WHAT IS...

Symplectic Geometry

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Symplectic structures are floppier than holomorphic functions or metrics.

In Euclidean geometry in a vector space over $\mathbb{R}$, lengths and angles are the fundamental measurements, and objects are rigid. In symplectic geometry, a two-dimensional area measurement is the key ingredient, and the complex numbers are the natural scalars. It turns out that symplectic structures are much floppier than holomorphic functions in complex geometry or metrics in Riemannian geometry.

The word "symplectic" is a calque introduced by Hermann Weyl in his textbook on the classical groups. That is, it is a root-by-root translation of the word "complex" from the Latin roots

$$com - plexus,$$

meaning "together – braided," into the Greek roots with the same meaning,

$$\sigma\upsilon\mu - \pi\lambda\epsilon\kappa\tau\iota\kappa\acute{o}\varsigma.$$

Weyl suggested this word to describe the Lie group that preserves a nondegenerate skew-symmetric bilinear form. Prior to this, that Lie group was called the "line complex group" or the "Abelian linear group" (after Abel, who studied the group).

A differential 2-form ω on (real) manifold M is a gadget that at any point $p \in M$ eats two tangent vectors and spits out a real number in a skew-symmetric, bilinear way. That is, it gives a family of skew-symmetric bilinear functions

$$\omega_p : T_pM \times T_pM \to \mathbb{R}$$

depending smoothly on the point $p \in M$. A 2-form $\omega \in \Omega^2(M)$ is **symplectic** if it is both closed (its exterior derivative satisfies $d\omega = 0$) and nondegenerate (each function ω_p is nondegenerate). Nondegeneracy is equivalent to the statement that for each nonzero tangent vector $v \in T_pM$, there is a symplectic buddy: a vector $w \in T_pM$ so that $\omega_p(v, w) = 1$. A **symplectic manifold** is a (real) manifold M equipped with a symplectic form ω.

Nondegeneracy has important consequences. Purely in terms of linear algebra, at any point $p \in M$ we may choose a basis of T_pM that is compatible with ω_p, using a skew-symmetric analogue of the Gram-Schmidt procedure. We start by choosing any nonzero vector v_1 and then finding a symplectic buddy w_1. These must be linearly independent by skew-symmetry. We then peel off the two-dimensional subspace that v_1 and w_1 span and continue recursively, eventually arriving at a basis

$$v_1, w_1, \dots, v_d, w_d,$$

which contains an even number of basis vectors. So symplectic manifolds are *even-dimensional.* This also allows us to think of each tangent space as a complex vector space where each v_i and w_i span a complex coordinate subspace. Moreover, the top wedge power, $\omega^d \in \Omega^{2d}(M)$, is nowhere-vanishing, since at each tangent space,

$$\omega^d(v_1, \dots, w_d) \neq 0.$$

In other words, ω^d is a volume form, and M is necessarily *orientable.*

Symplectic geometry enjoys connections to algebraic combinatorics, algebraic geometry, dynamics, mathematical physics, and representation theory. The key examples underlying these connections include:

(1) $M = S^2 = \mathbb{C}P^1$ with $\omega_p(v, w) =$ signed area of the parallelogram spanned by v and w;
(2) M any Riemann surface as in Figure 1 with the area for ω as in (1);

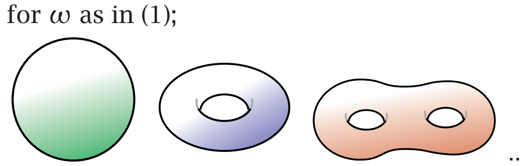

Figure 1. The area form on a Riemannian surface defines its symplectic geometry.

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(3) $M = \mathbb{R}^{2d}$ with $\omega_{std} = \sum dx_i \wedge dy_i$;
(4) $M = T^*X$, the cotangent bundle of any manifold X, thought of as a phase space, with p-coordinates coming from X being positions, q-coordinates in the cotangent directions representing momentum directions, and $\omega = \sum dp_i \wedge dq_i$;
(5) M any smooth complex projective variety with ω induced from the Fubini-Study form (this includes smooth normal toric varieties, for example);
(6) $M = \mathcal{O}_\lambda$ a coadjoint orbit of a compact connected semisimple Lie group G, equipped with the Kostant-Kirillov-Souriau form ω. For the group $G = SU(n)$, this class of examples includes complex projective space $\mathbb{C}P^{n-1}$, Grassmannians $\mathcal{G}r_k(\mathbb{C}^n)$, the full flag variety $\mathcal{F}\ell(\mathbb{C}^n)$, and all other partial flag varieties.

