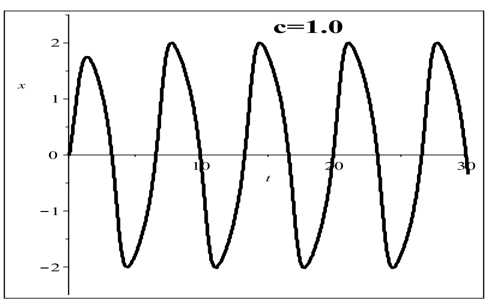
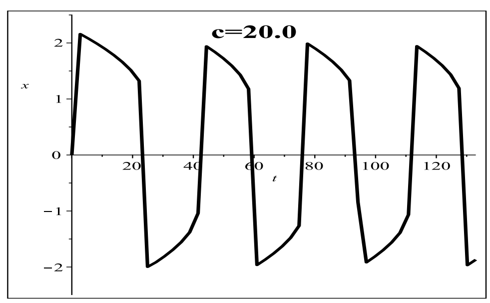
# Relaxation Oscillators and Neural Networks

In Section 3 of Chapter 5, the van der Pol equation

$$x'' + c(x^2 - 1)x' + x = 0, \tag{1}$$

is used to introduce *limit cycles*. This equation was originally derived in 1926 by the Dutch physicist and electrical engineer Balthasar van der Pol, to model an electrical circuit, with nonlinear resistance, in a vacuum tube. At that time, radios were made with vacuum tubes, since the transistor had not yet been invented. When the constant $c$ in the equation is small, the solutions all approach a limit cycle, and the graph of $x(t)$ approaches a smooth periodic sine-like curve (left graph below). When the constant $c$ is large, say $c = 20$, the graph of the limit cycle has a very different non-sine-like appearance (right graph below). The oscillation in the circuit with large $c$ is called a "relaxation oscillation". The phrase "relaxation oscillation" was introduced by van der Pol in 1926.




The van der Pol equation (1) can be written as a system of two first-order ODEs:

$$\begin{aligned} x' &= c(y - f(x)) \equiv F(x, y) \\ y' &= -\frac{1}{c}x \equiv G(x, y). \end{aligned} \tag{2}$$

The function $f$ is a cubic in $x$. Using the information in Chapter 5, we can then do a phase plane analysis on this system in order to see why the change in behavior occurs.

Finding answers to the four problems below will help you understand the behavior of the system (2):

1. Show that the function $f(x) = \frac{x^3}{3} - x$ makes the system (2) and the second-order equation (1) equivalent. Remember that $\frac{d}{dt}\left(f(x(t))\right) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt}$.

2. Draw a phase plane for (2) and draw the x and y nullclines; that is, all curves where $x' = 0$, or $y' = 0$, respectively. Show that the nullclines intersect only at $(\bar{x}, \bar{y}) = (0, 0)$, so that $(0, 0)$ is the unique fixed point for the system.

3. Find the partial derivatives $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}$, and write out the Jacobian matrix $J(x, y)$ for the system. Evaluate $J(0, 0)$.

4. Determine all bifurcation values of the system for $c > 0$. How does the type of the equilibrium at $(0, 0)$ change as $c$ increases?

If you have any trouble with these problems, the answers to 2, 3, and 4 can be found at the end of this article.

Around the same time, other scientists and engineers were beginning to be interested in modelling the electrical activity in nerve cells (neurons) in order to "understand" the behavior of the human brain. In the case of engineers, they were hoping to be able to produce a model that could be used to program a computer capable of "learning" to do human-like tasks, such as translating foreign languages, recognizing objects in a scene, etc. One of the first biological models was a system of differential equations for the electrical activity in a giant squid axon. It won its authors A. L. Hodgkin and A. F. Huxley the 1963 Nobel prize in Physiology and Medicine. Their model approximated voltages across the cell membranes, and involved a number of parameters which had to be determined to make the behavior of the model mimic the behavior of the voltage in the cell. Its complexity, and the fact that it was specific to the squid axon, made it not the type of model the computer engineers were looking for.

Later in the 1960's, a simplification of this model was proposed by Richard FitzHugh and J. Nagumo. It consisted of the following two first-order differential equations:

$$\begin{aligned} \frac{dV}{dt} &= V - V^3 - w + I \\ \tau \frac{dw}{dt} &= V - a - bw. \end{aligned} \tag{3}$$

In this model, $V(t)$ is a membrane-like voltage, $I(t)$ is an input current, $w(t)$ is a general gate voltage, and $a, b$, and $\tau$ are parameters. The value of $\tau$ is assumed to be positive and strictly less than one.

