RINGS COMPLEMENTED BY ANNIHILATORS

1. Annihilator-complemented left ideals

Throughout this section, A will denote a ring, no necessarily unital, L a left ideal of A, and R a right ideal. By $\mathrm{rann}(L)$ we will mean the right annihilator of L. Similarly, $\mathrm{lann}(R)$ will stand for the left annihilator of R.

The correspondences $L \mapsto \mathrm{rann}(L)$ and $R \mapsto \mathrm{lann}(R)$ define a Galois connection between the lattices of left ideals and right ideals of A, with associated closure operations $L \mapsto \overline{L} = \mathrm{lann}(\mathrm{rann}(L))$ and $R \mapsto \overline{R} = \mathrm{rann}(\mathrm{lann}(R))$.

Definition 1.1. A left ideal L of A is said to be *annihilator-complemented* if there exists a right ideal R such that $A = L \oplus \mathrm{lann}(R) = R \oplus \mathrm{rann}(L)$.

Example 1.2. Let $e \in A$ be an idempotent. Then the left ideal $L = Ae$ is annihilator-complemented. Indeed, take $R = eA$. We have $\mathrm{lann}(R) = A(1-e)$, $\mathrm{rann}(L) = (1-e)A$, and $A = Ae \oplus A(1-e) = eA \oplus (1-e)A$.

Lemma 1.3. *Suppose that* $\mathrm{lann}(A) = 0$. *Then every annihilator-complemented left ideal* L *of a ring* A *is closed.*

Proof. By the Modular Law, $\overline{L} = \overline{L} \cap (L + \mathrm{lann}(R)) = L \oplus (\overline{L} \cap \mathrm{lann}(R)) = L$, since

$$(\overline{L} \cap \mathrm{lann}(R))A = (\mathrm{lann}(\mathrm{rann}(L)) \cap \mathrm{lann}(R))(R + \mathrm{rann}(L)) = 0$$

and hence $\overline{L} \cap \mathrm{lann}(R) \subset \mathrm{lann}(A) = 0$ by assumption. □

2. Simple rings with minimal one-sided ideals

• Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a pair of dual vector spaces over a division ring Δ. Denote by X^* the dual vector space of X. Then the map $y \mapsto \hat{y}$, where for any $y \in Y$, $\hat{y}$ is the linear functional of X defined by $(x)\hat{y} = \langle x, y \rangle$, $x \in X$, is an isomorphism of Y into X^*. So (X, Y) is the canonical pair (X, X^*) if and only if the isomorphism $y \mapsto \hat{y}$ is onto.

• A linear map $a : X \to X$ is *adjointable* if there exists a linear map $a^{\#} : Y \to Y$, necessarily unique and called the *adjoin of* a, such that $\langle xa, y \rangle = \langle x, a^{\#} y \rangle$ for all $x \in X$, $y \in Y$. Note that we write the maps of a left vector space on the right (thus composing them from left to right), and the maps of a right vector space on the left (thus composing them from right to left), so by our conventions if a, b are adjointable so is ab with (careful!) $(ab)^\# = a^\# b^\#$.

• Denote by $\mathcal{L}_Y(X)$ the ring *of adjointable linear maps* of X and by $\mathcal{F}_Y(X)$ the ideal of those maps having finite rank. We will simply write $\mathcal{L}(X)$, $\mathcal{F}(X)$ instead of $\mathcal{L}_{X^*}(X)$, $\mathcal{F}_{X^*}(X)$ with respect to the canonical pair (X, X^*).

• For $x \in X$, $y \in Y$, write $y \otimes x$ to denote the linear map defined by

$$(x')y \otimes x = \langle x', y\rangle x$$

for all $x' \in X$. Then $y \otimes x \in \mathcal{F}_Y(X)$, with adjoint $(y \otimes x)^\# y' = y\langle x, y'\rangle$ for all $y' \in Y$.

• Given $V \leq X$ and $W \leq Y$ we denote by $W \otimes V$ the subgroup of the abelian group $(\mathcal{F}_Y(X), +)$ generated by the set $\{w \otimes v : w \in W, v \in V\}$.

Lemma 2.1. [2, IV.Theorem 16.1] *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a pair of dual vector spaces over a division ring. Then:*

1. *Any left ideal of the ring $\mathcal{F}_Y(X)$ is of the form $Y \otimes V$ where $V \leq X$, and any right ideal is of the form $W \otimes X$ where $W \leq Y$.*
2. *The correspondences $V \mapsto Y \otimes V$ and $W \mapsto W \otimes X$ are lattice isomorphisms.*
3. $\operatorname{rann}(Y \otimes V) = V^\perp \otimes X$ *and* $\operatorname{lann}(W \otimes X) = Y \otimes W^\perp$,

where $V^\perp = \{y \in Y : \langle V, y\rangle = 0\}$, and similarly, $W^\perp = \{x \in X : \langle x, W\rangle = 0\}$.