Plenty of orientable manifolds do not admit a symplectic structure. For example, even-dimensional spheres that are at least four-dimensional are not symplectic. The reason is that on a compact manifold, Stokes' theorem guarantees that $[\omega] \neq 0 \in H^2(M;\mathbb{R})$. In other words, compact symplectic manifolds must exhibit nonzero topology in degree 2 cohomology. The only sphere with this property is S^2.

Example (3) gains particular importance because of the nineteenth century work of Jean Gaston Darboux on the structure of differential forms. A consequence of his work is

Darboux's Theorem. *Let M be a two-dimensional symplectic manifold with symplectic form ω. Then for every point $p \in M$, there exists a coordinate chart U about p with coordinates $x_1, \dots, x_d, y_1, \dots, y_d$ so that on this chart,*

$$\omega = \sum_{i=1}^{d} dx_i \wedge dy_i = \omega_{std}.$$

> *Tools from the 1970s and 1980s set the stage for dramatic progress in symplectic geometry and topology.*

This makes precise the notion that symplectic geometry is floppy. In Riemannian geometry there are local invariants such as curvature that distinguish metrics. Darboux's theorem says that symplectic forms are all locally identical. What remains, then, are global topological questions such as, What is the cohomology ring of a particular symplectic manifold? and more subtle symplectic questions such as, How large can the Darboux charts be for a particular symplectic manifold?

Two tools developed in the 1970s and 1980s set the stage for dramatic progress in symplectic geometry and topology. Marsden and Weinstein, Atiyah, and Guillemin and Sternberg established properties of the **momentum map**, resolving questions of the first type. Gromov introduced **pseudoholomorphic curves** to probe questions of the second type. Let us briefly examine results of each kind.

If a symplectic manifold exhibits a certain flavor of symmetries in the form of a Lie group action, then it admits a momentum map. This gives rise to conserved quantities such as angular momentum, whence the name. The first example of a momentum map is the height function on a 2-sphere (Figure 2).

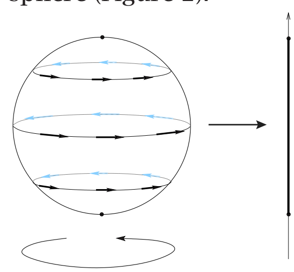

Figure 2. The momentum map for S^1 acting on S^2 by rotation.

In this case, the conserved quantity is angular momentum, and the height function is the simplest example of a **perfect Morse function** on S^2. When the Lie group is a product of circles $T = S^1 \times \cdots \times S^1$, we say that the manifold is a **Hamiltonian T-space**, and the momentum map is denoted $\Phi : M \to \mathbb{R}^n$. In 1982 Atiyah and independently Guillemin and Sternberg proved the Convexity Theorem (see Figure 3):

Convexity Theorem. *If M is a compact Hamiltonian T-space, then $\Phi(M)$ is a convex polytope. It is the convex hull of the images $\Phi(M^T)$ of the T-fixed points.*

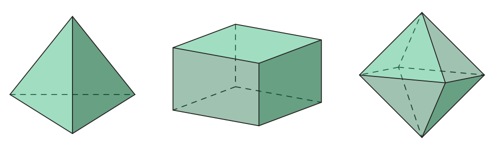

Figure 3. Atiyah and Guillemin-Sternberg proved that if a symplectic manifold has certain symmetries, the image of its momentum map is a convex polytope.

This provides a deep connection between symplectic and algebraic geometry on the one hand and discrete geometry and combinatorics on the other. Atiyah's proof demonstrates that the momentum map provides Morse functions on M (in the sense of Bott), bringing the power of differential topology to bear on global topological questions about M. Momentum maps are also used to construct **symplectic quotients**. Lisa Jeffrey will discuss the cohomology of symplectic quotients in her 2017 Noether Lecture at the Joint Mathematics Meetings, and many more researchers will delve into these topics during the special session Jeffrey and I are organizing.

