that φ preserves the product. For $x, y \in A$, we have $\varphi(\overline{x} \cdot_{a^2} \overline{y}) = \varphi(\overline{xa^2y}) = axa^2ya = (axa)(aya) = \varphi(\overline{x})\varphi(\overline{y}.)$ □

Lemma 9.7. *Let A be prime, $B \supset A$ a $\mathcal{C}$-algebra, and $e \in B$ a division idempotent of B. Then any nonzero element $a \in eBe \cap A$ is square-cancellable. Moreover, if eBe is finite-dimensional, then $a^2 \in \mathrm{PI}(A)$.*

Proof. Let $0 \neq a \in eBe \cap A$ be such that $a^2x = 0$ for some $x \in A$. Since eBe is a division algebra (with e as unit element), there exists $b \in eBe$ such that $ba = e$. Hence

$ba^2x = eax = ax = 0$. Similarly, $xa^2 = 0 \Rightarrow xa = 0$, so a is square cancellable. Finally, if eBe is finite-dimensional over $\mathcal{C}$ (and therefore PI), then $aAa \subset eBe$ is PI. Hence, by Proposition 9.6, $A_{a^2} \cong aAa$ is PI, i.e. $a^2 \in \mathrm{PI}(A)$. □

Martindale's theorem of prime algebras satisfying a GPI can now be rephrased as follows.

Teorema 9.8. *Let A be a prime associative algebra with extended centroid $\mathcal{C}(A)$ and central closure $\tilde{A}$. Then $\mathrm{PI}(A) = \mathrm{PI}(\tilde{A}) \cap A$. Moreover, the following conditions are equivalent:*

(i) $\mathrm{PI}(A) \neq 0$,
(ii) *$\tilde{A}$ has nonzero socle and its division algebra is finite-dimensional over $\mathcal{C}(A)$.*

In this case,

(iii) $\mathcal{C}(A) \cong \mathcal{C}(A_a) \cong Z(A_a)^{-1}Z(A_a)$, *for any $0 \neq a \in \mathrm{PI}(A)$, and*
(iv) $\mathrm{PI}(A) = \mathrm{Soc}(\tilde{A}) \cap A$.

Proof. Let $a \in \mathrm{PI}(A)$, Since $\tilde{A}$ is generated by A as a $\mathcal{C}(A)$-vector space, $\tilde{A}_a$ is generated by A_a as a $\mathcal{C}(A)$-vector space, so $\tilde{A}_a$ is PI if and only if so is A_a. Thus $\mathrm{PI}(A) = \mathrm{PI}(\tilde{A}) \cap A$.

(i) ⇒ (ii). Let $0 \neq a \in \mathrm{PI}(A) \subset \mathrm{PI}(\tilde{A})$. Since $\tilde{A}$ is a centrally closed prime algebra over $\mathcal{C}(A)$, we have by Proposition 9.2 that $\tilde{A}$ has nonzero socle and its associated division algebra is finite-dimensional over $\mathcal{C}$.

(ii) ⇒ (i). Let e be a division idempotent of $\tilde{A}$. Then there exists a nonzero ideal I of A such such $0 \neq eIe \subset eBe \cap A$. Now Lemma 9.7 implies that A contains a nonzero PI-element.

(iii) Arguing as in the proof of Proposition 2.6, we prove that $\mathcal{C}(A_a)$ is a field extension of $\mathcal{C}(A)$. Conversely, by [3, Theorem 8.6.5], $Z(A_a) \neq 0$. Hence, by [4, Propostion 1.20], $\mathcal{C}(A_a) \cong Z(A_a)^{-1}Z(A_a)$. Then

$$\mathcal{C}(A) \leq \mathcal{C}(A_a) \cong Z(A_a)^{-1}Z(A_a) \leq \mathcal{C}(A)$$

since $Z(A_a) \leq Z(\tilde{A}_a) = \mathcal{C}(A)$ [4, Lemma 1.22].

(iv) As just seen, $a \in \mathrm{PI}(A) \subset \mathrm{PI}(\tilde{A})$ implies $a \in \mathrm{Soc}(\tilde{A})$. Conversely, let $a \in A \cap \mathrm{Soc}(\tilde{A})$. Then $\tilde{A}_a$ is a simple Artinian algebra whose division algebra is finite-dimensional. Then $A_a \leq \tilde{A}_a$ is PI. □

Exercises

Ex 9.9. Let A be an associative algebra and $0 \neq a \in A$. Show that the following conditions are equivalent.

