William P. Thurston, 1946–2012

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Yair Minsky

Screenshot from video by Clay Mathematics Institute. Courtesy of Martin Bridgeman.

William P. Thurston died on August 21, 2012, at the age of 65. This is the second part of a two-part obituary; the first appeared in the December 2015 issue of the *Notices*.

Some of my earliest memories of Bill Thurston—hands perched over a table in the Fine Hall common room, fingers moving to indicate the Teichmüller space of the n-times-punctured sphere—are filled with his sense of visual intuition and conceptual depth. I began working with Bill as an undergraduate more or less by accident. I had the notion that I could contribute more to mathematics at that point by programming computers, and people told me "talk to Thurston. He's been using computers to draw pictures of fractals." In the end I did not do a programming project; Bill suggested I work out a naive version of his theorem for critically finite rational maps, using only the Brouwer fixed point theorem. As I struggled with this, I began to be exposed to Bill's style of doing mathematics, the importance of clear geometric intuition, and the beauty of topology. He did not push me hard or insist that I learn a lot of background, but provided a kind of gentle guidance and encouragement that I found very helpful. I remember feeling somewhat cowed by fellow students who had mastered a lot of mathematical machinery and receiving a very clear signal from Bill that this is not all there is to doing good mathematical work.

After a brief flirtation with computer science graduate school, I came back to mathematics and back to working with Bill in a more systematic way. His unique style of talking about and explaining mathematics affected my whole approach to the subject, for better and worse. Most inspiring was his insistence on understanding everything in as intuitive and immediate a way as possible. A clear mental image of a mathematical construction or proof was worth much more than a formalization or a calculation. This worked well for him and for a lot of the mathematics he inspired, but it had its downsides. In his lectures there was a kind of 0–1 law: If you were able to follow the images and the structure he was exploring, you obtained beautiful insights; but if you got lost, you were left with nothing: there was nothing on the board or in your notes at the end of the lecture but the scribbled overlay from a dozen geometric arguments.

I cannot remember many details from my meetings with Bill, but I recall the sense that he was drawing insights from a vast conceptual landscape and the struggle as a student to obtain a hint of these intuitions. When he talked his eyes would be half-closed and a slight smile would be on his face as he described something that he could see very clearly.

Of course, in the end it was all mathematics, not mysticism, and the insights came down to proofs, clear statements, and sometimes even a rare formula. When I didn't understand a proof, Bill would rarely repeat the details. Instead he would often just come up with another proof. This gave a window into a system of interconnected structure.

Bill's writing had what I came to think of as the "unique correct completion property". He wrote clearly but sparsely, and upon first (or second) reading it was hard to know how to fill in the details in his arguments. I repeatedly had the experience of thinking for a long time and realizing finally that there was a real missing point or condition that needed to be mentioned. Then, upon reconsulting the text, I would discover that the point had been written in there the whole time.

I volunteered one time in Bill's course to look up the proof of something; to my detriment I cannot remember what it was. I dutifully studied the proof, understood the sequence of calculations, and reported back to the class, where I began working through them on the board. Bill looked on with a pained expression on his face and eventually said, "I didn't mean a *formula*.… "

There was a year that Bill and Dennis Sullivan combined their groups of graduate students in a joint seminar that alternated between Princeton and the CUNY Graduate Center. We learned a lot of mathematics from these seminars and from the conversations that took place around them and on the train rides. Bill's interaction

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Courtesy of Rachel Findley.

Kitchen duties, with Dylan and Nathaniel Thurston.

with Dennis always gave me the feeling that a vast and deep universe is out there, with structure that they both understood intimately.

Bill's influence continued long past graduate school. The questions he raised and the techniques he introduced have continued to be a central part of the way I do mathematics. I still do not feel I have really understood something or can be confident about a proof until I have been able to scratch out some kind of geometric doodle that encapsulates the argument. Bill cared a lot about mathematical communication and the process of doing mathematics, and I try to keep in mind his insistence that mathematics is not just about theorems but really about patterns and structure.

Lee Mosher

I first encountered Bill Thurston's name as an undergraduate senior at Michigan State when my mentor, Richie Miller, told me to "go to Princeton and study with Thurston." So I did, starting in 1979.

Being Bill's student during that time was a heady experience, during which I learned how exciting mathematics could be. A stream of visitors came through who became mentors and colleagues in future years, including David Epstein, Michael Handel, Ulrich Oertel, Daryl Cooper, and many others. Bill's mathematical ideas spread through papers, lecture notes, courses, seminars, discussions, and many other forms of activity. I witnessed and experienced firsthand the benefit of how Bill practices mathematics, as he would later describe in a summary sentence from his essay "On proof and progress in mathematics" [*Bull Amer. Math. Soc., (N.S.)* **30** (1994), 161–177] : "The measure of our success is whether what we do enables *people* to understand and think more clearly about mathematics."

In that first year (as well as subsequent years) I attended Bill's course, which was full of ideas and techniques and theorems and tricks and which imparted more than just mathematics, but a way of thinking about mathematics. I also attended the graduate student seminar, where I learned a fair amount of background from the older students, including David Gabai and Robert Meyerhoff. I had not interacted much at all with Thurston during my first semester when, one day, while wandering down the corridor of the new Fine Hall, Thurston came wandering towards me in the opposite direction, looking at me with a smiling and expectant demeanor. I realized with a slight panic that I ought to say something, so I blurted out some half-formed thoughts about simple closed curves based on things I had learned in the graduate seminar. He took me up to his office and gave me a copy of the book in which his work on this topic, including his magnificent theory of pseudo-Anosov homeomorphisms of surfaces, had been written up by the French seminar led by Fathi, Laudenbach, and Poenaru. Over the years I often found Bill to be similarly generous with his time and his mathematical ideas.

During my Princeton years Thurston and Rachel Findley, his wife at that time, were also gracious and generous with their personal time. Matt Grayson and I hung out a lot with Bill and Rachel and their children, Nathaniel, Dylan, and Emily. During our last two years at Princeton, Matt lived with them, I lived nearby, and Bill, Rachel, Matt, and I formed a kind of food co-op in which we jointly shopped for food, cooked meals at their house, and attempted, with limited success, to keep some semblance of order in the kitchen.

