Electrical Resistor Networks

Thomas Lam

We discuss the inverse problem for electrical networks on surfaces and the action of the electrical Lie group on electrical networks.

The physical axioms governing the behavior of electricity have been known for over a century. Nevertheless, many fundamental mathematical problems related to the simplest electrical networks remain unsolved. An *electrical network* $G = (V, E)$ is a finite (undirected) graph, as in Figure 1. The edges of the network represent resistors, and each edge $e \in E$ is given a positive real weight c_e equal to the conductance (inverse of resistance) of that resistor. We distinguish a subset $V_B \subset V$ of *boundary vertices* and call the complement $V \backslash V_B$ the set of *interior vertices*. When voltages are assigned to the boundary vertices, currents will flow in or out of the network at these boundary vertices. By definition, the *response matrix* $L(G)$ of G is the linear map $L(G) : \mathbb{R}^{V_B} \to \mathbb{R}^{V_B}$ given by

$$L(G): \text{ boundary voltage vector} \longmapsto \text{ boundary current vector.}$$

The matrix $L(G)$ can be computed from physical axioms discovered by Gustav Kirchhoff and Georg Simon Ohm.

Kirchhoff's Law: For each interior vertex $v \in V \backslash V_B$, the total current flowing into v equals the total current flowing out of v.

Ohm's Law: For each edge $e \in E$, the current flowing across e equals the product of the conductance of e with the difference in voltages of the endpoints of e.

Imagine that we are given a black box inside of which is an electrical network of interest. Some wires of the electrical network are sticking out of the box, but the rest of the network is hidden inside. Can we recover the electrical network by performing experiments using these wires? This is an informal version of the following inverse problem.

Inverse Problem: Can we recover G from $L(G)$?

Equivalence Problem: For which graphs G and G' do we have $L(G) = L(G')$?

The inverse problem above is closely related to *electrical impedance tomography*, a medical imaging technique.

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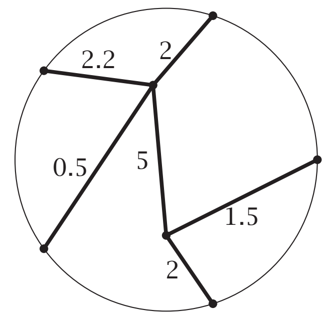

Figure 1. A planar electrical network with edges labeled by conductances.

Electrical Networks Embedded in Surfaces

A *planar electrical network*, as in Figure 1, is an electrical network with an embedding into a disk such that the boundary vertices lie on the boundary of the disk. Colin de Verdière-Gitler-Vertigan (1996) and Curtis-Ingerman-Morrow (1998) solved the inverse and equivalence problems for planar electrical networks. An electrical network is *critical* if it has the minimal number of edges among networks with the same response matrix.

Theorem 1.

(1) *Two planar electrical networks G and G' have the same response matrix if and only if they are related by local equivalences (see Figure 2) in the interior of the network.*
(2) *For a critical network G, the conductances can be uniquely recovered from the response matrix $L(G)$.*

It is natural to extend these results by considering electrical networks embedded in a topological surface S with boundary. Such topological inverse and equivalence problems have not received much attention in full generality, but some work has been done in the case that S is a cylinder or a torus. Critical electrical networks on surfaces have nontrivial *monodromy* (unlike in the planar case): it is possible to perform a sequence of local transformations that alters the conductances but preserves the underlying unweighted graph as well as the response matrix.

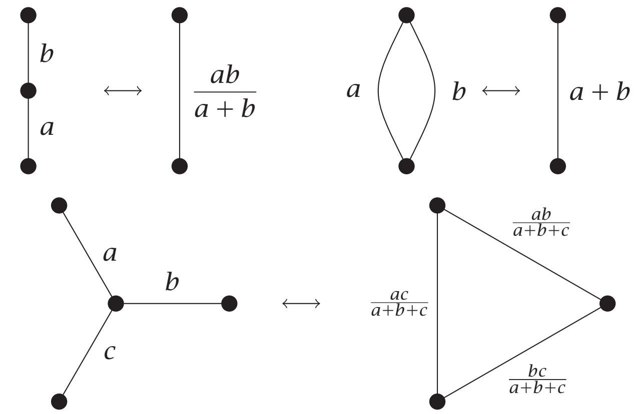

Figure 2. Some local equivalences. The series and parallel equivalences are likely familiar to many readers. The most interesting equivalence, the star-triangle or Y–Δ relation, was discovered by Edwin Kennelly in 1899.

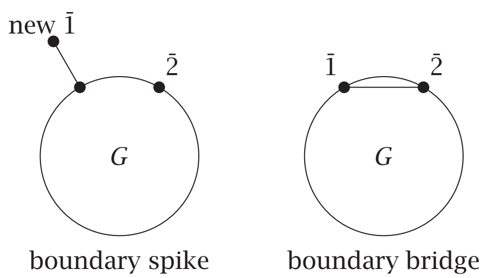

Figure 3. The electrical Lie group has two types of generators. The first generator adds a new edge to a boundary vertex, and the new endpoint is now considered the boundary vertex. The second generator adds a new edge between two adjacent boundary vertices.

Electrical Lie Groups

It is tempting to think of the local equivalences of Figure 2 as relations in an algebraic structure. To that end, Pylyavskyy and I defined the *electrical Lie algebra* $\mathfrak{el}_n$, a deformation of the nilpotent subalgebra $\mathfrak{n}^+$ of $\mathfrak{sl}_{n+1}$. This deformation is obtained by replacing Serre's relation for the generators of $\mathfrak{n}^+$ by the electrical Serre relation:

$$\text{Serre relation: } [e,[e,e']] = 0,$$

$$\text{electrical Serre relation: } [e,[e,e']] = -2e.$$

The corresponding electrical Lie group (or more precisely, its "positive" subsemigroup) acts on the space of planar electrical networks via the combinatorial generators of Figure 3.

Credit

Author photo courtesy of Charlotte Chan.

ABOUT THE AUTHOR

Thomas Lam enjoys rock climbing and playing Go.

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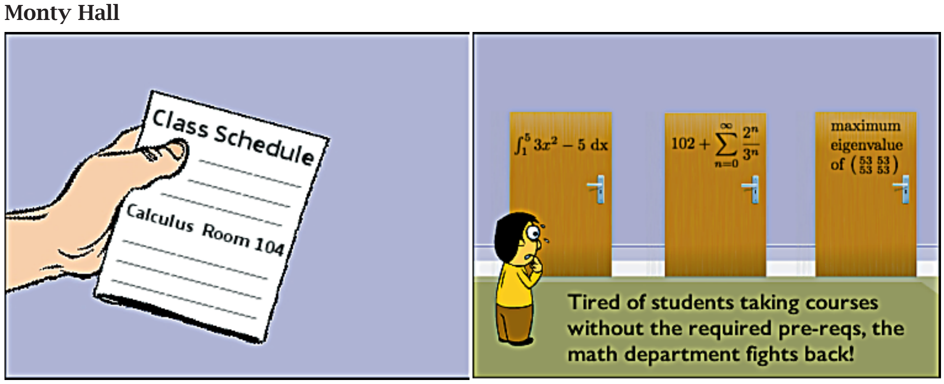

by John Bihn