(i) $a \in a^2Aa^2$.
(ii) There exists $b \in A$ such that $ab = ba$, $a = aba$ and $b = bab$.
(iii) There exists an idempotent $e \in A$ such that a is invertible in eAe.

In this case, the elements b and e are uniquely determined, with b being the inverse of a in eAe and calling the *group inverse* of A. An element a satisfying these equivalent conditions will be called *locally invertible.*

Ex 9.10. Show that every locally invertible element is square cancellable and that if A is semiprime and $a \in \mathrm{Soc}(A)$ is square cancellable, then a is locally invertible.

10. Finitary associative algebras

Definition 10.1. Let A be an associative algebra over a field $\mathbb{F}$. Then A is said to be *finitary* (over $\mathbb{F}$) if it is isomorphic to a subalgebra of the algebra $\mathcal{F}(X)$ of all finite rank linear maps of a vector space X over $\mathbb{F}$.

Teorema 10.2. *Let A be an associative algebra over a field $\mathbb{F}$. Then the following conditions are equivalent:*

(i) *A is simple and finitary.*
(ii) *A is simple and for every $a \in A$, the local algebra A_a of A at a is finite-dimensional.*
(iii) *A is simple and contains a nonzero finite-dimensional* J*-inner ideal.*
(iv) *A is simple and for every division idempotent $e \in A$ the division algebra eAe is finite-dimensional.*
(v) *A is isomorphic to the algebra $\mathcal{F}_Y(X)$ of finite rank continuous linear operators relative to a pair (X, Y) of dual vector spaces over a division associative algebra Δ which is finite-dimensional over $\mathbb{F}$.*

Proof. (i) $\Rightarrow$ (ii). Let $A \leq \mathcal{F}(X)$ be simple. Then for any $a \in A$,

$$aAa \subset a\mathcal{F}(X)a = (a^{\#}X^*)^*(Xa)$$

is finite-dimensional and hence A_a is finite-dimensional, since, by Lemma 4.2, A_a is isomorphic to aAa as a vector space.

(ii) $\Rightarrow$ (iii). For any nonzero $a \in A$, A_a is finite-dimensional. Hence, again by the linear isomorphism $A_a \cong aAa$, aAa is a nonzero finite-dimensional J-inner ideal.

(iii) $\Rightarrow$ (iv). Suppose that A contains a nonzero finite-dimensional J-inner ideal. Then A contains a finite-dimensional minimal J-inner ideal, say aAa. Let e be a division idempotent in A. By Proposition 2.3 and Corollary 4.6, $eAe \cong A_e \cong A_a$ is finite-dimensional.

(iv) $\Rightarrow$ (v). Let $e \in A$ be a division idempotent such that eAe is finite-dimensional. By Theorem 8.1, A is isomorphic to $\mathcal{F}_Y(X)$, where (X, Y) is the pair of dual vector spaces defined by the division idempotent e.

(v) $\Rightarrow$ (i). Let $A = \mathcal{F}_Y(X)$, where (X, Y) is a pair of dual vector spaces over a division algebra Δ which is finite-dimensional over $\mathbb{F}$. Then A can be regarded as a subalgebra of $\mathcal{F}({}_{\mathbb{F}}X)$ and therefore it is finitary over $\mathbb{F}$. □

Teorema 10.3. *Let A be an associative algebra a field $\mathbb{F}$. Then the following conditions are equivalent:*

(i) *A is semiprime and every principal inner ideal of A is finite-dimensional.*
(ii) *A is the direct sum of ideals each of which is a simple finitary associative algebra.*

Proof. (i) $\Rightarrow$ (ii). Suppose that for any $a \in A$ the principal inner ideal aAa is finite-dimensional. Then, by Proposition 2.5 and Lemma 4.2, A_a is a semiprime finite-dimensional associative algebra over $\mathbb{F}$. In particular, A_a is right Artinian and hence, by Theorem 4.10, $a \in \mathrm{Soc}(A)$. Moreover, since for every division idempotent $e \in A$, eAe is finite-dimensional, we have by Corollary 8.2 that every simple component M_i of the socle of A is an algebra $\mathcal{F}_{Y_i}(X_i)$ of finite-rank continuous linear maps with respect to a pair (X_i, Y_I) of dual vector spaces over a division algebra $\Delta_i \cong e_iAe_i$ (Exercise 6.12) which is finite-dimensional over $\mathbb{F}$. This implies that each M_i is finitary over $\mathcal{F}$, and that if A is in fact simple, then $A \cong \mathcal{F}_Y(X)$, where (X, Y) is a pair of dual vector spaces over a division associative algebra Δ which is finite-dimensional over $\mathbb{F}$.