Proposition 2.2. *Assume that A is the ring $\mathcal{F}_Y(X)$ relative to a pair of dual vector spaces $(X, Y, \langle\cdot,\cdot\rangle)$ over a division ring Δ. Then for any left ideal $L = V \otimes X$, $V \leq X$, of A, the following conditions are equivalent:*

1. *L is an annihilator-complemented left ideal of A.*
2. *There exists $W \leq Y$ such that $X = V \oplus W^\perp$ and $Y = W \oplus V^\perp$.*
3. *$L = Ae$, where e is an idempotent of the ring $\mathcal{L}_Y(X)$.*

Moreover, in (2), $V = V^{\perp\perp}$ and (V, W) is a dual subpair of (X, Y).

Proof. (1) ⇒ (2). Given $V \leq X$, consider the left ideal $L = Y \otimes V$. Then there exists a right ideal R such that $A = L \oplus \operatorname{lann}(R) = R \oplus \operatorname{rann}(L)$. By Lemma 2.1, $R = W \otimes X$ for some $W \leq Y$, $\operatorname{lann}(R) = Y \otimes W^\perp$, and $\operatorname{rann}(L) = V^\perp \otimes X$. Hence (2) follows easily.

(2) ⇒ (3). Given $L = Y \otimes V$, let e be the projection of X on V relative to the decomposition $X = V \oplus W^\perp$, and similarly, let f be the projection of Y onto W relative to the decomposition $Y = W \oplus V^\perp$. We claim that $e \in \mathcal{L}_Y(X)$, with adjoint $e^\# = f$. Indeed, for any $v \in V,\ x \in W^\perp,\ w \in W,\ y \in V^\perp$, we have

$$\langle (v + x)e, w + y\rangle = \langle v, w\rangle = \langle v + x, f(w + y)\rangle.$$

It is clear that $Ae = (Y \otimes X)e = Y \otimes (Xe) = Y \otimes V = L$.

(3) $\Rightarrow$ (1). Let $L = Ae$ for some $e \in \mathcal{L}_Y(X)$. Taking $R = eA = e(Y \otimes X) = (e^{\#}Y) \otimes X$, we have as in Example 1.2 that L is annihilator-complemented.

Suppose that we are in (2). Then we have $X = V \oplus W^{\perp}$. Hence, by the Modular Law, $V^{\perp\perp} = V \cap (V^{\perp\perp} \cap W^{\perp}) = V$, since it follows from $Y = W \oplus V^{\perp}$ that $V^{\perp\perp} \cap W^{\perp} = 0$. Thus V is closed. Similarly it can be proved that (V, W) is a dual subpair of (X, Y). $\square$

- The ring $\mathcal{L}_Y(X)$ is the *symmetric rings of quotients* $Q_s(\mathcal{L}_Y(X))$ [1, Theorem 4.38].

3. Simple rings complemented by annihilators

Theorem 3.1. *Let A be a simple ring. Then the following conditions are equivalent:*

(1) *Every left ideal of A is annihilator-complemented.*
(2) *$A = \mathrm{Soc}(A)$ and $L = \mathrm{lann}(\mathrm{rann}(L))$ for any left ideal L of A.*
(3) *$A = \mathcal{F}(X)$, where X is a left vector space over a division ring Δ.*
(4) *Any left ideal of A is of the form $L = Ae$ for some idempotent $e \in Q_s(A)$.*

Proof. (1) $\Rightarrow$ (2). By [2, IV.Theorem 1.2], A is completely reducible as a left A-module, so A coincides with its socle and therefore it can be regarded as a ring $\mathcal{F}_Y(X)$, relative to a pair of dual vector spaces over a division ring Δ. That any left ideal of A is closed follows from Lemma 1.3.

(2) $\Rightarrow$ (3). Regarded A as the ring $\mathcal{F}_Y(X)$, let $0 \neq \varphi \in X^*$. Then $V = \mathrm{Ker}(\varphi)$ is a hyperplane of X, i.e. $X = V \oplus \Delta x_0$, where we may assume $(x_0)\varphi = 1$. Denote by L the left ideal $Y \otimes V$. Since L is closed, $V = V^{\perp\perp}$, which implies that $V^{\perp}$ is a nonzero subspace of Y. Then there exists a unique $y \in V^{\perp}$ such that $\langle x_0, y \rangle = 1$. It is clear that $\varphi = \hat{y}$.

(3) $\Rightarrow$ (4). Let L be a left ideal of $A = \mathcal{F}(X)$, $L = X^* \otimes V$ for some $V \leq X$. Take a complement $U \leq X$ of V, i.e. $X = V \oplus U$, and consider the projection $e \in \mathcal{L}_Y(X)$ of X onto V relative to this decomposition. Then $L = X^* \otimes (Xe) = (X^* \otimes X)e = Ae$. As pointed out above, $\mathcal{L}_Y(X) = Q_s(A)$.

(4) $\Rightarrow$ (1). Let $A = \mathcal{F}(X)$ and $L = Ae$ for some idempotent $e \in \mathcal{L}_Y(X) = Q_s(A)$. Then L is annihilator complemented with $R = eA = e(X^* \otimes X) = (e^{\#}X^*) \otimes X$. $\square$

References

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Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, Spain
E-mail address: emalfer@uma.es