Bill's enthusiasm during the early stages of mathematical discovery was infectious. Once, while sitting in his living room, Bill said to me, "I can do this group with grep," which was sort of strange to hear at first. But being his student I knew just enough computerese to have an inkling of what he was saying: he was able to compute in that group with the UNIX utility for processing regular expressions using finite deterministic automata. From there, it was exciting to observe the quick unfolding of the theory of automatic groups.

Conversation with Bill could be hard, either because of strange statements that would stop you in your tracks or what would seem to be his lack of attention. Matt Grayson tells a story about the encounter between his father and Thurston at our 1983 Princeton graduation. After talking with Thurston, Mr. Grayson came to Matt and said that the conversation was difficult because Thurston's attention seemed to wander. Matt advised his father: "He thinks about many things at the same time. Try saying just every other word." Mr. Grayson came back later and said, "You were right! That worked."

Bill was also generous with his deep knowledge of mathematics. In the spring of 1995 at MSRI, while Thurston was director, Benson Farb and I were quixotically attempting to prove that a certain class of groups, the solvable Baumslag-Solitar groups, were all quasi-isometric to each other. We had trouble constructing certain quasi-isometries of the hyperbolic plane—when we pushed them this way, they sprung out that way, and so on—and we began to suspect that these quasi-isometries were much more rigid than

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we had at first thought. We explained our troubles to Bill, who told us about a related but somewhat obscure phenomenon long known to complex analysts: a rigidity property for certain conformal maps of the hyperbolic plane, maps closely related to the quasi-isometries we were vainly attempting to construct. That was just the idea that we needed, and our project flipped over instead to developing a rigidity property for those groups.

In our graduate student years Matt Grayson and I would marvel at the mysteriousness of Bill's ideas, saying to each other, "He's so weird." We thought that Bill was obscurely acknowledging this point when, during our third year of graduate school, as part of Bill's entourage visiting the University of Colorado, a bumper sticker appeared on his blue van with the word "Alien" superimposed over the mountain silhouette familar from the Colorado license plate. Of course we also figured that it was really just an antichauvinistic response to the popular "Native" bumper sticker of the time.

The marvel of Bill's ideas continued to his last years. His recent article "Entropy in dimension one" contains a wonderful theorem characterizing the entropies of free group outer automorphisms. He includes an example that continues to boggle my mind of a free group outer automorphism with entropy 3, in sharp contrast to the irrationality of the entropies of surface group outer automorphisms.

The world is a richer place for Bill Thurston having been in it.

Jeff Weeks

Bill treated everybody with equal respect. Whether he was talking with the university president or with the janitor, it didn't matter; he treated all people with kindness. This may or may not have been a direct result of Bill's Quaker beliefs; it was certainly consistent with them.

During my years as his graduate student, Bill gave me much mathematical advice. Most of it I have forgotten, and much of it I didn't fully grasp even then. Yet one piece of advice made an impression on me at the time and has stuck with me ever since: "Don't make arbitrary choices. Do only what you're forced to do." In other words, if you're trying to prove a theorem and you find you need to make an arbitrary choice, then you're probably looking at things the wrong way. You should resist the temptation to simply make the choice and push on. Instead, you should stop, take a step back, and try to rethink the problem in a way that requires no arbitrary choice. That advice from Bill has served me well countless times over the years, and I've passed it on to all students willing to listen. It applies not only for proving theorems but also for designing software: if a particular algorithm requires you to make an arbitrary choice, it's time to stop, step back, and find a better and more natural algorithm.

Bill's gift, of course, was his vision, both in the direct sense of seeing geometrical structures that nobody had seen before and in the extended sense of seeing new ways to understand things. While many excellent mathematicians might understand a complicated situation, Bill could look at the same complicated situation and find simplicity. For example, his method for putting hyperbolic structures on knot complements is a straightforward exercise in cut-and-paste topology. When I first saw that method I was struck by its simplicity. My fellow graduate students and I could have discovered it years earlier. We had all the tools we needed, and the method was easy, almost obvious. But we didn't see it. Bill did.

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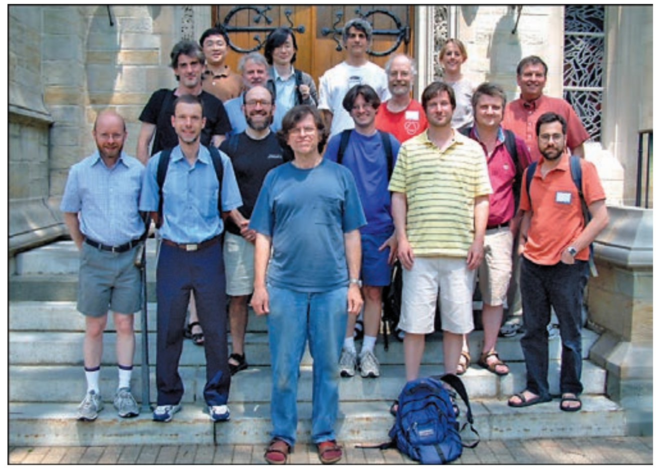

Courtesy of Princeton University.

Thurston with some of his students, 2007.

Benson Farb

Being a Thurston student was inspiring and frustrating—often both at once. At our second meeting I told Bill that I had decided to work on understanding fundamental groups of negatively curved manifolds with cusps. In response I was introduced to the famous "Thurston squint", whereby he looked at you, squinted his eyes, gave you a puzzled look, then gazed into the distance (still with the squint). After two minutes of this he turned to me and said, "Oh, I see, it's like a froth of bubbles, and the bubbles have a bounded amount of interaction." Being a diligent graduate student, I dutifully wrote down in my notes: "Froth of bubbles. Bounded interaction." After our meeting I ran to the library to begin work on the problem. I looked at the notes. Froth? Bubbles? Is that what he said? What does that mean? I was stuck.

Three agonizing years of work later I solved the problem. It's a lot to explain in detail, but if I were forced to summarize my thesis in five words or less, I'd go with "Froth of bubbles. Bounded interaction."

