WHAT IS... a Quandle?

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A *quandle* is an algebraic structure analogous to a group. While the group axioms are motivated by symmetry—every symmetry is invertible, the composition of symmetries is associative, and the identity is a symmetry—the quandle axioms are motivated by the Reidemeister moves in knot theory.

A *knot diagram* is a projection of a knot in $\mathbb{R}^3$ on a plane where we draw the undercrossing strands broken, as in Figure 1.

Figure 1. A knot diagram.

The portions of a knot diagram running from one undercrossing to the next are called *arcs*.

Two knots $K, K' \subset \mathbb{R}^3$ are considered "the same" if there is an *ambient isotopy*, i.e. a continuous map

$$H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$$

with $H(S^1, 0) = K$, $H(S^1, 1) = K'$, with $H(\vec{x}, t)$ injective for every $t \in [0,1]$. We can think of H as a movie continuously deforming K onto K', with the injectivity condition preventing the knot from passing through itself.

In the late 1920s Kurt Reidemeister proved that two knot diagrams represent ambient isotopic knots if and only if there is a sequence of the *Reidemeister moves*, depicted in Figure 2, changing one diagram to the other, together with planar isotopy. These moves come in three types, called I, II, and III according to the numbers of crossings involved; these are local moves, meaning that the portion of the knot diagram outside the pictured area is fixed by the move. Move I involves introducing or removing a kink, move II involves crossing or uncrossing one strand over a neighboring strand, and move III involves dragging a top strand past a crossing in the two lower strands beneath it.

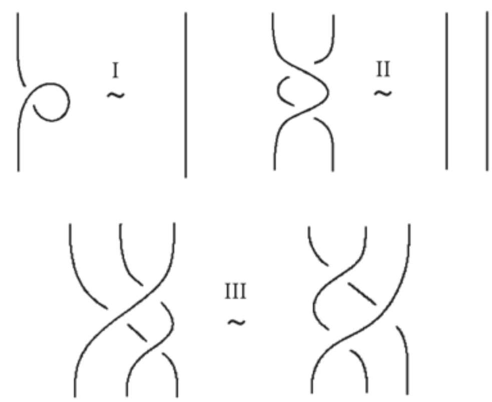

Figure 2. The three Reidemeister moves relate all equivalent knot diagrams.

Let X be a set whose elements we will use as labels or *colors* for the arcs in an oriented knot diagram. Then when three labeled arcs meet to form a crossing, we define an operation $\triangleright : X \times X \to X$ on the labels as pictured in Figure 3. That is, when the strand labeled x crosses under the strand labeled y, the result is a strand labeled $x \triangleright y$. The Reidemeister moves then translate into the quandle axioms, namely:

Definition 1. A *quandle* is a set X with a binary operation $\triangleright$ such that for all $x, y, z \in X$,

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Figure 3. When the strand labeled x crosses under the strand labeled y, the result is a strand labeled $x \triangleright y$.

(i) $x \triangleright x = x$ (idempotence),
(ii) the map $\beta_y : X \to X$ by $\beta_y(x) = x \triangleright y$ is invertible, and
(iii) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ (self-distributivity).

Axiom (ii) is sometimes called *right-invertibility*, since it means that x can always be recovered from $x \triangleright y$, though in general y cannot; thus, the $\triangleright$ operation is invertible from the right but not the left.

Many familiar mathematical objects have quandle structures:

- An abelian group is a quandle with
$$x \triangleright y = 2y - x,$$
called a *cyclic quandle* or *dihedral quandle* (since the set of reflections in a dihedral group forms this kind of quandle).
- A group is a quandle with
$$x \triangleright y = y^{-n} x y^n$$
for each $n \in \mathbb{Z}$, called an *n-fold conjugation quandle*.
- A module over the ring $\mathbb{Z}[t^{\pm 1}]$ of Laurent polynomials is a quandle with
$$x \triangleright y = tx + (1-t)y,$$
called an *Alexander quandle*.
- A vector space with an antisymmetric bilinear form $\langle , \rangle$ is a quandle with
$$\vec{x} \triangleright \vec{y} = \vec{x} + \langle \vec{x}, \vec{y} \rangle \vec{y},$$
called a *symplectic quandle*.
- A symmetric space, i.e. a manifold X with a point reflection $S_y : X \to X$ for each point $y \in X$, is a quandle with
$$x \triangleright y = S_y(x).$$
For instance, the sphere S^n is a quandle with quandle operation $x \triangleright y$ defined by reflecting x across y along the geodesic connecting x and y, as in Figure 4.

