Errata for **Geometric Relativity** by Dan A. Lee, as of March 8, 2024.

- On page 14, in Exercise 1.15, the coefficient of r^2 in the given formula should be $\frac{1}{6(n+2)}R(p)$ instead of $\frac{6}{(n+2)}R(p)$. Also, in this exercise, ω_n is defined to be the volume the n-dimensional unit ball, but everywhere else in the text, it is used to denote the volume of the n-dimensional sphere.
- On page 73, in Exercise 3.12, the constant in front of the integral should be $\frac{-2}{(n-2)\omega_{n-1}}$ rather than $\frac{-2}{(n-1)\omega_{n-1}}$.
- On page 86, in the second displayed equation, the $=$ should be $\leq$.
- On page 96, the second line of the displayed computation should be

$$= 0 + \frac{2}{(n-2)\omega_{n-1}} \int_M -\Delta_{g_t} u_t \, d\mu_{g_t},$$

  and then the integral in the next line should also be integrated with respect to $d\mu_{g_t}$. This should give rise to a term involving $\frac{d}{dt} d\mu_{g_t}\big|_{t=0}$ in the computation that follows, but it does not contribute anything since $R_{g_0} = 0$.
- On page 102, Exercise 3.49 is false, and as a result, the entire proof of Lemma 3.48 is incorrect. Since this is a substantial error, we include a complete proof of Lemma 3.48 at the end of these Errata. We thank Ryan Unger for pointing out this error, as well as some others.
- On page 162, in the proof of Proposition 5.4, where it says $(vw)x(wv)^{-1}$, it should say $(vw)x(vw)^{-1}$,
- On page 165, there is a sign error in the computation of $g'(0)$. The displayed equation should be $g'(0) = -\omega^i_j(X) e_i \otimes \theta^j = \frac{1}{2}\omega^i_j A^j_i$. Unfortunately, this leads to many other sign errors. In particular, on that same page, we should have $\tilde{g}'(0) = \frac{1}{4}\omega^i_j(X) e_i e_j$ and $\nabla_X s = -\frac{1}{4}\sum_{i,j=1}^n \omega^i_j(X) e_i e_j s$, but most importantly, equation (5.2) should be

$$\nabla\psi = -\frac{1}{4}\sum_{i,j=1}^n \omega^i_j(X) e_i e_j \psi.$$

- On page 166, the first displayed equation is correct under the assumption that ϕ and ψ are constant with respect to a given frame. In general, the correct formula is $\langle \nabla\phi, \psi\rangle + \langle \phi, \nabla\psi\rangle = \nabla\langle\phi,\psi\rangle$.
- On pages 167, there are a few incorrect signs due to the sign error in equation (5.2) on page 165. We should have

$$\begin{aligned}
R^S\psi &= \nabla(\nabla\psi) \\
&= \nabla\left(-\frac{1}{4}\sum_{i,j=1}^n \omega^i_j e_i e_j \cdot \psi\right) \\
&= -\frac{1}{4}\sum_{i,j=1}^n d\omega^i_j e_i e_j \cdot \psi + \omega^i_j \wedge e_i e_j \cdot \nabla\psi \\
&= -\frac{1}{4}\sum_{i,j=1}^n d\omega^i_j e_i e_j \cdot \psi + \frac{1}{16}\sum_{i,j,k,\ell=1}^n \omega^i_j \wedge \omega^k_\ell e_i e_j e_k e_\ell \cdot \psi.
\end{aligned}$$

Continuing on page 168, there are a couple more sign errors (that effectively cancel out the original sign error made on page 165). We should have

$$R^S = -\frac{1}{4}\sum_{i,j=1}^{n} (d\omega + \omega\wedge\omega)^i_j e_i e_j$$
$$= -\frac{1}{4}\sum_{i,j=1}^{n} \mathrm{Riem}(\cdot,\cdot,e_i,e_j)e_i e_j.$$

All other equations in the proof of Theoren 5.10 have the correct signs.

- On page 171, the first displayed equation should be

$$e_i = \partial_i - \tfrac{1}{2}h_{ij}\partial_j + o_1(|x|^{-2q}),$$

where repeated index summation is used. Also, the description of how to find this frame is incorrect. Applying Gram–Schmidt directly to the coordinate frame $\partial_1,\dots,\partial_n$ will not yield this formula. Instead, one should apply Gram–Schmidt to the frame $u_i = \partial_i - \frac{1}{2}h_{ij}\partial_j$, which has the property that $\langle u_i, u_j\rangle = \delta_{ij} + o_1(|x|^{-2q})$.