(ii) $\Rightarrow$ (i). Suppose that $A = \oplus M_i$ is a direct sum of ideals each of which is a simple finitary algebra. For every $a = \sum a_i \in A$, with $a_i \in M_i$ for all i and where only finitely many a_i are nonzero, we have by the theorem above that $aAa = \bigoplus a_iM_ia_i$ is finite-dimensional. □

Remark 10.4. If A is prime and contains a nonzero PI-element, then the socle of its central closure is a central simple finitary algebra.

11. Algebras with involution and nonzero socle

Throughout this section we will assume that 2 is invertible in the underlying ring of scalars Φ. Recall that for a *ring involution* of an associative algebra A we mean an involution of A regarded as a ring, that is, we ignore how $*$ behaves with respect to the scalar product.

Proposition 11.1. *Let A be a prime associative algebra with nonzero socle. For a ring involution $*$ of A the following conditions are equivalent:*

(i) *A contains a symmetric division element a such that $a^2 \neq 0$.*
(ii) *A contains a division element b such that $bb^* \neq 0$.*
(iii) *A contains a symmetric division idempotent e.*

Proof. To prove (i) $\Rightarrow$ (ii) take $b = a$, and for (iii) $\Rightarrow$ (i) take $a = e$. So we have just to show (ii) $\Rightarrow$ (iii). Let $b \in A$ be a division element such that $bb^* \neq 0$. It follows from Corollary 4.9 that $b^*Ab = eAe$ for a division idempotent e. Since the division algebra b^*Ab is $*$-invariant, $e = e^*$. □

Definition 11.2. Let A be a prime associative algebra with nonzero socle. A ring involution $*$ of A is of *transpose type* if it satisfies the equivalent conditions of Proposition 11.1. Otherwise $*$ is said to be of *symplectic type*.

Let Δ be a division algebra with a ring involution $\alpha \mapsto \overline{\alpha}$ and let X be a left vector space over Δ. Recall that a *Hermitian form* on X is a map $(x, y) \mapsto \langle y, x\rangle$ of $X \times X \to \Delta$ satisfying the following conditions:

(i) It is linear in the first variable,

(ii) $\langle y, x\rangle = \overline{\langle x, y\rangle}$ for all $x, y \in X$.

If instead of (ii), the form $\langle\, ,\rangle$ satisfies $\langle y, x\rangle = -\overline{\langle x, y\rangle}$, then it is called *skew Hermitian.* In the particular case that the ring involution of Δ is the identity (and therefore, Δ is a field), the Hermitian (resp. skew Hermitian) form is called *symmetric* (resp. *alternate*). A Hermitian or skew Hermitian form $\langle\, ,\rangle$ is said to be *nondegenerate* if $\langle x, X\rangle = 0 \Rightarrow x = 0$, equivalently, $\langle X, x\rangle = 0 \Rightarrow x = 0$.

Every left vector space X with a nondegenerate Hermitian or skew Hermitian form $\langle\, ,\rangle$ becomes a pair of dual vector spaces $(X, \overline{X}, \langle\, ,\rangle)$, where $\overline{X}$ coincides with X as an abelian group but it is regarded as a right vector space over Δ by defining $x \cdot \alpha = \overline{\alpha}x$, for all $\alpha \in \Delta$, $x \in X$. Set $\mathcal{L}_X(X) := \mathcal{L}_{\overline{X}}(X)$ and $\mathcal{F}_X(X) := \mathcal{F}_{\overline{X}}(X)$ and observe that the adjoint map $a \mapsto a^{\#}$ is actually a ring involution on $\mathcal{L}_X(X)$, which will be denoted by $a \mapsto a^*$.

The following lemma, whose proof is left to the reader, shows how finite linear maps behave with respect to the adjoint involution.

