A PDE used in cooking, plus more work with Bessel functions

The following very interesting partial differential equation was brought to the author's attention by Steve Gifford, a computer consultant in San Francisco, California. In a paper, located on the web at

www.douglasbaldwin.com/Baldwin-UGFS-Preprint.pdf,

a method of cooking called "Sous-vide" is described. It involves heating food at a fixed temperature in vacuum-sealed plastic bags for possibly a very long time. The temperature T of the food in the bag is assumed to satisfy the following version of the one-dimensional heat equation:

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial r^2} + \frac{\beta}{r}\frac{\partial T}{\partial r} \right), \ T(r,0) = T_0,$$

with boundary conditions

$$\frac{\partial T}{\partial r}(0,t) = 0, \ k\frac{\partial T}{\partial r}(R,t) = h(T_{water} - T(R,t));$$

where $0 \le r \le R$ is the distance from the center of the bag, $t \ge 0$ is time, and $0 \le \beta \le 2$ is a geometric factor that makes it possible to adjust for a bag of arbitrary shape, from a large slab ($\beta = 0$) to a long cylinder ($\beta = 1$) to a sphere ($\beta = 2$). T_0 is the initial temperature of the food in the bag, and T_{water} is the constant temperature of the water bath in which it is immersed. The three constants α, k, and h specify the physical properties of the object being cooked. Some characteristic values for the physical constants are given in Appendix A of the pdf cited above.

The object of this project is to see if we can find a series solution for this PDE, and then compare its values with results obtained from a numerical solution. Along the way you will learn some interesting things about Bessel functions.

Exercise 1. Show that if we replace the function $T(r,t)$ by the function $U(r,t) \equiv T(r,t) - T_{water}$, then the problem in terms of U becomes:

$$\boxed{\begin{gathered} \frac{\partial U}{\partial t} = \alpha \left(\frac{\partial^2 U}{\partial r^2} + \frac{\beta}{r}\frac{\partial U}{\partial r} \right), \ U(r,0) = T(r,0) - T_{water} = T_0 - T_{water} \\ \frac{\partial U}{\partial r}(0,t) = 0, \ \frac{\partial U}{\partial r}(R,t) + \frac{h}{k}U(R,t) = 0, \end{gathered}}$$

which is a parabolic PDE with *homogeneous* boundary conditions. □

It should now be possible to solve the problem by separation of variables. Letting $U(r,t) = X(r)Y(t)$, the PDE becomes

$$XY' = \alpha\left(X''Y + \frac{\beta}{r}X'Y\right).$$

Exercise 2.
Separate the variables by dividing both sides of this equation by αXY, and show that the resulting ODEs for X and Y are

$$\boxed{Y'(t) = -\alpha\lambda Y(t) \text{ and } X''(r) + \frac{\beta}{r}X'(r) + \lambda X(r) = 0.}$$

$\square$

If the equation

$$X''(r) + \frac{\beta}{r}X'(r) + \lambda X(r) = 0 \tag{1}$$

is multiplied by r^β, it can be seen to be a **Sturm-Liouville equation**

$$r^\beta X'' + r^\beta \frac{\beta}{r}X' + r^\beta \lambda X = \frac{d}{dr}(r^\beta X') + \lambda r^\beta X = 0$$

with weight factor $w(r) = r^\beta$; therefore, if the eigenvalues λ_n and the corresponding eigenfunctions $X_n(r)$ are found, we know that the family of functions $\{X_n(r)\}_{n=1}^\infty$ is an orthogonal family on $0 \le r \le R$. This means that

$$\boxed{\int_0^R r^\beta X_j(r)X_k(r)dr = 0 \text{ whenever } j \neq k.}$$

The two boundary conditions on the temperature function $U(r,t) \equiv X(r)Y(t)$ can be used to find boundary conditions on $X(r)$.

