WHAT IS...

a Varifold?

The Editors

A varifold is one definition of a generalized m-dimensional surface in say R^n. The full class of varifolds is a very general class of measures. To encode information about both position and tangent direction, a varifold is defined as a measure on the cross product of R^n with the space of possible tangent planes.

To understand how to view an m-dimensional surface in R^n as a measure, you need to know that there is a nice m-dimensional "Hausdorff" measure H^m on R^n. It assigns to every smooth m-dimensional submanifold its standard m-dimensional area, but it is defined on a much wider class of sets, including of course all closed sets. Given a nice m-dimensional surface S, consider the associated measure

$$\mu_S(A) = H^m(S \cap A). \tag{1}$$

By definition the surface S determines the measure μ_S, but it also goes the other way: the measure μ_S determines the nice surface S, at least up to sets of measure 0.

To directly encode information about position and tangent plane, a varifold is defined as a measure on the Cartesian product of R^n with the set of all possible tangent planes. For an integral varifold, the tangent planes are coordinated with the surface. In a general varifold, the tangent planes are independent. Such a varifold can model an infinitesimally corrugated diagonal slice of a crystal which favors horizontal and vertical directions, having both horizontal and vertical tangent planes, each with probability one-half, as in Figure 1.

As a space of measures, the space of varifolds has a nice topology and nice compactness properties. Most important are the integral varifolds, which correspond to smooth submanifolds and infinite sums of such. There's also a deep compactness theorem for integral varifolds, but it needs a bound not only on total measure but also on curvature and boundary, to avoid nonintegral varifolds in the limit, perhaps as in Figure 1, where the encoded tangent planes are not the geometric ones.

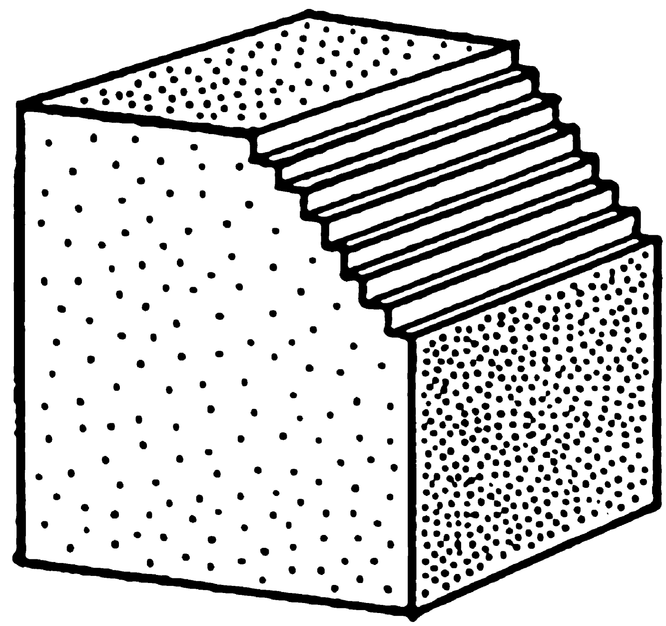

Figure 1. If you cut an edge off a cubical crystal, the exposed surface forms tiny steps, well modeled by a varifold.

For more about varifolds and current applications, see "The Concept of a Varifold" in this issue of *Notices* (page 1148).

Photo Credit

Figure 1 drawn by J. Bredt, from F. Morgan's *Geometric Measure Theory*, courtesy of Frank Morgan.

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