A Thurston lecture would typically begin by Bill drawing a genus 4 surface, slowly erasing a hole, adding it back in, futzing with the lines, and generally delaying things while he quickly thought up the lecture he hadn't prepared. Why did we all still attend? The answer is that once in

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a while we would receive a beautiful insight that was absolutely unavailable via any other source.

Here's an example. Consider a Tinker Toy set of rigid unit-length rods, bolts, and hinges. Rods can have one end bolted to a table and can be hinged to each other. For any given Tinker Toy T, bolted down on a table at one point, we have the space $C(T)$ of all possible configurations of T. If T is a single rod, then $C(T)$ is a circle. If one adds a hinged rod on the end of T, the resulting configuration space is the torus. What other smooth, compact manifolds can you get with this method? I still remember the communal thrill when Bill explained to us how to obtain all compact, smooth manifolds as a component of some $C(T)$. Further, every smooth map between manifolds can actually be represented via some rods connecting the two associated Tinker Toys.

Thurston completely transformed several areas of mathematics, including 3-manifold theory, foliation theory, geometric group theory, and the theory of rational maps. His papers contain a dizzying array of deep, original, influential ideas. All of this is well known. However, in my opinion Thurston's influence is underrated; it goes far beyond the (enormous) content of his mathematics. As Bill wrote in his paper "On proof and progress in mathematics" [*Bull. Amer. Math. Soc. (N.S.)* **30** (1994), 161–177]: "What mathematicians most wanted and needed from me was to learn my ways of thinking, and not in fact to learn my proof of the geometrization conjecture for Haken manifolds."

We did learn his ways of thinking—or at least some approximation of them. Bill changed our idea of what it means to "encounter" and "interact with" a mathematical object. The phrase "I understand X" has taken on a whole new meaning. Mathematical symbols and even pictures are not sufficient for true understanding, especially in geometry and topology. We must strive to live somehow inside the objects we study, to experience them as three-dimensional beings. I think that this change is now almost invisible; it has become a structural feature of the way many of us do mathematics.[1] This kind of pervasive influence can be likened to the way that Grothendieck changed the way many people think about mathematics, even on topics Grothendieck himself never touched.

The change in viewpoint described above was taken beyond topology by many of Thurston's students, who went out and "Thurstonized" a number of other areas of mathematics, changing those areas in a notable way. Oded Schramm's work is a case in point. Early in his career Schramm solved many of the major open problems about circle packings. This theory gives a way to really understand (in the Thurstonian sense) the Riemann Mapping Theorem as the limit of an iterative process. Schramm then moved on to apply his geometric insight to understand the scaling limit for many two-dimensional lattice models in statistical physics. The Schramm-Loewner evolution gives a geometric, "what it looks like" understanding of these limits.

Bill was probably the best geometric thinker in the history of mathematics. Thus it came as a surprise when I found out that he had no stereoscopic vision, that is, no depth perception. Perhaps the latter was responsible somehow for the former? I once mentioned this theory to Bill. He disagreed with it, claiming that all of his skill arose from his decision, apparently as a first-grader, to "practice visualizing things" every day.

I think that there is a fundamental misunderstanding that many people have about Thurston's work. In particular, the completeness of the proofs in his later work has sometimes been questioned. Such complaints are not justified. One can point to Thurston's occasional lack of proper attributions and to some brevity in his mathematical arguments, but, for the most part, he gave complete, albeit concise, proofs. The skepticism seems to stem from the frustration one can feel in not understanding what Bill was trying to communicate and the desire for more detail, only to realize, after understanding things, that the details were there all along.

I had an uneven relationship with Bill. However, like so many other people, my mathematical viewpoint was shaped by his way of thinking. In interacting with other mathematical greats, one gets the feeling that these people are like us but just 100 (OK, 500) times better. In contrast, Thurston was a singular mind. He was an alien. There is no multiplicative factor here; Thurston was simply orthogonal to everyone. Mathematics loses a dimension with his death.

[1] *This reminds me of the story of the old fish who passes by two young fish and says, "Morning, boys. How's the water?" The two young fish look at each other, and one asks the other, "What the heck is water?"*

Danny Calegari

I counted Bill as my friend, as well as my mentor, and I have many vivid and happy memories of time I spent with him.

I remember seeing Bill for the first time when I arrived at Berkeley in 1995. At the start of the academic year, all the incoming graduate students were ushered into the colloquium room to meet some of the senior personnel. Bill was there in his capacity as director of MSRI (the Mathematical Sciences Research Institute). He was wearing jeans with big holes at the knees. He made a speech about MSRI, inviting us all to come up the hill and interact with the visitors there. He also encouraged us to pronounce it as "emissary" rather than "misery". It didn't work; we all called it "misery" (and still do).

I remember actually taking the bus up the hill (maybe a few months later?) in the vague hope of running into Bill and asking him to be my advisor (people had warned me against this, saying that Bill "wasn't taking students," because he was too busy running MSRI). I don't think I had a very clear plan about how this was going to work out. I walked in and saw Bill chatting with Richard Kenyon about the entropy of dimer tilings and hyperbolic volume. At this point I basically froze, turned around, and walked out again.

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I remember Bill running the "very informal foliations seminar" at MSRI with Dave Gabai, Joe Christy, and a few other people. This seminar was not advertised; I basically wandered in off the street into the middle of a three-hour lecture by Bill, explaining his new ideas about universal circles and how they might be used to approach the geometrization conjecture for 3-manifolds with taut foliations. By the time he was done, I had decided I wanted to work on foliations, and I more or less had my thesis problem.

I remember when Bill moved to Davis. This was the only time I ever saw him in his office at Berkeley, when he was cleaning it out. I remember the little photo that used to be on the door—the one that's on the cover of *More Mathematical People* [Harcourt Brace Jovanovich, Boston, 1990]—of Bill as a child working at a desk. He saw me watching him carrying his boxes out of his office and looking at the photo and gave a slightly embarrassed smile.