Also, using the corrected version of equation (5.2) from page 165 results in a sign change in equation (5.4), and consequently, a sign change in the computation at the top of page 172. After this sign correction, it follows that Proposition 5.14 is correct as stated.

- On page 172, the limit statement in Corollary 5.15 should be for *some* sequence of radii ρ rather than any sequence of radii. The reason is that $\xi = \psi - \psi_0$ and its first derivatives are only assumed to decay in an integral sense rather than a pointwise sense. The integrals of last two terms of equation (5.5) over S_ρ, viewed as functions of ρ, need not decay in a pointwise sense, but they will decay in an integral sense, which is good enough to find a sequence of ρ's for which the limits of the S_ρ integrals vanish.

The revised statement of Corollary 5.15 is clearly good enough for our desired application in the proof of Theorem 5.12. (In fact, under the assumptions of nonnegative scalar curvature and $\mathcal{D}\psi = 0$, the S_ρ integral in Corollary 5.15 is actually monotone nondecreasing in ρ because of Corollary 5.13, and thus in this case, the conclusion of Corollary 5.15 is correct as printed.)

- On page 174, the argument starting with, "To prove better regularity. . . ," is incorrect. (The reason is that choosing test functions $\varphi \in W^{1,2}_{-q}$ is insufficient for proving weak convergence in L^2.) Instead, one correct way to argue why $\xi \in W^{1,2}_{-q}$ is to invoke elliptic regularity of the operator $\mathcal{D}$. Essentially, we need a version of Theorem A.32 that applies to *systems* of first-order elliptic operators (on spinor sections) asymptotic to the "Euclidean" Dirac operator, rather than second-order elliptic operators (on scalar functions) asymptotic to the Euclidean Laplacian. Specifically, given that $\xi \in L^2_{-q-1} \subset L^2_{-q}$ and ξ weakly solves the equation $\mathcal{D}\xi = \eta \in L^2_{-q-1}$, the appropriate analog of Theorem A.32 for the Dirac operator implies that $\xi \in W^{1,2}_{-q}$.

Alternatively, one can view $\mathcal{D}^2\omega = \eta$ as a system of second-order elliptic operators that are asymptotic to the Euclidean Laplacian in the "diagonal" components and asymptotic to zero in the "off-diagonal" components (with respect to a trivialization of the spinor bundle near infinity). In this perspective, it is more straightforward to see why a systems version of Theorem A.32 should apply to $\mathcal{D}^2$. Given that $\omega \in W^{1,2}_{-q} \subset L^2_{-q+1}$ and ω weakly solves $\mathcal{D}^2\omega = \eta \in L^2_{-q-1}$, a systems analog of Theorem A.32 implies that $\omega \in W^{2,2}_{-q+1}$, and thus $\xi \in W^{1,2}_{-q}$.

Using either approach, we can establish that for any compactly supported $\eta \in L^2$, there exists a $\xi \in W^{1,2}_{-q}$ such that $\mathcal{D}\xi = \eta$. To prove the result for general $\eta \in L^2$, we can use the argument on page 174 as written (starting with, "Finally,...").

We thank Michael Lin for pointing out this mistake. We note that the same mistake appears in [**LL15**].

- On page 177, the sign error in equation (5.2) on page 165 means that the big computation of $\nu \cdot (\hat{\mathcal{D}} - \mathcal{D})\psi$ has the wrong sign starting with the third line of the computation. Consequently, the last line of page 177 should have a minus sign in front of the integral, which is exactly what we need for $H_{\text{out}} \leq H_{\text{in}}$ to imply nonnegativity of the mass. In other words, after the correction, it follows that Theorem 5.18 is correct as stated.
- On page 225, in Definition 7.17, the required decay rate for k should be $-q-1$ rather than $-q$. That is,