Lemma 11.3. *Let X be a left vector space with a nondegenerate Hermitian (resp. skew Hermitian) form $\langle\, ,\rangle$ over $(\Delta, -)$, $\mathrm{char}(\Delta) \neq 2$. For any $\alpha \in \Delta$, $x, y \in X$, we have*

(1) *$(\alpha x)^* y = x^* \overline{\alpha} y$ and $(x^* y)^* = y^* x$ (resp. $(x^* y)^* = -y^* x$). Hence*

(2) *the linear map $[x, y] := x^* y - y^* x$ (resp. $[x, y] := x^* y + y^* x$) belongs to $\mathrm{Skew}(\mathcal{F}_X(X), *)$ and it is called a skew trace.*

If V, W are subspaces of X, we write $[V, W]$ to denote the set of all finite sums of the skew traces $[v_i, w_i]$, for all $v_i \in V$, $w_i \in W$. With this notation, we have

(3) $\mathrm{Skew}(\mathcal{F}(X), *) = [X, X]$.

(4) *If $\langle\, ,\rangle$ is skew-Hermitian, then $x^* x = [(1/2)x, x]$ is a skew-trace for any $x \in X$. In fact, for any skew-trace $[x, y]$ we have, $[x, y] = (x + y)^*(x + y) - x^* x - y^* y$.*

(5) *If $\langle\, ,\rangle$ is Hermitian, then $(\alpha x)^* x$ is a skew-trace if and only if $\overline{\alpha} = -\alpha$.*

Proposition 11.4. *Let X be a left vector space with a nondegenerate Hermitian (resp. alternate) form $\langle\, ,\rangle$ over a division algebra Δ with a ring involution, $\mathrm{char}(\Delta) \neq 2$. Then the adjoint involution of the prime associative algebra $\mathcal{L}_X(X)$ is of transpose (resp. symplectic) type.*

Proof. If $\langle\, ,\rangle$ is Hermitian, we can choose a non-isotropic vector $x \in X$. Then $a := x^* x \in \mathcal{F}_X(X)$ is a symmetric division element with $a^2 = x^*\langle x, x\rangle x \neq 0$, so the adjoint involution $*$ is of transpose type.

If $\langle\, ,\rangle$ is alternate, then any $x \in X$ is isotropic, and since the division elements a of $\mathcal{L}_X(X)$ are precisely the rank-one linear maps, i.e. $a = y^* x$, x, y nonzero vectors of X, $aa^* = (y^* x)(y^* x)^* = -y^*\langle x, x\rangle y = 0$, which proves that the adjoint involution $*$ is of symplectic type. □

The following definition is borrowed from the terminology of Jordan pairs.

Definition 11.5. Let A be an associative algebra over Φ. A pair (a,b) of elements of A will be called a *Jordan pair idempotent* if $a = aba$ and $b = bab$.

If (a,b) is a Jordan pair idempotent, then a and b are von Neumann regular. Conversely, every von Neumann regular element $a = axa$ can be extended to a Jordan pair idempotent (a,b) by taking $b = xax$. Moreover, if A has an involution $*$, $\frac{1}{2} \in \Phi$, and u is symmetric (resp. skew-symmetric), replacing x by $\frac{1}{2}(x+x^*)$ (resp. $\frac{1}{2}(x-x^*)$, we get both components of the Jordan pair idempotent (a,b) symmetric (resp. skew-symmetric).

The following lemma is a useful tool to prove the converse of Proposition 11.4.

Lemma 11.6. *Let A be a prime associative algebra with a ring involution $*$ and let $e \in A$ be a division idempotent. We have:*

(i) *For any nonzero element $u \in eAe^*$ there exists $v \in e^*Ae$ such that (u,v) is a Jordan pair idempotent. In this case, $uv = e$ and $vu = e^*$.*

(ii) *If $0 \neq u \in eAe^*$ is symmetric (resp. skew-symmetric), then u can be extended to a Jordan pair idempotent (u,v), where $v \in e^*Ae$ is symmetric (resp. skew-symmetric).*

Let $(u,v) \in eAe^* \times e^*Ae$ be a nonzero Jordan pair idempotent where both u,v are symmetric (resp. skew-symmetric). Denote by Δ the division algebra eAe and by X the left vector eA space over Δ. Then:

(iii) *The map $\tau : \Delta \to \Delta$ defined by $\alpha^\tau = u\alpha^* v$ for all $\alpha \in \Delta$ is a ring involution.*

(iv) *The map $\langle\,,\rangle : X \times X \to \Delta$ defined by $\langle x,y\rangle = xy^*v$, $x,y \in X$, is a nondegenerate Hermitian (or skew-Hermitian) form.*

Proof. (i) Let $u \in eAe^*$ be nonzero. By Corollary 4.9, eAe^* is a minimal J-inner ideal, so there exists $x \in A$ such that $u = uxu = u(e^*xe)u$. Taking $v := (e^*xe)u(e^*xe)$ we have that (u,v) is the required Jordan pair idempotent. Since $uv \in eAe$ is an idempotent, $uv = e$ (because eAe is a division algebra). Similarly, $vu = e^*$.