I remember emailing Bill in early 1998 to explain a few of my tentative ideas about foliations, which had been inspired by his slitherings paper. He invited me to come out to visit him at Davis and talk to him in person. Over the next year or so I drove out there perhaps a couple of times per month, struggling up the freeway in my third-hand lemon, with the wind rushing in through the bad seals in the door frame. We would have conversations that lasted for hours, stopping occasionally for lunch and coffee. Bill basically became my "unofficial advisor" (my official advisor, Andrew Casson, was moving to Yale), and perhaps because he did not have many "real" students at Davis at the time, I got a lot of his attention. We spent a lot of time working through the theory of universal circles; I learned a huge amount of mathematics, not only stuff obviously connected to foliations (or even low-dimensional topology) but combinatorics, analysis, group theory, and so on. And yet, Bill listened very carefully to my ideas and always gave them his full attention and consideration. At the time I don't think I appreciated how rare this attitude is in a senior mathematician towards a graduate student.

I remember when we were trying to work out the details of some construction, Bill got very enthusiastic, and we went to the campus store to buy some enormous sheets of paper and a few sets of colored pencils, bringing them all back to Bill's office and laying the paper out on the floor. Bill was really excited by this episode; he remarked that he used to do this sort of thing "all the time" when he was at Princeton. I got the impression he hadn't done it for a while.

I remember working to try to get a project finished in the week before Bill's daughter was born (we didn't make it in time). My wife and I were thinking about having kids at the time, and I shyly asked him about the experience. He became very emotional and tender and talked about what it was like to hold a newborn and have them lie in your arms, trusting you completely.

I remember visiting Bill in the winter of 2008. At the time my family and I were on a vegan kick, and I remember discussing veganism and Colin Campbell's book *The China Study* with Bill while waiting for the cafeteria people to make us our vegan burritos for lunch. Bill's wife happened to be very sick that week, and in addition they were moving house, so Bill was very distracted. When I left at the end of my visit, Bill apologized for being distracted with so many other things but hoped that I'd visit again soon. Of course I told him not to apologize, that I'd had a great visit (which was true), and that I hoped I would come again soon when we both had more free time. That was the last time I saw him.

Ian Agol

My first encounter with Thurston was during a graduate student workshop at MSRI. He demonstrated to us how he could count in binary on his fingers and told us how many steps we had taken on the hike to the picnic, which he had counted out using his method while hiking. Later during the workshop, when I expressed my interest to him in finding a solution to the unknotting problem using grid diagrams, he immediately dismissed my approach and took me up to the computer lab to demonstrate the program SnapPea (written by his former student Jeff Weeks and others). This made an impression on me, since his geometric approach clearly gave a more powerful way to study knots than the intrinsic three-dimensional route I was taking (it's interesting to note, however, that grid diagrams have become an important tool recently in the study of knots and their invariants). Moreover, SnapPea enabled Thurston to take a deep mathematical construction (namely, his proof of the geometrization conjecture for Haken manifolds) and make it into a very concrete output that could be appreciated by nonexperts. I learned later that his approach to mathematics was to start with a simple model where things could be understood very explicitly, and once this model was understood well, it would help him understand the more general case. For example, he told me that he re-proved Andreev's theorem for Haken reflection groups using the techniques that would generalize to his proof of the geometrization conjecture for Haken manifolds. This principle has guided my own research, where I always ask myself when first considering a problem, "What is the simplest nontrivial special case of a given problem?"

My first meeting with Thurston at Davis was a bit of déja vu: I began to explain a result from my thesis about volumes of Haken hyperbolic 3-manifolds using the Gromov norm, and he immediately began thinking of an alternative approach using the new technology developed by Besson-Courtois-Gallot. Although this approach didn't work initially, it eventually led to a collaboration several years later when the more powerful techniques of Perelman became available.

I sat in on a graduate seminar run by Thurston each quarter called Experimental Mathematics, where he would ask for questions that interested the students and then would attempt to investigate the problem

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Neil Michel (Axiom), Special Collections, University of California Library, Davis.

Bill Thurston and Ian Agol teaching a class at UC Davis.

using Mathematica or other programs. I remember being impressed with a solution to a knight's tour problem on a chessboard, which he solved on the fly essentially by folding the chessboard up into smaller sizes, until the solution became trivial. I learned a lot of tricks for using math programs from this class, which ended up being helpful in my research.

We also co-taught a version of the class Geometry and the Imagination, where I probably learned much more than the students. One class, we gave them a series of "exercises in imagining" problems, including one to show that pentagons glued three to a vertex would form a dodecahedron. While they were doing this, we tried our own three-dimensional version of the problem to visualize gluing four dodecahedra to a vertex, making the 120-cell. Mathematics becomes much more vivid and personal in this way and enables access to the vast computational and memory power of the visual cortex. At one point he had read a book about vision and the brain, and explained to us how the image from the eye projected onto the brain is approximately a conformal mapping. I learned also that he thought anyone could do mathematics using visualization and that he really didn't want to bias the students as to the outcome of a particular exercise, but wanted them to make their own discoveries.

After leaving Davis, I had only occasional interactions with Thurston by email or at conferences, but I found he could usually digest my interim research accomplishments with about five minutes of explanation. This was another thing I learned from him: that it is much easier to absorb mathematics talking to experts than by reading papers. This is especially true within the field of low-dimensional topology, where the right picture can replace the many pages of obscure notation sometimes needed to translate it into rigorous mathematics.

When Dani Wise was lecturing on his work in New York a couple of years ago, Thurston happened to be in New York for treatment and came by the talks a few times. In spite of his degenerating health, he asked questions enthusiastically about the talks. He asked me at one point if he were to give me $1,000 to find a mistake in Wise's arguments, where would I be most likely to find an error?

I'm glad to have been introduced to Thurston's mathematical playground, and I hope to be able to inspire others with his vision and insights.