Notice the similarities and differences between this system and the Van der Pol system (2). The $V$-isocline $w = V - V^3 + I$ is again a cubic, which can be moved up or down in the $(V, w)$ phase plane by changing the value of the input current $I$. The $w$ isocline $V = a + bw$ is a straight line that can be moved to any position by the values of the parameters $a$ and $b$. This means that for certain values of the parameters the neuron will act like a relaxation oscillator. Several investigators have used different versions of this model to study the behavior of the output of groups of interacting neurons. See the article: H. Ryu and S.A. Campbell. Geometric Analysis of Synchronization in Neuronal Networks with Global Inhibition and Coupling Delays, to appear to Philosophical Transactions of the Royal Society A, 2019.

Many models for the behavior of neurons have been suggested. One of the simplest ways to model the behavior of a neuron is called the "integrate and fire" model. Basically it is assumed that in a network of N neurons, some or all of which are interconnected, each neuron receives a weighted sum of inputs from the other neurons to which it is connected, and produces an output if the sum is greater than some specified threshold value. The weights between the neurons are usually varied over time, giving the network a way in which to adapt, or "learn". The most popular such network, called a **backpropagation network**, is one in which the weights between cells are varied to minimize the difference between the output of the network and a desired output specified by the problem solver. This is referred to as *supervised learning*. Backpropagation's popularity has experienced a recent resurgence given the widespread adoption of **deep neural networks** for problems such as image recognition. This method is rather far removed from the actual behavior of neurons in the brain, but performs surprisingly well when extremely large computer resources are available.

A more realistic approach to learning might consist of allowing each neuron to collect signals from a subset of the neurons in the system. The weights between them could then be augmented in terms of the concurrence of activity in the two connected cells. One model in which this was tried[1], for a fully connected system of $N$ cells, uses the equations

$$\begin{aligned} x_i'(t) &= x_i(t)(1 - cx_i(t)) - \sum_{j \neq i} A_{ij}(t)\sigma(x_j(t))x_i(t), i = 1..N \\ A_{ij}'(t) &= \frac{1}{T}[x_i(t)\sigma(x_j(t)) - A_{ij}(t)], 1 \leq i, j \leq N, j \neq i. \end{aligned} \tag{4}$$

The function $\sigma(z)$ is either a Heaviside function $H(z - \theta)$ with threshold $\theta$, or a smoothed version of $H(z - \theta)$. Several differentiable approximations to $H$ were mentioned in the text. The differential equation for the weight $A_{ij}$ is derived by assuming that $A_{ij}$ is the average, over the past, of the interaction between cells $i$ and $j$; that is

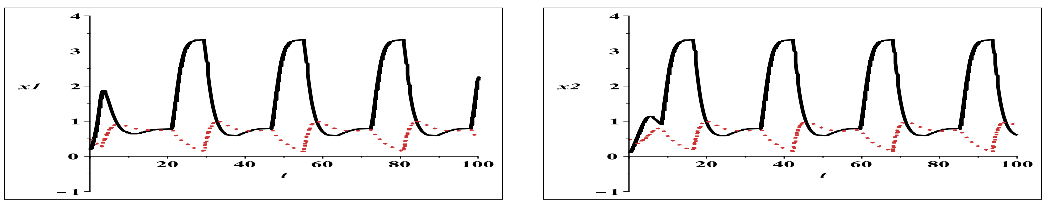
$$A_{i,j}(t) = \frac{1}{T} \int_{-\infty}^{t} e^{(u-t)/T} x_i(u)\sigma(x_j(u))du.$$

Note that if the weights $A_{ij}$ are positive, the action of cells on each other in (4) is *inhibitory*; that is, the larger the value of $A_{ij}$, the greater the negative effect of cell $j$ is on cell $i$. The value of a weight increases if, and only if, both cells $i$ and $j$ are active at the same time. It would probably be more realistic to have

[1]Noonburg, V. W., (1997). Threshold-dependent limit cycles in a nonlinear network model. *Neural Networks*, **10**, 639-648.

the feedback term $\sigma(x_j(t))$ evaluated at time $t - \Delta t$, which would make (4) be a system of differential-delay equations. This has not been tried, but even without the delay this system can be shown to produce periodic solutions in which each neuron acts as a relaxation oscillator. As an example, consider a two cell system (4) with $N = 2, c = 0.3, T = 5.0$, and $\sigma(z) = H(z - \theta), \theta = 0.768$. The value of the threshold $\theta$ is critical; it must be very close, but not equal, to $\frac{1}{1+c} \approx 0.7692$. The initial values used are $x_1(0) = 0.2, x_2(0) = 0.1, A_{12}(0) = 0.5, A_{21}(0) = 0.5$. The initial values for $x_1$ and $x_2$ must be between $0$ and $\frac{1}{c}$ (the cell's "carrying capacity").