$$k_{ij} = O_1(|x|^{-q-1}).$$

- On page 231, there is a missing comma in the first displayed equation in the statement of Theorem 7.23. Also on page 231, there is a missing right parenthesis in the first line of the group of equations at the bottom of the page.
- On page 240, in the second sentence of the second paragraph, there is a θ_+ which should be θ^+.
- On page 243, in the second equation of Definition 7.35, the ∇u should be $\nabla\varphi$.
- On page 277, in the first set of displayed equations in the proof of Theorem 8.21, each line has a $(\operatorname{tr} k)^2$ term whose sign should be reversed. Similarly, in the first set of displayed equations at the top of page 278, there are two occurrences of $|k|^2$ whose signs should be reversed. With these corrections, combining the two calculations will yield the equation in Theorem 8.21.
- On page 279, Corollary 8.26 should be corrected in the exact same way as we described in the erratum to Corollary 5.15 on page 172.
- On page 280, the first sentence of the proof of Proposition 8.27 is incorrect. Specifically, since we do not assume nonnegative scalar curvature, Proposition 5.16 does not apply, and $\mathcal{D} : W^{1,2}_{-q} \longrightarrow L^2_{-q-1}$ need not be an isomorphism. However, it is still true that both $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are Fredholm operators of index zero from $W^{1,2}_{-q}$ and L^2_{-q-1}. This fact does not require the dominant energy condition and is true for more general PDE reasons: There is an analog of Corollary A.42 for *systems* of first-order elliptic operators that are asymptotic to the "Euclidean" Dirac operator, with the difference being that the "exceptional set" is now related to the Euclidean Dirac operator rather than Euclidean Laplacian. The proof is essentially the same, ultimately relying on a Dirac version of Theorem A.35, which can

be proved using the fundamental solution of the Euclidean Dirac system. In particular, it is true that for $1-n<s<0$ and $p>1$,

$$\mathcal{D} : W^{1,p}_s \longrightarrow L^p_{s-1}$$

is a Fredholm operator with index zero, and the same is true for $\tilde{\mathcal{D}}$. (Failure occurs here when $s=1-n$, rather than $s=2-n$ for the Laplacian, because that is the decay rate of the fundamental solution.)

We also note that Proposition 8.27 can be proved in a manner more analogous to our (corrected) proof of Proposition 5.16, but here it takes more work to prove an injectivity estimate for $\tilde{\mathcal{D}}$. See Lemma 5.5 of [**PT82**].

- On page 287 on the fourth line (in the statement of Proposition 9.4), the sign on $\langle V, \operatorname{div} \pi\rangle_g$ should be a plus sign rather than a minus sign. Furthermore, there is a factor of $\frac{1}{2}$ missing in the definition of $a \odot b$. It should be

$$(a \odot b)^{ij} = \tfrac{1}{2}(a^i b^j + a^j b^i).$$

- On page 288, the formula $\tilde{\pi} := u^{s/2}(\pi + \mathfrak{L}_g Y)$ should actually be

$$\tilde{\pi}_{ij} := u^{s/2}(\pi_{ij} + (\mathfrak{L}_g Y)_{ij}),$$

or in other words, the formula $\tilde{\pi} := u^{s/2}(\pi + \mathfrak{L}_g Y)$ is only correct if one thinks of π as a $(0,2)$ tensor, contrary to the fact that we defined π as a $(2,0)$ tensor on page 223 and treat it as a $(2,0)$ tensor throughout Chapter 9. An equivalent way to write the correct formula is

$$\tilde{\pi}^{ij} := u^{-3s/2}(\pi^{ij} + (\mathfrak{L}_g Y)^{ij}),$$

or in other words, since π is a $(2,0)$ tensor, we can write this as

$$\tilde{\pi} := u^{-3s/2}(\pi + \mathfrak{L}_g Y).$$

Unfortunately, this error repeats throughout Chapter 9, and the corresponding correction should be made on pages 289, 293–295, and 298–299.

Here we would also like to note that the *reason* for including a power of the conformal factor u in the formula for $\tilde{\pi}$ is for natural scaling purposes. However, all of the *analysis* in Chapter 9 works just as well regardless of what power of u is used. In particular, we *could have* defined $\tilde{\pi} := \pi + \mathfrak{L}_g Y$, and all of the arguments would work just as well, but the details of various formulas would be different.

