?WHAT IS...

a Scheme?

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Schemes pop up everywhere in mathematics. But getting a grasp on this ubiquitous concept is not so easy. And the bare definition might leave you wondering why a scheme is defined the way it is. The following interpretation introduces important elements in the definition of a scheme and helped me understand the basic idea. I hope it can help you too.

The idea of a scheme borrows from the idea of a manifold.

Schemes have played a fundamental role in algebraic geometry ever since they were introduced by Alexander Grothendieck [Gr]. The idea of a scheme borrows from the idea of a manifold in differential topology. Recall that a manifold is a topological space locally diffeomorphic to an open subset of $\mathbb{R}^n$, subject to some additional topological considerations to avoid pathologies. Now, if you have a function defined on an open neighborhood of a point in $\mathbb{R}^n$, you can take the derivative of that function at that point. So a manifold is a topological space that locally looks like "the place to do differential calculus." In a similar way, a scheme is a topological space that locally looks like "the place to do algebra/find zeros of functions." But that's a pretty long name, so let's call "the place to do algebra/find zeros of functions" an *affine scheme* instead. (Note for advanced readers: There's actually more to it than just a topological space. There is a *sheaf* of functions associated to the space as well, but we won't go into that here.)

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But what exactly is an affine scheme? Affine schemes are the full manifestation of a simple idea: If A is a commutative ring, we should view the elements of that ring as functions instead of points. If you're thinking about rings like $\mathbb{Q}(i)$ or $\mathbb{Z}/6\mathbb{Z}$, this can be a little weird, but if you do the construction by first imagining rings like $\mathbb{R}[x, y]$ it might be more clear. But the beautiful part is, this idea works for any ring, meaning that any ring can be viewed as functions on some space.

For the remainder of this article, fix a commutative ring A. You might ask, "What space could this ring A be a function on?" To motivate this, let's do an example from basic high school algebra. If you want to see where the polynomial $p(x) := x^3 + x^2 + x$ vanishes, your natural instinct would be to factor: $p(x) = x(x^2 + x + 1)$. Then if $p(x) = 0$, you know that either $x = 0$ or $x^2+x+1 = 0$. Put in slightly more abstract terms, if the function $p(x)$ vanishes at some point, then either the function x vanishes at that point or the function $(x^2 + x + 1)$ vanishes at that point. This is an important property we'd like our topological space to satisfy:

The "Vanishing" Property: If $f, g \in A$ and the product fg vanishes at a point p in the topological space, then either f vanishes at p or g vanishes at p.

This definition looks suspiciously close to the definition of a prime ideal of a ring. Recall that a prime ideal $\mathfrak{p}$ of a ring A is an ideal where if $f, g \in A$ and $fg \in \mathfrak{p}$, then either $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. So if we assume that the "points" of our special topological space are prime ideals of the ring A, then we have an obvious choice for what it might mean for a function $f \in A$ to vanish at the point $\mathfrak{p}$. We'll say f *vanishes* at $\mathfrak{p}$ if $f \in \mathfrak{p}$ (or equivalently, it is zero in the ring $A/\mathfrak{p}$).

Given a ring A, define the *spectrum* of A, written $Spec(A)$ to be the set of prime ideals of A. (Again, for advanced readers: We've just defined a scheme in the category of sets, but you can put additional structure on your spectrum so that it becomes an object in the category of *locally ringed spaces*.) Now I've promised a topology on this space, and any good topology in a subject relating to "vanishing" should have the topology related to vanishing. And luckily for us, it does. We define a set of points (which are prime ideals, but we're thinking of them as "points") to be closed if it is of the form $V(S) = \{\mathfrak{q} \in Spec(A) : S \subset \mathfrak{q}\}$, for some set of "functions" $S \subset A$ (meaning, "every element of S vanishes at $\mathfrak{q}$.") Checking this forms a topology on $Spec(A)$ is not too hard and is a good exercise.