Through example (2), we see that two-dimensional symplectic geometry boils down to area-preserving geometry. Because symplectic forms induce volume forms, a natural question in higher dimensions is whether symplectic geometry is as floppy as volume-preserving geometry: can a manifold be stretched and squeezed in any which way so

Gromov proved that symplectic maps are more rigid than volume-preserving ones.

long as volume is preserved? Using pseudoholomorphic curves, Gromov dismissed this possibility, proving that symplectic maps are more rigid than volume-preserving ones. In $\mathbb{R}^{2d}$ we let $B^{2d}(r)$ denote the ball of radius r. In 1985 Gromov proved the nonsqueezing theorem (see Figure 4):

Nonsqueezing Theorem. *There is an embedding*

$$B^{2d}(R) \hookrightarrow B^2(r) \times \mathbb{R}^{2d-2}$$

preserving ω_{std} if and only if $R \leq r$.

One direction is straightforward: if $R \leq r$, then $B^{2d}(R) \subseteq B^2(r) \times \mathbb{R}^{2d-2}$. To find an obstruction to the existence of such a map, Gromov used a pseudoholomorphic curve in $B^2(r) \times \mathbb{R}^{2d-2}$ and the symplectic embedding to produce a minimal surface in $B^{2d}(R)$, forcing $R \leq r$.

On the other hand, a volume-preserving map exists for any r and R. Colloquially, you cannot squeeze a symplectic camel through the eye of a needle.

Gromov's work has led to many rich theories of symplectic invariants with pseudoholomorphic curves

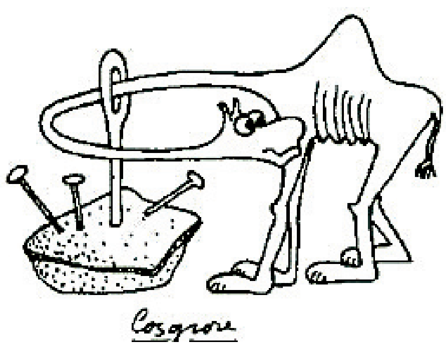

Figure 4. The Nonsqueezing Theorem gives geometric meaning to the aphorism on the impossibility of passing a camel through the eye of a needle: A symplectic manifold cannot fit inside a space with a narrow two-dimensional obstruction, no matter how big the target is in other dimensions.
This cartoon originally appeared in the article "The Symplectic Camel" by Ian Stewart—published in the September 1987 issue of *Nature*—we thank him for his permission to use it. Cosgrove is a well-known cartoonist loosely affiliated with mathematicians. He did drawings for *Manifold* for a few years and most recently did a few sketches for the curious cookbook *Simple Scoff*: `https://www2.warwick.ac.uk/newsandevents/pressreleases/50th_anniversary_cookery`.

the common underlying tool. The constructions rely on subtle arguments in complex analysis and Fredholm theory. These techniques are essential to current work in symplectic topology and mirror symmetry, and they provide an important alternative perspective on invariants of four-dimensional manifolds.

Further details on momentum maps may be found in [CdS], and [McD-S] gives an account of pseudoholomorphic curves.

References

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[McD-S] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Oxford Mathematical Monographs, Oxford University Press, New York, 1995. MR 1373431

ABOUT THE AUTHOR

In addition to mathematics, Tara Holm enjoys gardening, cooking with her family, and exploring the Finger Lakes. This column was based in part on her AMS–MAA Invited Address at MathFest 2016 held in August.

Photo by Melissa Totman.

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A Decade Ago in the *Notices*: Pseudoholomorphic Curves

"WHAT IS…Symplectic Geometry?" discusses Gromov's "nonsqueezing theorem," a key result in symplectic geometry. Gromov proved the theorem using the notion of a *pseudoholomorphic curve*, which he introduced in 1986.

Readers interested in these topics might also wish to read "WHAT IS…a Pseudoholomorphic Curve?" by Simon Donaldson, which appeared in the October 2005 issue of the *Notices*. "The notion [of pseudoholomorphic curve] has transformed the field of symplectic topology and has a bearing on many other areas such as algebraic geometry, string theory and 4-manifold theory," Donaldson writes. Starting with the basic notion of a plane curve, he gives a highly accessible explanation of what pseudoholomorphic curves are. He notes that they have been used as a tool in symplectic topology in two main ways: "First, as geometric probes to explore symplectic manifolds: for example in Gromov's result, later extended by Taubes, on the uniqueness of the symplectic structure on the complex projective plane…Second, as the source of numerical invariants: Gromov–Witten invariants."

Another related piece "WHAT IS…a Toric Variety?" by Ezra Miller appeared in the May 2008 issue of the *Notices*.