- On page 288, in Exercise 9.7, the second part of the formula for $T(u,Y)$, corresponding to $\tilde{J}$, should be

$$\tilde{J}^i = u^{-3s/2}\left[(\operatorname{div}_g \mathfrak{L}_g Y + \operatorname{div}_g \pi)^i + \frac{s}{2}(n-1)(\pi + \mathfrak{L}_g Y)^{ij} u^{-1} u_{,j} - \frac{s}{2}\operatorname{tr}_g(\pi + \mathfrak{L}_g Y) g^{ij} u^{-1} u_{,j}\right],$$

and consequently, the second part of the formula for $DT|_{(1,0)}(v,Z)$, corresponding to the linearization of $\tilde{J}$, should be

$$(\operatorname{div}_g \mathfrak{L}_g Z)^i + \frac{s}{2}(n-1)\pi^{ij} v_{,j} - \frac{s}{2}(\operatorname{tr}_g \pi)\nabla^i v - \frac{3s}{2} J^i v.$$

The formula for $\tilde{J}$ above is used in the proof of Lemma 9.8 on page 289, so the second equation appearing in that proof should be

$$u^{3s/2} J^i = \overline{\Delta} Y^i + \frac{s}{2}(n-1)(\overline{\mathfrak{L}} Y)^{ij} u^{-1} u_{,j} - \frac{s}{2}\overline{\operatorname{tr}}(\overline{\mathfrak{L}} Y) u^{-1} \partial^i u.$$

- On page 289, near the bottom of the page where it says, "the decay rate of $\max(-2q-2, -n-1-\delta)$ is less than $2-n$," it should read, "the decay rate of $\max(-2q-2, -n-1-\delta)$ is less than $-n$."
- On page 290, the two formulas for $\operatorname{Hess}\xi$ should include $\pi * \nabla V$ terms, but after taking the trace, the $\Delta_g \xi$ will have the form described in equation (9.1), which is what is used on page 291.
- On page 292, the function h that appears near the end of the proof of Theorem 9.9 should be F.
- On page 294, in the second paragraph, where it says $D\Phi|_{g,\pi}(K_2)$, it should say $D\Phi|_{(g,\pi)}(K_2)$.
- On page 298, in equation (9.4) there should be a factor of 2 in $t(f+|J|_g)$, because of the factor of 2 in the definition of Φ. Also, farther down the page there is a displayed equation for $\overline{T}_\lambda(u,Y)$. The λ subscripts appearing in this equation should not be there. (There is no λ in this proof.)
- On page 305, in the definition of $A(v)$ in the middle of the page, the fraction $\frac{1}{2}$ should be inside the parentheses, so that the correct definition is $A(v) := \int_M \left(\frac{1}{2}|\nabla v|^2 + fv\right) d\mu_g$. Also, the text says that one can use the Lax-Milgram Theorem to prove existence of a minimizer, but technically, the Lax-Milgram Theorem is a statement about bilinear forms, and this A is not a bilinear form. Instead, one can produce a minimizer using the direct method and taking a weak limit in $W_0^{1,2}$, using the Poincaré inequality as needed. We leave this as an exercise.
- On page 336, in the first paragraph of the proof of Theorem A.40, it is implicitly assumed, without loss of generality, that the sequence u_i is bounded in $C_s^{2,\alpha}$ in order for the argument to be correct. We leave it as an exercise to prove this. This proof uses the fact that Δ_g has finite dimensional kernel, which we also leave as an exercise.

Lemma 3.48 (Density lemma for nonnegative scalar curvature)**.** *Let (M^n, g) be a complete asymptotically flat manifold with nonnegative scalar curvature. Let $p > n/2$ and $q < n-2$ such that q is less than the asymptotic decay rate of g in Definition 3.5. Then for any $\epsilon > 0$, there exists a complete asymptotically flat metric $\tilde{g}$ with nonnegative scalar curvature on M that is harmonically flat outside a compact set, such that $\|\tilde{g} - g\|_{W^{2,p}_{-q}} < \epsilon$ and $\|R_{\tilde{g}} - R_g\|_{L^1} < \epsilon$.*

Note that if we take $p > n$ and $q > \frac{n-2}{2}$, then Lemma 3.35 tells us that in the conclusion of Lemma , we can also demand that

$$|m_{\mathrm{ADM}}(\tilde{g}) - m_{\mathrm{ADM}}(g)| < \epsilon.$$

Proof. The proof starts off the same way as in Lemma 3.34, with the same definition of g_λ, which converges to g in $W^{2,p}_{-q}$. The difference is that instead of trying to make a conformal factor that deforms g_λ to a scalar-flat metric (which would result in a large deformation since g is not already scalar-flat), we make a conformal change to return the scalar curvature to something close to the original R_g but with compact support.