The two figures show $x_1$ with its weight $A_{12}$ (dotted) and $x_2$ with $A_{21}$. It is clear




that the cells are behaving like relaxation oscillators.

## IDEAS FOR FURTHER INVESTIGATION

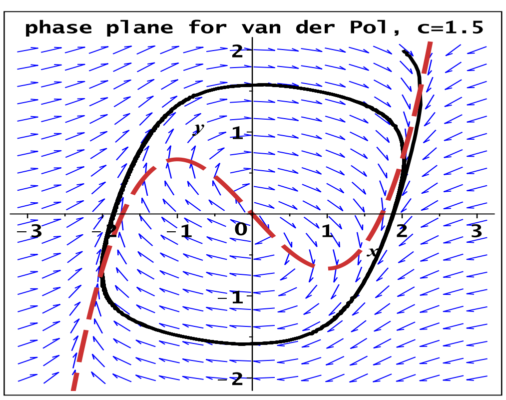
1. This one is for the computer experts. Read the file entitled "An Introduction to Backpropagation Networks" and write a computer program that learns to recognize 3 or 4 different patterns in a $4 \times 4$ matrix. This means that your input layer must contain $16$ cells and the output layer contains $4$ (or $5$, if you want a unique response when the input is not close to any of the four patterns). Train the system with sample inputs and then test it on a random set of binary matrices (i.e. matrices containing only 1's and 0's).

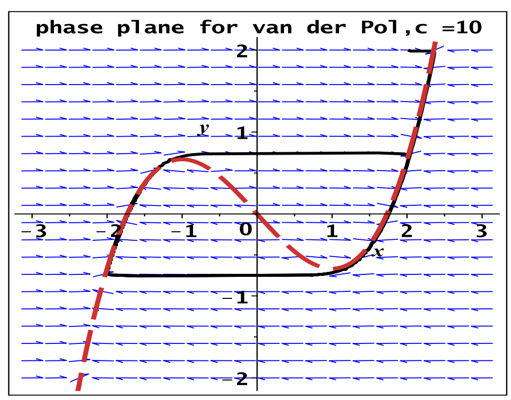
2. For the system (3), write out the Jacobian $\mathbf{J}(\mathbf{V}, \mathbf{w})$, assuming the input $I$ is zero. and $\tau = 0.1$. Find the equilibria, and determine the type of each for arbitrary values of $a$ and $b$. What happens if you let the input $I(t)$ be a periodic function? See if you can find a set of parameters such that the system produces relaxation oscillations. What other types of behavior can occur?

3. Try to design a system (4) with three cells $x_1, x_2$, and $x_3$. How many weights will there be? Find all critical points of this system. Try to set the parameters so that the system produces relaxation oscillations.

## ANSWERS TO PROBLEM 2 - 4

The two graphs below show the phase planes for the van der Pol equation when $c = 1.5$ and $c = 10$. The $x'$ nullcline is the curve $y = \frac{x^3}{3} - x$, and the $y'$ nullcline is the $y$ axis, $x = 0$. These intersect only at the critical point $(0, 0)$. Notice the way in which trajectories behave around $(0, 0)$. The point $(0, 0)$ is a **spiral source** when $c < 2$ and is a **source** for $c > 2$. When $(0, 0)$ is a source, one gets the behavior characteristic of a relaxation oscillator. The trajectories move along the cubic to the max or min point, and then shoot across to the other side at high speed. To see why, look at the arrows.




The Jacobian for the system is the matrix

$$J(x, y) = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix} = \begin{pmatrix} c(-x^2 + 1) & c \\ -\frac{1}{c} & 0 \end{pmatrix}.$$

At the origin, $J(0, 0) = \begin{pmatrix} c & c \\ -\frac{1}{c} & 0 \end{pmatrix}$, so its determinant is 1 and the trace is $c$. If you put the point $(c, 1)$ into the trace-determinant plane, you will see that since the determinant is always equal to 1, the point $(c, 1)$ crosses the parabola $det = (trace)^2/4$ when $c$ is equal to 2. This implies that the **type** of the equilibrium at $(0, 0)$ changes from a spiral source to a source when $c$ passes through 2.