Let X be a quandle and consider the maps $\beta_y : X \to X$ defined by $\beta_y(x) = x \triangleright y$. Quandle axiom (i) says $\beta_x(x) = x \triangleright x = x$ so each $x \in X$ is a fixed point of its map β_x, and quandle axioms (ii) and (iii) together imply that the β_x maps are quandle automorphisms:

$$\begin{aligned} \beta_z(x \triangleright y) &= (x \triangleright y) \triangleright z \\ &= (x \triangleright z) \triangleright (y \triangleright z) \\ &= \beta_z(x) \triangleright \beta_z(y). \end{aligned}$$

Thus, we can define quandles without reference to knot theory by saying a quandle is an algebraic structure such that right multiplication is an automorphism for each element, with each element a fixed point of its action.

Just as the group axioms can be strengthened with additional axioms or weakened by removing axioms to obtain structures like abelian groups or monoids, there are many related algebraic structures with extra axioms (*involutory quandles*, *medial quandles*), with reduced axioms (*racks*, *shelves*), or with additional operations (*biquandles*, *virtual quandles*). Each of these corresponds to a type of knot theory: involutory quandles are the appropriate labeling objects for unoriented knots, racks are the appropriate objects for framed knots, and so on.

In 1980, David Joyce associated to each oriented knot $K \subset S^3$ a quandle $\mathcal{Q}(K)$, called the *fundamental quandle* of K, and proved that $\mathcal{Q}(K)$ determines K up to ambient homeomorphism. That is, if two knots K and K' have isomorphic fundamental quandles, then K is ambient isotopic either to K' or to its mirror image. Many classical knot invariants can be obtained from the knot quandle in a straightforward way; for example, the Alexander polynomial is a minor of a presentation matrix for the Alexander quandle of the knot.

To compare fundamental quandles of knots, we can fix a finite quandle X and count *homomorphisms* (i.e., maps satisfying $f(x \triangleright y) = f(x) \triangleright f(y)$) from $\mathcal{Q}(K)$ to X, obtaining an integer-valued knot invariant called the *quandle counting invariant* $|\mathrm{Hom}(\mathcal{Q}(K), X)|$. Such homomorphisms can be visualized and even computed as colorings of the arcs of our knot diagram by elements of X, making these invariants suitable for research by undergrads and even motivated high school students. For example, let X be the cyclic quandle on $\mathbb{Z}_3 = \{0, 1, 2\}$; then the trefoil knot has counting invariant value $|\mathrm{Hom}(\mathcal{Q}(K), X)| = 9$, as depicted in Figure 5.

Quandle theory is an excellent source of easily computable yet powerful knot and link invariants. For example, Carter, Jelsovsky, Kamada, Langford, and Saito used quandle cocycle invariants to settle the question of whether the 2-twist spun trefoil, a knotted orientable surface in $\mathbb{R}^4$, is equivalent to itself with the opposite orientation. They do this by finding a quandle-based invariant which distinguished the two, demonstrating that they are not ambient isotopic.

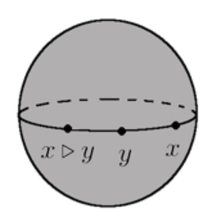

Figure 4. The sphere S^n is a quandle with quandle operation $x \triangleright y$ defined by reflecting x across y along the geodesic connecting x and y.

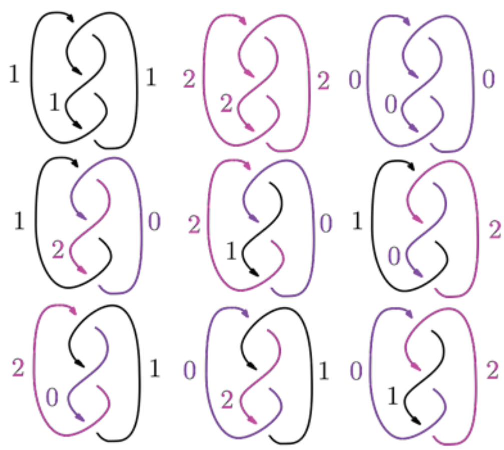

Figure 5. The trefoil knot has counting invariant value 9.

Sam Nelson currently teaches at Claremont McKenna College and recently co-authored the first textbook on quandle theory. When not doing mathematics, he spins Industrial dance tracks at SoCal discotheques as DJ AbsintheMinded.

Further Reading

1. D. Joyce, A classifying invariant of knots, the knot quandle, *J. Pure Appl. Algebra* **23** (1982), 37–65. MR 638121 (83m:57007)
2. S. V. Matveev, Distributive groupoids in knot theory, *Mat. Sb. (N.S.)* **119** (1982), 78–88. MR 672410 (84e:57008)