Genevieve Walsh

I want to try to give an impression of what it was like to be Bill Thurston's student at Davis. When I first arrived at the University of California, Davis, in 1997, Bill was a new professor, teaching Differential Geometry. He was quite famous by this time, and his class was packed. I still have some of my notes from this class. They are filled with pictures of turning the plane minus a point into a cylinder, pictures of a cube folded along its faces, descriptions of the space of all hexagons, with plane fields and Lie brackets and frame bundles, with computations of the fundamental groups of knot complements, with descriptions of "group negative curvature" and quasi-isometries, with laminations and Gabai's Ubiquity Theorem, and with pictures of what it is like to live in the different geometries. For example, there is a picture of a very skinny person with an arc connecting the head and feet. Next to it is an explanation: "In $\mathbb{H}^2 \times \mathbb{E}$, you look tall and skinny. (Not many fit in vertical circles; a lot fit in horizontal circles.)"

By the end of the year, there were three students left in the class. I was hooked.

I began to work with Bill Thurston, and my notebooks from this period contain many pictures of cusps and bending laminations and computations to figure out what the knot is from a gluing diagram of its complement. I was also lucky to have many like-minded mathematicians in the department to help with clarification. Greg Kuperberg, Ian Agol, Abby Thompson, and Joel Hass were all very useful resources. Still, I was often confused. After our meetings, I would try to write up quickly what had happened so I would not forget what was going on. Here are a few random excerpts: "I'm not sure exactly why we started talking about this, but I asked about bending and he started telling me about complex projective structures…."

"This is kind of a way to 'see' the link with the Hopf fibration and the Clifford torus. Figure out how this works."

"I made a little bit of progress over the weekend, mostly by reading the paper by Hatcher and Thurston, which I still do not totally understand and which Thurston doesn't want to explain to me."

"I asked about whether or not quadratic differentials can be identified with the tangent space of Teichmueller space. He said, yes they can, but sometimes it's better to use the dual. He said to look it up in some book, he might get it mixed up."

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"In this meeting, we mainly talked about the paper by Cooper and Long, which shows that most Dehn fillings are virtually Haken. I'm writing this up on paper because there were a lot of pictures in this meeting."

"So I got to tell him that the Schwarzian derivative is $f'''/f' - (3/2)(f''/f')^2$."

Apart from trying to understand and picture many different things, and figuring out formulas which he steadfastly refused to put in his papers, I was also supposed to be working on some project. The problem was, I wasn't completely sure what this was. One day, instead of meeting in his office, he went out and brought us some coffee and some great circle links. Of course, I was thrilled to be out talking math, having coffee, and walking around with a bag full of perfectly round key rings. But I still didn't know what I was supposed to DO with them. I asked Joel Hass for some mathematical direction one time, to which he replied, "Do whatever it is that Bill tells you to do!" Great circle links did turn out to be beautiful examples of several phenomena, but I didn't figure that out until later. They are also just plain fascinating, as are so many of his suggestions.

Years later I went to Cornell to give a talk and had the opportunity to meet with Bill Thurston again. He was very ill and seemed tired as we talked in his small house one afternoon by the fire. Bill was wearing a big wool sweater that dwarfed him, and he could barely speak. But Bill asked what I was doing mathematically, and I wanted to talk about it. So I asked him about the space of acute triangulations of the 2-sphere that are combinatorially equivalent to some specific triangulation. In particular, is this space connected? We talked about it, and he turned the problem over in his head, thinking about it from different points of view. He thought it was interesting, and at one point he nodded and said, "I think it probably is."

Carol Wood

Bigger than life and smarter than I could have imagined, Bill Thurston brought a generation of mathematicians to see things that they might otherwise have missed: mathematical things, of course, but not only. In remembering Bill, I would like to point to things he saw and did at MSRI, resulting in changes in the fabric of the institution.

It has been my privilege over time to meet all the MSRI directors and their teams. Each has put an imprint on MSRI, and each has worked hard and smart, to the benefit of mathematics and mathematicians. My perspective is that of someone who was a member in 1989–90, a member of Bill's team in 1996–97, and, subsequently, a program organizer and trustee. Bill was director of MSRI from 1992 to 1997; his deputies were Bob Osserman, Lenore Blum, T. Y. Lam, and, for the final year, me. Bill thought deeply about things. He formulated a vision of the role of MSRI and of himself as director, some of which was articulated in a document called "Possibilities". I cannot assign individual credit to each event of Bill's years as director, 1992–97, although anyone involved with MSRI in that era would know that the genius touch for public events was Bob Osserman's.

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Courtesy of Carol Wood.

Thurston and Bob Osserman at MSRI, 1997. Sculpture by Helaman Ferguson.

Under Bill's leadership, MSRI's reach broadened dramatically. Here are some examples:

- Mathematical Conversations, in which mathematics teachers and researchers meet as equals, as well as workshops and conferences about mathematics education, such as one on the "calculus wars".
- Programs for students and young researchers, including introductory workshops and the admission of graduate students to research programs. (In 1989–90 we had to smuggle students into the building!)
- Public events, most famously the Fermat Fest in San Francisco in October 1993, but several other events aimed at bringing mathematically relevant topics to the general public.
- Introduction of the latest forms of technology and communication, including video streaming, Mbone, and Elmo projection. I recall a demonstration of 3D printing in 1997, something only now poised for widespread use.
- Promoting diversity, with the establishment of the Human Resources Advisory Committee, by hosting the first-ever CAARMS (Conference for African-American Researchers in the Mathematical Sciences), by coorganizing the Julia Robinson Conference with the Association for Women in Mathematics.
- Opening sponsorship to departments beyond the eight in the original proposal, so that there were thirty in 1997. (At present there are over ninety sponsors.)

One change, a cosmetic one, was to trade the MSRI nickname "misery", with its negative connotations, for "emissary", to indicate the outward-looking institute it was to become. Ironically, this new nickname did not take except as the name of the newsletter: some of the old guard persist with "misery", but most often the four initials are articulated, in the mouthful "em-ess-are-eye". I admit I never cottoned to either nickname, so I'm happy with the end result.

Most, if not all, of the changes Bill achieved may seem less original or revolutionary today, but they were sea changes in the mathematical culture of the day, all the more so since they took place in an institution devoted to the best of mathematics, and these changes put MSRI well ahead of the curve. Bill's initiatives were developed in subsequent years under the leadership of David Eisenbud and Robert Bryant, who added their own ideas as well. Today, mathematics students, educators, departments, and researchers worldwide claim ownership of MSRI. The visionary Bill Thurston had indeed thrown open the doors, and good luck to anyone who tries to pull them shut again!