Exercise 3. Show that the two boundary conditions on $X(r)$ are

$$X'(0) = 0,\ \ X'(R) + \frac{h}{k}X(R) = 0. \qquad \square.$$

To find the general solution of (1), make the substitutions $\tau = \sqrt{\lambda}r$ and $Z(\tau) \equiv X(r)$. Then (1) becomes

$$\lambda\left(Z''(\tau) + \frac{\beta}{\tau}Z'(\tau) + Z(\tau)\right) = 0.$$

This is not quite Bessel's equation of order 0, because of the parameter β; but using Maple, the general solution is found to be

$$Z(\tau) = C_1 \tau^{\frac{1}{2}(1-\beta)} \mathcal{J}_{\frac{1}{2}(\beta-1)}(\tau) + C_2 \tau^{\frac{1}{2}(1-\beta)} \mathcal{Y}_{\frac{1}{2}(\beta-1)}(\tau),$$

where $\mathcal{J}_\nu$ and $\mathcal{Y}_\nu$ are Bessel functions of the first kind. As $\tau \to 0^+$, the function $\mathcal{Y}_\nu(\tau)$ tends to $-\infty$ for any order ν, so in order to make the temperature finite at $r = 0$, it is necessary to set $C_2 = 0$. Thus the required solution of (1) is any constant multiple of

$$\boxed{X(r) = Z(\tau) = Z(\sqrt{\lambda}r) = (\sqrt{\lambda}r)^{\frac{1}{2}(1-\beta)} \mathcal{J}_{\frac{1}{2}(\beta-1)}(\sqrt{\lambda}r).}$$

To find the eigenvalues λ_n, we need to make $X(r)$ satisfy the two boundary conditions. Letting $p = \frac{1}{2}(\beta - 1)$ and $\tau = \sqrt{\lambda}r$, we can use the known series for the Bessel function $\mathcal{J}_p$ to write

$$X(r) = \tau^{-p}\mathcal{J}_p(\tau) = \tau^{-p} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\tau}{2})^{2n+p}}{n!\Gamma(1+n+p)} = \sum_{n=0}^{\infty} \frac{(-1)^n \tau^{2n}}{2^{2n+p}\, n!\Gamma(1+n+p)},$$

and since this is a Taylor series in powers of τ^2, it can be seen that $X'(0) = 0$ as required. The series can also be used to compute $X(0) = \frac{1}{2^p\Gamma(1+p)}$.

> The Gamma Function $\Gamma(z)$
>
> The Gamma function is a continuous extension of the factorial function. For any positive real number z it is defined by
> $$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx.$$
> For $z > 0$ it satisfies the equation $\Gamma(z+1) = z \cdot \Gamma(z)$. One very useful value is $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

The second boundary condition requires that $\frac{X'(R)}{X(R)} = -\frac{h}{k}$. To find the derivative of the function $X(r)$, we can use a known formula which states that for any order p, $\mathcal{J}_p'(\tau) = \frac{1}{2}(\mathcal{J}_{p-1}(\tau) - \mathcal{J}_{p+1}(\tau))$. Using this, with $\tau = \sqrt{\lambda}r$ and $p = \frac{1}{2}(\beta - 1)$, the chain rule for differentiation implies that

$$X'(r) = \frac{d}{dr}\left(\tau^{-p}\mathcal{J}_p(\tau)\right) = \frac{d}{d\tau}\left(\tau^{-p}\mathcal{J}_p(\tau)\right)\frac{d\tau}{dr}.$$

Therefore, by the product rule,

$$X'(r) = \left(-p\tau^{-p-1}\mathcal{J}_p(\tau) + \tau^{-p}\left(\frac{\mathcal{J}_{p-1}(\tau) - \mathcal{J}_{p+1}(\tau)}{2}\right)\right)\sqrt{\lambda}$$

and

$$\frac{X'(r)}{X(r)} = \frac{X'(r)}{\tau^{-p}\mathcal{J}_p(\tau)} = \left(\frac{-p}{\tau} + \frac{1}{2}\left(\frac{\mathcal{J}_{p-1}(\tau) - \mathcal{J}_{p+1}(\tau)}{\mathcal{J}_p(\tau)}\right)\right)\sqrt{\lambda}.$$

We can then set

$$\frac{X'(R)}{X(R)} = \left(\frac{-p}{\sqrt{\lambda}R} + \frac{1}{2} \left(\frac{\mathcal{J}_{p-1}(\sqrt{\lambda}R) - \mathcal{J}_{p+1}(\sqrt{\lambda}R)}{\mathcal{J}_p(\sqrt{\lambda}R)} \right) \right) \sqrt{\lambda} = -\frac{h}{k}.$$

Exercise 4. Use algebra to show that this results in the requirement that

$$\frac{1}{2} \left(\frac{\mathcal{J}_{p-1}(z) - \mathcal{J}_{p+1}(z)}{\mathcal{J}_p(z)} \right) = \left(p - \frac{h}{k} R \right) \frac{1}{z}, \text{ where } z = \sqrt{\lambda}R$$