(ii) Let $0 \neq u \in eAe^*$ be symmetric (resp. skew symmetric). By Definition 11.5, we can choose the two components of (u,v) symmetric (resp. skew-symmetric). *We will assume that this is the case in what follows.*

(iii) Let $\alpha, \beta \in \Delta := eAe$, and let $\tau : \Delta \to \Delta$ be the map defined by $\alpha^\tau = u\alpha^*v$. Clearly, $(\alpha+\beta)^\tau = \alpha^\tau + \beta^\tau$. We also have

$$(\alpha\beta)^\tau = u(\alpha\beta)^*v = u\beta^*\alpha^*v = u\beta^*e^*\alpha^*v = u\beta^*vu\alpha^*v = \beta^\tau\alpha^\tau,$$

and $\alpha^{\tau^2} = u(u\alpha^*v)^*v = u(v^*\alpha u^*)v = u(v\alpha u)v = e\alpha e = \alpha$, which proves that τ is a ring involution of the division algebra Δ.

(iv) Set $X = eA$, regarded as a left vector space over the division algebra $\Delta = eAe$, and define $\langle x,y\rangle := xy^*v$ for all $x,y \in X$. Clearly, $\langle\,,\rangle$ is linear in the first variable and

$$\langle x,y\rangle^\tau = u(xy^*v)^*v = uv^*yx^*v = \pm eyx^*v = \pm yx^*v = \pm\langle y,x\rangle,$$

which proves that $\langle\,,\rangle$ is a Hermitian if $v^* = v$), (skew Hermitian if $v^* = -v$) form. Since A is prime, $\langle X, y\rangle = eAy^*v = 0$ implies $y^* = y^*e^* = y^*vu = 0$, so $y = 0$, which proves that $\langle\,,\rangle$ is nondegenerate. □

Remark 11.7. Let Δ be a division algebra with a ring involution $\alpha \mapsto \overline{\alpha}$, and let X a left vector space with Hermitian (resp. skew Hermitian) form $\langle\,,\rangle$ over $(\Delta, -)$. If there exists $0 \neq \xi \in \mathrm{Skew}(\Delta, -)$, then the map $\alpha \mapsto \widetilde{\alpha}$ defined by $\widetilde{\alpha} := \xi^{-1}\overline{\alpha}\xi$, $\alpha \in \Delta$, is also a ring involution of Δ. Moreover, $\langle x, y\rangle^{\xi} := \langle x, y\rangle\xi$, $x, y \in X$, defines a skew Hermitian (resp. Hermitian) form on X over Δ with the *widetilde* ring involution, without changing the adjoint involution in $\mathcal{L}_X(X)$. So, when dealing with a ring involutions $*$ of a prime associative algebra with nonzero socle, we may suppose, according to our convenience, that $*$ comes from a nondegenerate Hermitian or alternate form, or that $*$ comes from a nondegenerate skew Hermitian or symmetric form.

Teorema 11.8. (Kaplansky) *The $*$-subalgebras A of $\mathcal{L}_X(X)$ containing $\mathcal{F}_X(X)$, relative to a nondegenerate Hermitian or alternate form, are precisely the prime algebras with ring involution and nonzero socle. Moreover, transpose involutions correspond to Hermitian forms and symplectic involutions to alternate forms*

Proof. Suppose that A is a prime associative algebra with nonzero socle and ring involution $*$ and let $e \in A$ be a division idempotent. By Lemma 11.6, there exists a Jordan pair idempotent $(u, v) \in eAe^* \times e^*Ae$, where both u, v are symmetric or skew-symmetric. Take Δ, τ, X and $\langle\,,\rangle$ as previously defined, where $\langle\,,\rangle$ is Hermitian if u, v are symmetric, and skew-Hermitian if they are skew-symmetric. Now let $\varphi : A \to \mathrm{End}({}_\Delta X)$ be the map given by $x\varphi_a = xa$ for all $x \in X$, $a \in A$. We have:

(i) for every $a \in A$ and $x, y \in X$, $\langle x\varphi_a, y\rangle = (xa)y^*v = x(ya^*)^*v = \langle x, ya^*\rangle = \langle x, y\varphi_{a^*}\rangle$, which proves that $\varphi_a \in \mathcal{L}_X(X)$ with $\varphi_a^* = \varphi_{a^*}$,