In 1996–97, Bill's final year as director of MSRI, I joined the team as deputy director, a job offered to me out of the blue. This was not the first time since we met in the early '90s that Bill had proposed a role for me that I could barely imagine for myself, but it was the biggest role by far. In conversations with others that spring about the job, I heard complaints, grumbles, even predictions of disaster for MSRI. I trusted Bill's judgment, buoyed by Alberto Grunbaum's saying he too thought I could do the job. When I arrived at MSRI, Bill greeted me, as he did all visitors at that time, by displaying his handmade tape model showing the great similarities between human and pig DNA. I witnessed then and many times thereafter his childlike capacity for sheer delight. That was my first hint that this daunting job was also going to be fun.

Bill was the kindest, least pretentious, and most affable person I have ever worked alongside. Moreover, he had assembled a staff remarkably talented and dedicated to the concept of an institute which served its community. My role was straightforward: to make timely decisions so that others could do their jobs. Bill was untidy and Bill was a perfectionist, a combination that sometimes challenged this practical, not-so-visionary deputy. However, during what he might easily have considered his "annus horribilis", I never heard a complaint from Bill nor an unkind word from him about anyone. (I cannot say the same for me.) Seeing that the MSRI train ran on time was—and still is—a 24-7 job. But I regretted then, and regret all the more with his untimely death, that I didn't stop more often just to enjoy being around Bill. When I did, it was delightful.

The last time I saw Bill was in Boston at the mathematics meetings in January 2012. He spoke of his plan to visit Berkeley to be near family. He intended to come to MSRI that year for the first time since 1997. I know Robert Bryant shares my regret that fate intervened. Bill's remarks in accepting the Steele Prize were poignant: he expressed gratitude to the mathematics community for its acceptance of him and his ideas. In our finest moments, mathematicians are tolerant of each other's quirks, a fact about which I have always felt proud. No one achieved this acceptance of others more effectively and more widely than Bill Thurston. For his leadership at MSRI, as well as for his mathematics, the mathematics community owes him an enormous debt of gratitude.

Tan Lei

I was visiting John Hubbard at Cornell during the winter of 1986. His student Ben Wittner and I were among the first generation of students exploring Thurston's newly found characterization theorem for rational maps. This powerful theorem gives a necessary and sufficient condition for a flexible object, dynamics on a branched covering, to represent a rigid object, the dynamics of a rational map. It is one of the fundamental theorems in the theory of iterations of rational maps.

Ben and I decided to make a one-day trip to Princeton to meet Bill Thurston in person. That was the first time I met Bill. I was very impressed by his highly animated seminar, scheduled at lunchtime, full of young people eating burgers and drinking cokes while discussing mathematics around him.

Many years later I mentioned this to Bill, and he said that a lunchtime seminar was not a common practice, but they were trying to get a more informal atmosphere to lower people's defenses so that they could discuss more naturally what they were actually thinking. Bill also expressed that he was spoiled in Princeton, where he was surrounded by a big group of graduate students and junior mathematicians, enabling him to take the role of queen bee. I think that around this time he was supervising several PhD students simultaneously.

Despite his fundamental role in the theory of holomorphic dynamics, Bill Thurston rarely participated in the activities of the community, probably because he was totally absorbed by his other interests (and what interests!). Most of the newcomers in the field did not have the chance to meet him in person.

Luckily for us, Bill renewed his interest in rational dynamics during the last two or three years of his life and started to participate in our conferences. The reason for this comeback was explained in his talk at the Banff conference in honor of John Milnor's eightieth birthday in February 2011: he was led by his investigation of realizing Perron numbers as various dynamical growth factors. A video record of this talk can be viewed on the conference webpage.[2]

This video, as well as many other video records of Bill's talks, illustrates very well Bill's working style: he definitely preferred geometrical visions to formulae, he used computers intensively to do calculations and illustrations for him, and in his descriptions of mathematical objects he often put himself inside, like the moment he was talking about projective space and looking up towards the line at infinity. Later on I experienced many face-to-face and email conversations with him in that style. I must say it wasn't always easy (actually quite often

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[2] `www.math.sunysb.edu/jackfest/Videos/Thurston/`

difficult) to understand him: either I would have the right intuition and get it all or I would get nothing. Sometimes I preferred email conversations, which would allow me to slowly build up correct intuitions represented by his descriptive words. And once you realized what he meant, the clouds would disappear and things became just clear and beautiful.

Bill's geometric insight was truly amazing. Even a classical result such as the Gauss-Lucas Theorem, when it passed through Bill's hands, gained immediately a deeper geometric meaning that I never saw before. Later on, Arnaud Chéritat and I wrote a tribute to Thurston (on the French website Images des mathématiques) illustrating his point of view of this theorem.

Here are some of Bill's own words about his being a geometer:

I've always taken a "lazy" attitude toward calculations. I've often ended up spending an inordinate amount of time trying to figure out an easy way to see something, preferably to see it in my head without having to write down a long chain of reasoning. I became convinced early on that it can make a huge difference to find ways to take a step-by-step proof or description and find a way to parallelize it, to see it all together all at once—but it often takes a lot of struggle to be able to do that. I think it's much more common for people to approach long case-by-case and step-by-step proofs and computations as tedious but necessary work, rather than something to figure out a way to avoid. By now, I've found lots of "big picture" ways to look at the things I understand, so it's not as hard.

Another deep conviction of Bill Thurston's was that all mathematics is connected: *The more you make connections, the more you see things as interconnected and the more you expect these connections.*

I once witnessed his exceptional connecting skills in action, a truly remarkable experience. In March 2011, Bill was investigating the space of iterated cubic polynomials. Each such polynomial can be combinatorially described by a finite lamination in the unit disc called "a primitive cubic major": it is either an equilateral triangle inscribed in the circle or a pair of chords that each cuts off a segment of angle-length $2\pi/3$. While trying to visualize the space of such laminations, he recognized a familiar pattern. Here is how he described the discovery on the 26th of March:

This figure can be embedded in S^3, which should somehow connect to the parameter space picture.... This (figure) is also a spine for the complement of the discriminant locus for cubic polynomials, but I'm not sure how that description fits in.