□

Notice that for $\beta = 0, 1, 2$ the constant $p = \frac{1}{2}(\beta - 1)$ has the three possible values $-\frac{1}{2}, 0, \frac{1}{2}$. The function $\frac{\mathcal{J}_{p-1}(z) - \mathcal{J}_{p+1}(z)}{\mathcal{J}_p(z)}$ has vertical asymptotes at the zeros of $\mathcal{J}_p(z)$, and these are known to approach the zeros of $\cos\left(z - \frac{p}{2}\pi - \frac{\pi}{4}\right)$ as $z \to \infty$; therefore, if it can be shown that the function $\mathcal{J}_p'$ is monotonically increasing or monotonically decreasing in each open interval between the asymptotes, there will be exactly one intersection z_n between $\left(n - \frac{3}{4} + \frac{p}{2}\right)\pi$ and $\left(n + \frac{1}{4} + \frac{p}{2}\right)\pi$ for each integer $n = 1, 2, \cdots$. These intersections can be found using the Maple command `fsolve`. The graph in Figure 1 shows the first 5 intersections, using the parameter values $\beta = 2 \to p = \frac{1}{2}, h = 100, k = 0.5, R = 0.04$. With $\beta = 2$, the first three intersections are $z_1 \approx 2.76536, z_2 \approx 5.60777, z_3 \approx 8.54057$.

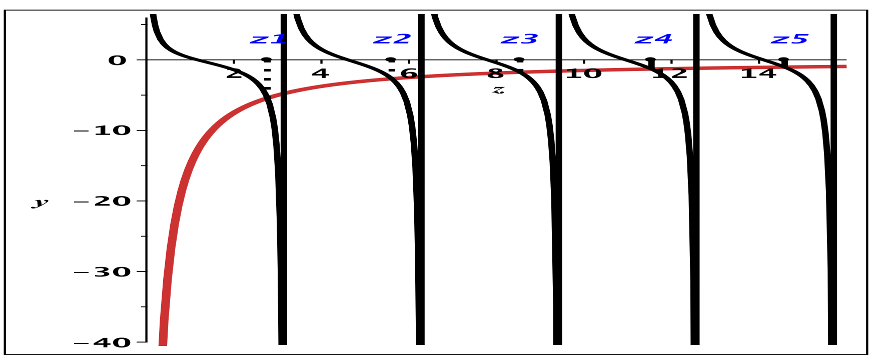


Figure 1: Intersections $z_1, \cdots, z_5$ with $\beta = 2$

Try to make an argument to show that the function $\mathcal{J}_p'$ is monotonically increasing or monotonically decreasing between zeros of $\mathcal{J}_p$. Think about the fact that $\mathcal{J}_p$ has the shape of a damped cosine function. If you would like to learn more about Bessel functions, you could try writing this up as a formal proof.

Once the z_n are found, we can set $\lambda_n = \left(\frac{z_n}{R}\right)^2$. For each $n = 1, 2, \cdots$ the functions $Y_n(t)$ can then be found by solving the equation $Y_n'(t) = -\alpha\lambda_n Y_n(t)$.

This first-order differential equation has solutions $Y_n(t) = Ce^{-\alpha\lambda_n t}$, and the product $U_n(r,t) = X_n(r)Y_n(t)$ is a solution of the partial differential equation for each integer $n = 1, 2, \cdots$. Since the pde is linear, its general solution can be written in the form of an infinite series:

$$\boxed{\begin{array}{l} U(r,t) = \sum_{n=1}^{\infty} A_n U_n(r,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha\lambda_n t}(\sqrt{\lambda_n}r)^{-p}\mathcal{J}_p(\sqrt{\lambda_n}r). \\ \\ \lambda_n = \left(\frac{z_n}{R}\right)^2, \quad p = \frac{1}{2}(\beta - 1) \end{array}}$$