(ii) φ is a monomorphism: $\varphi_a = 0 \Rightarrow eAa = 0 \Rightarrow a = 0$, by primeness of A,

(iii) for any x', $x = ea$, $y = eb \in X = eA$, we have

$$(x')y^*x = \langle x', y\rangle x = x'(eb)^*v(ea) = x'\varphi_{b^*e^*vea} = x'\varphi_{b^*va},$$

since $e^*ve = (vu)v(uv) = v$. Hence

$$\mathcal{F}_X(X) = X^*X = \varphi_{AvA} = \varphi_{\mathrm{Soc}(A)}.$$

By the remark above, we can take $\langle\,,\rangle$ to be Hermitian or alternate. The converse follows from Proposition 11.4. □

Proposition 11.9. *Let A be a centrally closed prime associative algebra over a field $\mathbb{F}$ of characteristic not 2, let $*$ be a ring involution of A, and let $u \in A$ be a nonzero symmetric element such that $uKu = 0$, where $K = \mathrm{Skew}(A, *)$. Then A_u is the field $\mathbb{F}$, the involution $\overline{x} \mapsto \overline{x}^*$ induced in A_u is the identity, and A can be regarded as a $*$-subalgebra of $\mathcal{L}_X(X)$ containing $\mathcal{F}_X(X)$, where X is vector space with a nondegenerate symmetric form over $\mathbb{F}$ and where $*$ is the adjoint involution.*

Proof. By Proposition 2.5, A_u is prime, and since $u(x - x^*)u = 0$ for every $x \in A$, the induced involution $\overline{x} \mapsto \overline{x}^*$ given in Lemma 2.2 is the identity on A_u. Thus A_u is commutative integral domain. Then we have by Lemma 2.7 that $A_u = \mathbb{F}\overline{v}$, with $\overline{v}$ being the unit element of A_u. In particular, u is a division element. Extend u to a Jordan pair idempotent (u, v), with $v = v^*$, and denote by e the idempotent uv. Clearly, $e^* = vu$ and $(u, v) \in eAe^* \times e^*Ae$. Following the notation of Lemma 11.6, we claim that $(A_u, \star)$ and (Δ, τ) are isomorphic as algebras with involution, $\Delta = eAe$ and $\tau(\alpha) = u\alpha^*v$, $\alpha \in eAe$:

Let $\varphi : A_u \to (uAu)^{(v)}$ and $\psi : (uAu)^{(v)} \to eAe$ be the isomorphisms of algebras (see the proof of Proposition 2.3) defined by $\varphi(\overline{x}) = uxu$ for all $x \in A$, and $\psi(uxu) = uxuv = uxe$ for all $uxu \in uAu$. Both isomorphisms are actually $*$-isomorphisms:

$$\varphi(\overline{x}^*) = ux^*u = (uxu)^* = \varphi(\overline{x})^*$$

and

$$(\psi(uxu))^\tau = (uxe)^\tau = u(uxe)^*v = ue^*x^*uv = ux^*e = \psi((uxu)^*).$$

Thus Δ is the field $\mathbb{F}$ and τ is the identity. Taking $X = eA$ as a left vector space over $\mathbb{F}$ and $\langle x, y\rangle = xy^*v$, we have by Corollary 11.8 that $\langle\, ,\rangle$ is a nondegenerate symmetric form and A is *-isomorphic to $\mathcal{F}_X(X)$ with the adjoint involution. □

Remark 11.10. By Litoff's Theorem (8.6), an associative algebra satisfying the conditions of the proposition above is locally finite. This result is the statement of [6, Lemma 9.1].

References

[1] P.N. Anh and L. Marki, *On Martindale's theorem*, Rocky Mountain J. Math. **26** (1966), 481-483

[2] K. I. Beidar, W. S. Martindale, III, and A. V. Mikhalev. *Rings with generalized identities*, volume 196 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker Inc., New York, 1996.

[3] P.M. Cohn, *Further Algebra and Applications*, Springer, 2003.

[4] Antonio Fernández López, *Jordan Structures in Lie Algebras*, Mathematical Surveys and Monographs, volume 240, Amer. Math. Soc. 2019.

[5] N. Jacobson, *Structure of Rings*, American Mathematical Society, 1956.

[6] E.I. Zelmanov, *Lie algebras with a finite grading*, Math. USSR Sbornik **52** (1985), 537-552.