Having sensed a connection between the new object and the old familiar one, he obviously set off in search of a genuine link. The complement of the discriminant locus consists of polynomials with no multiple roots. However, it is not obvious at all how such a polynomial might appear from a pair of disc chords, but they should be connected somehow. At some moment Bill decided to try the opposite direction: why not try to put oneself in the familiar object and try to reach the new one from there? Why not try first to construct a pair of chords from a cubic polynomial with no multiple roots? This idea put him on the right track. He soon saw that the construction could easily be made, be reversed, and even worked for any degree. Here is what he wrote on April 1:

Take any degree d polynomial P with no multiple zeros, and look at log(P), thought of as a map from $\mathbb{C}\setminus\{roots\}$ to an infinite cylinder. For each critical point, draw the two separatrices going upward (i.e., this is the curve through each critical point of P that maps by P to a vertical half-line on the cylinder, a ray in $\mathbb{C}$ pointed opposite the direction to the origin).

Then make the finite lamination in a disk that joins the ending angles of these separatrices. This is a degree-d major set. Conversely, for each major set, there is a contractible family of polynomials whose separatrices end at the corresponding pairs of angles.

To pick a canonical representative of each of these families: look at polynomials whose critical values are on the unit circle. This forms a spine for the complement of the discriminant locus for degree d polynomials.

This is it! A beautiful theorem is born; a surprising bridge is built.

Bill was obviously very proud of this discovery, as we can see by how he presented it one year later.[3]

It turns out that the set of all "critical" degree d polynomials can be approximately described by collections of arcs of the disk whose endpoints have angle between them of the form k/d. It took me a while to realize that the set of all such arcs, along with the limiting cases where some endpoints coincide and there are additional implicit arcs, describe[s] a spine for sets of d disjoint points in $\mathbb{C}$, that is, its fundamental group is the braid group and higher homotopy groups are trivial.

People who had the chance to meet Bill Thurston know that he was a very caring and friendly person. You could feel his respect for others and his keenness to make everybody comfortable. I remember once, over a conference dinner table, he asked us one by one, with a gentle smile and with full attention, "So how did you come to mathematics?" One of us, Pascale Roesch, said that she initially hesitated between psychology and mathematics, and that response provoked a vivid conversation around the table for a long while.

At some point Bill was invited to a brain study about creative people. He got his brain tested, scanned, and modeled. During one of my visits to Cornell, while I was deeply absorbed by some hard thinking, he checked his email and said, out of nowhere, "My brain will arrive tomorrow." After a long moment of total confusion, I finally realized that he was only talking about the clay model of his brain which was sent to him.

Later on Bill showed off proudly his "brain" to everybody, and I must say it provoked quite a sensation in me, holding in my hands this exceptional "brain".

Sadly, we have since then lost this brain, leader of thoughts, and the many theorems he was about to prove. But, above all, we have lost a very dear friend. Bill, William Paul Thurston, will always be missed.

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[3] *Topology Festival, Cornell, May 2012, from Kathryn Lindsey.*

Curtis McMullen

Simple Curves

Thurston was a master at finding fresh and novel ways of looking at things.

What could be simpler than a loop on a surface? But in Thurston's hands, the collection of all simple loops (once completed) became a geometric object in its own right—the space $\mathbb{P}\mathcal{ML}$ of *projective measured laminations*. It now plays a central role as the boundary of Teichmüller space and stands as one of Thurston's most widely used inventions/discoveries (anticipated in the work of Nielsen). Yet it emerges from elementary topology as directly as the real numbers emerge from $\mathbb{Q}$.

The accompanying figure, taken from his Princeton notes *The Geometry and Topology of 3-Manifolds* (1979), gives part of the construction of stereographic coordinates on $\mathbb{P}\mathcal{ML}$, showing this space is a sphere (of dimension $6g - 7$ for a surface of genus $g > 1$).

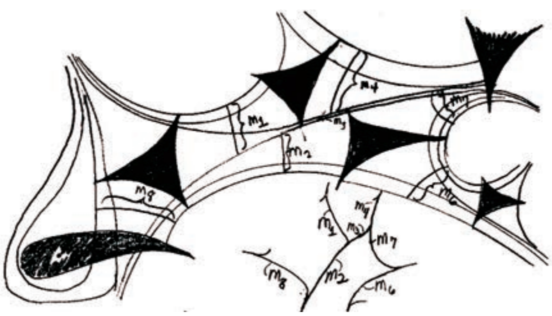

Figure 1.

View from Harvard, 1980s

I first heard about Thurston's work when I was a graduate student at Harvard, 1981–85. He had recently discovered a characterization of the branched covers of S^2 that arise in the dynamics of rational maps, but no detailed write-up was available. In this exciting atmosphere, Hubbard (who was visiting for a semester) lectured to a small group on the audacious idea of the proof: an iterative scheme on Teichmüller space whose fixed point would give the desired rational map. In those days Thurston's mimeographed Princeton notes, whose typed pages were covered with drawings that sometimes crossed the text itself, sat on the library reserve shelf next to more forbidding texts that assumed background in schemes, L-functions, symbols of operators, etc., and were often unencumbered by pictures, intuitions, or even examples. One sometimes heard that Thurston's notes were full of "great ideas," with a hint that rigorous arguments were missing and the work might be impounded for mathematical misdemeanor.

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Thurston at Princeton

While a professor at Princeton, Thurston served as my NSF postdoctoral supervisor, 1987–90. As I soon realized, Thurston actually had an uncanny ability to turn his insights into transparent logical proofs which integrated, rather than disguised, the underlying geometric intuition.

The audience for Thurston's course at Princeton included Peter Doyle and me in the front row, a coterie of graduate students, and a back row of visitors from the IAS. During my first year he lectured on his geometrization theorem for Haken 3-manifolds, a tour de force with another iteration on Teichmüller space as its engine: the *famous skinning map*.