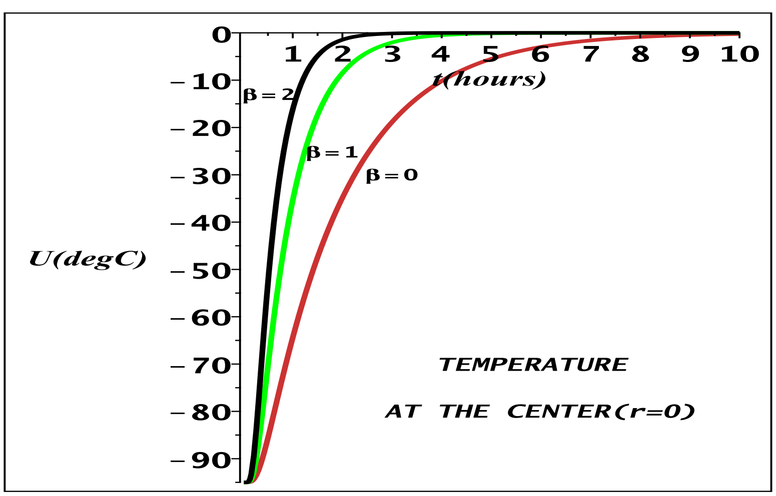
The initial condition on $U(r,t)$ required that, at time $t = 0$,

$$U(r,0) = \sum_{n=1}^{\infty} A_n X_n(r) Y_n(0) = \sum_{n=1}^{\infty} A_n(\sqrt{\lambda_n}r)^{-p}\mathcal{J}_p(\sqrt{\lambda_n}r) = T_0 - T_{water}.$$

Since the functions $X_n(r)$ form an orthogonal set on $[0, R]$ with weight function r^β, the coefficients are

$$\boxed{A_n = \frac{\int_0^R U(r,0)X_n(r)r^\beta dr}{\int_0^R (X_n(r))^2 r^\beta dr} = \frac{\int_0^R (T_0 - T_{water})(\sqrt{\lambda_n}r)^{-p}\mathcal{J}_p(\sqrt{\lambda_n}r)r^\beta dr}{\int_0^R ((\sqrt{\lambda_n}r)^{-p}\mathcal{J}_p(\sqrt{\lambda_n}r))^2 r^\beta dr}.}$$

Example 1 Using the parameter values $h = 100W/m^2K, k = 0.5W/mK, R = 0.04m$, and $\alpha = 1.4 \times 10^{-7}m^2/sec.$, compare the time it takes for the temperature in the center of the food package to rise close to the temperature of the water bath for $\beta = 0, 1$, and 2.



The graphs in the above figure were generated by the Maple program shown below. Remember that the actual temperature of the food is $T(r,t) = U(r,t) + T_{water}$. Realistic values for the heat transfer coefficient h and the thermal conductivity of the food k were taken from Appendix A in the web article cited on page 1. The radius of the package was arbitrarily chosen to be $R = 0.04m \approx 1.5in$. Note that the constant $p - \frac{h}{k}R$ is dimensionless, since R is in meters. To have time t in hours, the thermal diffusivity $\alpha = 1.4 \times 10^{-7}\mathrm{m}^2/\mathrm{sec}$ was multiplied by 3600. The temperatures used are $T_0 = 5^0C$ and $T_{water} = 100^0C$. It can be seen that a spherically shaped package requires the least amount of time for the temperature in the center to reach the desired value.

SERIES SOLUTION FOR $U(r,t)$

```
h:=100: k:=0.5: alpha:=1.4E-7*3600: R:=0.04: T0:=5: Twater:=100:
beta:=2; p:=(beta-1.0)/2.0: Nterms:=50:
for n from 1 to Nterms do
  z[n]:=fsolve((BesselJ(p-1,z)-BesselJ(p+1,z))/BesselJ(p,z)=
    2.0*(p-h*R/k)/z,z=(n-0.75+p/2.0)*Pi..(n+0.25+p/2.0)*Pi);
  lam[n]:=(z[n]/R)^2;
  A[n]:=int((T0-Twater)*(z[n]*r/R)^(-p)*BesselJ(p,z[n]*r/R)*
    r^(beta),r=0..R)/
    int(((z[n]*r/R)^(-p)*BesselJ(p,z[n]*r/R))^2
      *r^(beta),r=0..R); od:

U:=proc(r,t) local S; global A,lam,alpha,p,z,R,Nterms;
  if r=0 then
    S:=sum(A[j]*exp(-alpha*lam[j]*t),j=1..Nterms)/
      (2.0^p*GAMMA(p+1.0));
  else S:=sum(A[j]*exp(-alpha*lam[j]*t)*(z[j]*r/R)^(-p))
    *BesselJ(p,z[j]*r/R),j=1..Nterms);
  fi: S:
end proc:

U(0,1) = -16.1542
U(0,2) = -1.45323
```

For small values of t, the series for U converges very slowly, and it is useful to compare values of $U(r,t)$ obtained by this method with those obtained using a numerical method. Problem 5 below will ask you to do this.