We *could* attempt to prescribe the scalar curvature *exactly*, but that would require solving a nonlinear equation: One way to do this is to use use the inverse function theorem, but it is not clear whether the relevant linearization is an isomorphism. This problem can be circumvented using surjectivity of the linearized scalar curvature operator (Corollary 6.11), at the cost of making a deformation that is no longer conformal. This is indeed the approach that we take in Chapter 9, where we prove density theorems for initial data sets, but for this proof of Lemma 3.48, we can use a simpler trick that requires only the linear theory.

Given a conformal factor u_λ, if we define $\tilde{g}_\lambda = u_\lambda^{\frac{4}{n-2}} g_\lambda$, then equation (1.8) tells us that $R_{\tilde{g}_\lambda} = u_\lambda^{-\frac{n+2}{n-2}} L_{g_\lambda} u_\lambda$. Rather than prescribing $R_{\tilde{g}_\lambda}$, we instead prescribe $L_{g_\lambda} u_\lambda$. This turns out to be good enough because in the limit as $\lambda \to \infty$, we expect u_λ to converge to 1 uniformly. Explicitly, we attempt to solve

$$-\tfrac{4(n-1)}{n-2}\Delta_{g_\lambda} u_\lambda + R_{g_\lambda} u_\lambda = \chi_\lambda R_g \tag{1}$$

with $u_\lambda(\infty) = 1$, where χ_λ is the cutoff function described in the proof of Lemma 3.34. As we have done before, set $v_\lambda = u_\lambda - 1$ so that this is equivalent to

$$-\tfrac{4(n-1)}{n-2}\Delta_{g_\lambda} v_\lambda + R_{g_\lambda} v_\lambda = \chi_\lambda R_g - R_{g_\lambda}$$

with $v_\lambda(\infty) = 0$. Next, we assume $q > 0$ without loss of generality and then show that

$$-\tfrac{4(n-1)}{n-2}\Delta_{g_\lambda} + R_{g_\lambda} : W^{2,p}_{-q} \longrightarrow L^p_{-q-2} \tag{2}$$

is an isomorphism. By the construction and asymptotic flatness of g, it is not hard to see that $R_{g_\lambda} \to R_g$ in L^∞_{-2}. This is good enough to see that the operator (2) is close enough to $-\frac{4(n-1)}{n-2}\Delta_g + R_g$ (which is an isomorphism as explained in the proof of Lemma 3.31) to be an isomorphism for large enough λ, with a uniform injectivity estimate independent of λ.

Hence, the desired solution u_λ exists, and it is smooth by elliptic regularity. Next observe that both $\chi_\lambda R_g$ and R_{g_λ} converge to R_g in L^p_{-q-2} as $\lambda \to \infty$, and thus their difference converges to zero in L^p_{-q-2}. By our definition of v_λ and the uniform injectivity estimate on the operators (2), we see that $v_\lambda \to 0$ in $W^{2,p}_{-q}$, and hence $u_\lambda \to 1$ in $W^{2,p}_{-q} \subset C^0_{-q}$, and thus $u_\lambda > 0$ for large λ. So $\tilde{g}_\lambda$ is a well-defined asymptotically flat metric, and we can see that $\tilde{g}_\lambda \to g$ in $W^{2,p}_{-q}$. Moreover, since u_λ is harmonic outside a compact set, we see that $\tilde{g}_\lambda$ is harmonically flat outside a compact set. Finally, from our construction, we have $R_{\tilde{g}_\lambda} = u_\lambda^{-\frac{n+2}{n-2}} \chi_\lambda R_g$. This is clearly nonnegative, and the only thing thing left to verify is that $R_{\tilde{g}_\lambda} \to R_g$ in L^1. Since $u_\lambda \to 1$ uniformly, this reduces to observing that $\chi_\lambda R_g \to R_g$ in L^1, which is a simple consequence of R_g being integrable. □

The earliest proof of Lemma 3.48 in the literature seems to be in [**LP87**], where it appears as part of the proof of Lemma 10.6. The proof given here is similar to the one there except that instead of solving (1), they use $\chi_\lambda R_g u_\lambda$ as the right-hand side.