The course began with a discussion of the boundary of the convex hull of the limit set of a Kleinian group: it is a hyperbolic surface bent along a measured lamination. Similar-looking pleated surfaces interpolate between the faces of the convex core, and the area of each surface is controlled by its Euler characteristic. These ideas then flowed into a series of compactness results (approximate Mostow rigidity) which underpin convergence of the skinning iteration: boundedness yields uniform contraction.

When asked a question, Thurston would usually fix his gaze on the middle-distance as if to grasp some private, kinesthetic-geometric model. He almost always worked things out live, on the spot, especially in class. His ideas seemed to come out of nowhere. It was as if Thurston had started off on a different track at an early age and had looked at everything since then with fresh eyes; to fully appreciate his work might require a complete reeducation.

To explain the idea of orbifolds, Thurston once brought two mirrors and a toy Smurf to a kindergarten class. By changing the angle between mirrors, the children could see first three, and then four, and then more copies of the original stuffed figure. He then added a third mirror, making a triangle. The children crowded around and peered into it from above, to see an infinite Smurf universe, with the figures repeating at different angles forever.

On another occasion the course started with a discussion of the "pentagon problem" from the 1986 high school math Olympiad. Numbers $(x_i)_{i=1}^5$ are assigned to the vertices of a pentagon, with $\sum x_i > 0$. A move consists of picking a vertex with $x_i < 0$, changing its sign, and then subtracting the same amount from its two neighbors. Will this process eventually make all the numbers positive? Soon the blackboard was covered with cone manifolds and polygons and butterfly moves. Although he didn't mention it in class, by the end of the semester Thurston had connected this Olympiad problem to the work of Picard and Deligne–Mostow on hypergeometric functions and rediscovered their constructions of nonarithmetic lattices in $\mathrm{PU}(1, n)$, $n > 1$.

Thurston at MSRI

Figure 2 is taken from "The theory of foliations of codimension greater than one" [*Comment Math. Helv.* **49** (1974), 214–231]. Thurston's work in this area is an example of the power of the h-principle: by exploiting the flexibility of smooth constructions with sufficient imagination, one can construct a foliation that realizes any given homotopy data. (The same methods can be used for sphere eversion, as illustrated in his movie *Outside In*.)

While director at MSRI (1992–97), Thurston once lectured on a special case: any manifold with zero Euler characteristic admits a smooth codimension-one foliation. The lecture involved parking garages and ramps running up and down, as well as a preliminary triangulation of the manifold. Bott was in the audience, somewhat dismayed by these hands-on constructions. He asked at the end: Can't one do this sort of thing using an evolution equation from differential geometry that would gradually deform a field of tangent hyperplanes until it becomes integrable? Thurston's answer was immediate: The solution to a parabolic equation would be real-analytic, but it is well known (Haefliger) that real-analytic foliations are much harder to construct than smooth ones (e.g., there are none of codimension one on S^3).

Thurston in Banff

The last illustration (Figure 3) comes from Thurston's final paper, "Entropy in dimension one" [Princeton Math. Soc., **51**, Princeton Univ. Press, 2014, pp. 339–384]; it shows the Galois conjugates in $\mathbb{C}$ of the expansion factors for critically finite quadratic maps of the interval. These expansion factors are simply the values of $\lambda > 1$ such that $x = 0$ has a finite forward orbit under the tent map $x \mapsto \lambda(1 - |x|)$.

This paper, which returns to the work begun by Milnor and Thurston in the 1970s, characterizes the entropies of multimodal maps and free-group automorphisms. The latter, when realized on train tracks, also become one-dimensional dynamical systems, and the essential unity between these two subjects and the mapping-class group of a surface emerges.

Thurston spoke on this work at a conference in Banff in honor of Milnor in 2011. By that time he had been energetically networking, via email and dropbox, with a group of younger mathematicians from around the world, many of whom were present. After Banff, and more than thirty years after his mimeographed notes had arrived, graduate students and postdocs at Harvard were holding a weekly reading group on this new paper by Thurston.

Bill's radical way of looking at things continues to shape mathematics as much as his deepest theorems.

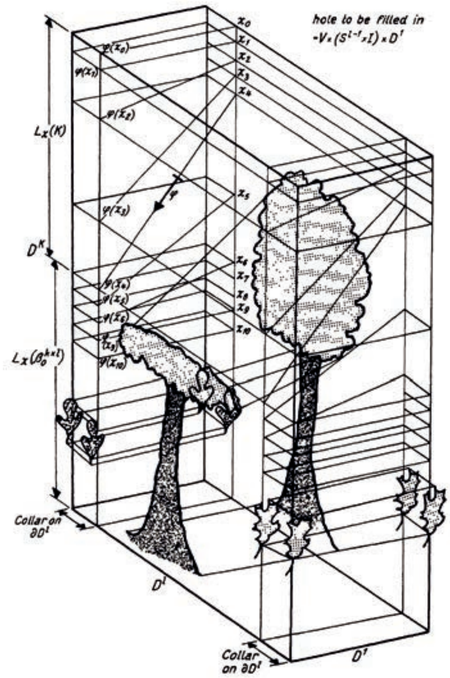

Figure 2. Opening the window.

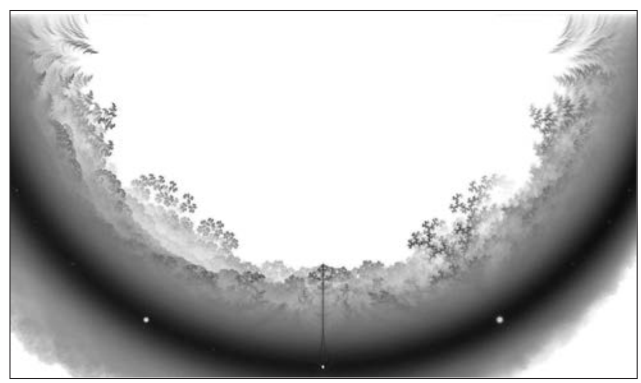

Figure 3.