Additional Exercises:

1. Using the parameter values from Example 1, run the Maple program to find the series solution $U(r,t)$. For each value $\beta = 0, 1$, and 2, compute the temperatures $U(0,1), U(0,2), U(0,4)$, and $U(0,6)$.

2. Define **cooking time** $G_\beta(X)$ to be the time in hours that it takes for a package of radius X inches to go from $T_0^\circ C$ to $(T_{water} - 2)^\circ C$, using a given set of parameters. Draw graphs of $G_\beta(X)$ for $\beta = 0, 1, 2$ on the interval $0.5in \leq X \leq 5in$. Choose your own set of parameters from the values given in Appendix A for some specific type of food.

3. The equation $\frac{\partial U}{\partial t} = \alpha\left(\frac{\partial^2 U}{\partial r^2} + \frac{\beta}{r}\frac{\partial U}{\partial r}\right)$ can be approximated by the difference equation
$$\frac{U(r, t+\Delta t) - U(r,t)}{\Delta t}$$
$$= \alpha\left[\frac{U(r+\Delta r, t) - 2U(r,t) + U(r-\Delta r, t)}{(\Delta r)^2} + \frac{\beta}{r}\left(\frac{U(r+\Delta r, t) - U(r,t)}{\Delta r}\right)\right].$$
Solve this equation for $U(r, t+\Delta t)$ in terms of values of U at time t.

4. How would you express the two boundary conditions $\frac{\partial U}{\partial r}(0,t) = 0$ and $\frac{\partial}{\partial r}U(R,t) + \frac{h}{k}U(R,t) = 0$ as difference formulas in U? Be careful at the end $r = 0$, since the PDE contains a term with r in the denominator.

5. Check that the Maple program below solves the difference equation in Problem 2 with the boundary conditions given in Problem 3. Explain how the program handles the two boundary conditions.

```
NUMERICAL SOLUTION FOR U(r,t)

h:=100: k:=0.5: alpha:=1.4E-7*3600: R:=0.04: T0:=5: Twater:=100:
beta:=2: (this constant needs to be set to 0, 1, or 2)
N:=16: delr:=R/N: delt:=0.005:
  C:=alpha*delt/delr^2; C2:=beta*delr/2.0: C3:=2.0*h*delr/k:
for i from 0 to N+1 do u[i,0]:=T0-Twater; od:
for j from 0 to 2000 do
  for i from 1 to N do
      u[i,j+1]:=u[i,j]+C*(u[i+1,j]-2*u[i,j]+u[i-1,j]+ (C2/(i*delr))*(u[i+1,j]-u[i-1,j]));
od;
  u[0,j+1]:=u[1,j+1];
  u[N+1,j+1]:=u[N-1,j+1]-C3*u[N,j+1];
od:
```

With $\beta = 2$, this program produced the values

$U(0,1) \equiv u(0,200) \approx -16.1105$

$U(0,2) \equiv u(0,400) \approx -1.44300$

6. Choose appropriate values for N (number of intervals in the partition of the r-axis) and Δt, and compute the numerical solution $u[i,j]$ on the interval $0 \leq t \leq 10$ hours. In order for the numerical method to be *stable*, the constant $C = \alpha\frac{delt}{delr^2}$ must be less than 0.5. For each $\beta = 0, 1,$ and 2, compare the values you get for $U(0,1), U(0,2), U(0,4)$, and $U(0,6)$ to

the values found in Exercise 1 (in each program, the values of $U(0,1)$ and $U(0,2)$, generated by the program with $\beta = 2$, are shown). Remember that in the numerical program $r = i * \Delta r$ and $t = j * \Delta t$. State what values you needed to use for N and Δt to get all of the results, for the two different types of solution, to agree in the first decimal place.

7. Assume you are a numerical analyst working for a company that makes sous-vide cookers. You have been assigned to write a procedure (numeric or analytic, your choice) that will produce the cooking time for food in the cooker, given the values of β, h, k, α, T_0, and T_{water}. This should be the simplest procedure that quickly produces the time it will take the temperature in the center of the food to reach $(T_{water} - 2)^0 C$. Justify all of the choices that you make.