This is a good definition for affine schemes (which, recall, was meant to be related to determining where functions vanish) because in some sense the definition is "maximal." That is, any set of elements in the ring A that can be the set of functions vanishing at a point is a prime ideal (since it has the "Vanishing" Property). Moreover, functions distinguish the points in the topological space. That is, if two points $\mathfrak{p}, \mathfrak{q}$ in the space are distinct, then there is a function that vanishes on one of the points, but not on the other. Seeing why this is true is a good check on your understanding of the concept. However, given two points $\mathfrak{p}, \mathfrak{q} \in Spec(A)$, it isn't necessarily the case that you can find a function $f \in A$ vanishing at $\mathfrak{p}$ but not at $\mathfrak{q}$. This is a good exercise to check your understanding of the material, and an example is also provided below.

Any ring can be viewed as functions on some space.

Example: Consider the ring $A := \mathbb{C}[x, y]$. What are some points in $Spec(A)$? Certainly if you choose some $a, b \in \mathbb{C}$, then $\mathfrak{p} := (x - a, y - b)$ is a maximal ideal, since the quotient field $A/\mathfrak{p}$ is $\mathbb{C}$, a field, and thus it is a prime ideal. A good way to think about points of the form $(x - a, y - b)$ is to think of the points of the form $(a, b) \in \mathbb{C}^2$ (equivalently, the places on the xy plane where $x - a = 0$ and $y - b = 0$). The function $p(x, y) := x^4 + x^3y - xy - y^2 \in A$ vanishes at $\mathfrak{p}$ since

$$p(x) = r(x, y)(x - 2) + (-x - y)(y - 8) \in \mathfrak{p}$$

where

$$r(x, y) = x^3 + 2x^2 + 4x + x^2y + 2xy + 4y.$$

Also, $p(x, y)$ vanishes in our normal sense because

$$p(2, 8) = 2^4 + 2^6 - 2^4 - 2^6 = 0.$$

So we can conclude that vanishing in our new sense generalizes vanishing in the old sense.

Now the question becomes, what else can we get from this scheme construction? One quick payoff from the definition of affine schemes is that we can talk about functions vanishing at places that aren't points in the traditional sense. For example, consider the polynomial $q(x, y) := x^3 - y$. It is an irreducible polynomial by Eisenstein's Criterion, so the ideal $\mathfrak{q} := (x^3 - y)$ is another point in $Spec(A)$. This has an obvious interpetation in $\mathbb{C}^2$. It's the curve $y = x^3$. And more amazingly, notice that $p(x, y) := x^4 + x^3y - xy - y^2 = (x^3 - y)(x + y)$, so $p(x, y) \in \mathfrak{q}$. We can interpret it to mean the function $p(x, y)$ vanishes on the entire curve. That is very pretty.

This is about half of the definition of an affine scheme. Continuing with the "rings are functions on some natural space" interpretation, we can develop a *sheaf* on the space $Spec(A)$ so that we can, for example, talk about the function $\frac{x^2+y^2}{x-3}$ on the places where $x - 3$ doesn't vanish. Sheaves are beyond the scope of this short piece, which is intended only to give you a good picture of what the topological space of an affine scheme looks like. And I hope it has also given you a new interpretation of rings!

Further Reading

[Gr] A. Grothendieck (with J. Dieudonne), *Eléments de Géométrie Algébrique*, Publ. Math. IHES, Part I:4 (1960); Part II:8 (1961); Part III:11 (1961), 17 (1963); Part IV:20 (1964), 24 (1965), 28 (1966), 32 (1967). MR 0217083

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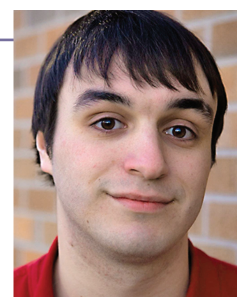
EDITOR'S NOTE. This article is a revision of Gannon's post "The 'Idea' of a Scheme" on the AMS Graduate Student Blog blogs.ams.org/mathgradblog. A related article "WHAT IS...an Anabelian Scheme?" by Kirsten Wickelgren appeared in the March 2016 issue of the *Notices*, www.ams.org/notices/201603/rnoti-p285z.pdf.

ABOUT THE AUTHOR

When not doing mathematics, Tom Gannon can often be found in the kitchen cooking new recipes, or can be found in the kitchen cooking old recipes.



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