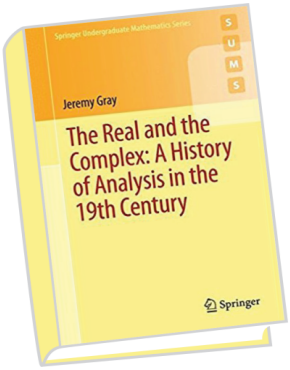
BOOK REVIEW

The Real and the Complex



A Review by Judith V. Grabiner

The Real and the Complex: A History of Analysis in the 19th Century
Jeremy Gray
Springer, 2015
Paperback, 350 pages
ISBN–13: 978-3-319-23714-5
eBook ISBN: 978-3-319-23715-2

Most basic formulas and techniques in what is often called advanced calculus were developed in the eighteenth century. More careful definitions and more rigorous proofs, new results that are too subtle to do without such proofs, and new concepts arising from these subtleties are products of the nineteenth century. The nineteenth century is also the source of the basics of complex analysis, from the Cauchy–Riemann equations to Riemannian surfaces and beyond. If you want to see how this all came about, you want to read *The Real and the Complex* by Jeremy Gray.

In the eighteenth century, most mathematicians acted as though symbolism and its heuristic power gave complete insight into the general structure of mathematical ideas. They moved back and forth between the real and the complex, the finite and the infinite. The Bernoullis, Taylor, Maclaurin, d'Alembert, Euler, Lagrange, and Laplace studied functions by means of their power-series representations, solved both ordinary and partial differential equations, and developed methods of integration. Integrals were defined as the inverse of derivatives. General statements about functions were accepted even if there were counterexamples, as long as the "exceptions" occurred only at a finite number of identifiable isolated points. Most mathematicians of the eighteenth century (Lagrange was an exception) regarded questions like "what is a limit?" or "how can you talk about infinity or the infinitely small?" as perhaps of philosophical interest but not essential to mathematical progress. Lagrange understandably, though mistakenly, believed that he had given the calculus a rigorous foundation based on what he thought of as the algebra of power series.

takes us on quite a ride through the ideas, results, and difficulties

But things turned out not to be so simple, and what happened next is the subject of Jeremy Gray's thorough, detailed, and fascinating book.[1] Gray's 29 chapters take us on quite a ride through the ideas, results, and difficulties of the major figures of nineteenth-century analysis. As the century began, analysts may have expected to succeed by continuing the calculational approach. But as they turned to Fourier series, elliptic integrals, foundations of calculus, and complex functions, earlier approaches did not suffice, and eighteenth-century problems and methods gave rise to nineteenth-century theories and standards of proof. We will hit some high points, hoping that readers will turn to the full text to experience the wealth and originality of the reasoning and ideas of the mathematicians of the nineteenth century.

Judith V. Grabiner is the Flora Sanborn Pitzer Professor of Mathematics Emerita at Pitzer College, Claremont, California. Her email address is `jgrabiner@pitzer.edu`.

For permission to reprint this article, please contact: `reprint-permission@ams.org`.

DOI: http://dx.doi.org/10.1090/noti1601

[1] *Gray intends this book as the second in a four-part series on nineteenth-century mathematics, all arising from courses he has taught to advanced undergraduates. The first,* Worlds Out of Nothing: A Course in the History of Geometry *(Springer, 2011) has appeared; the last two are to cover differential equations and algebra.*

Augustin-Louis Cauchy (1789–1857) began to treat real-variable calculus with a new level of rigor. *The Real and the Complex* lets readers follow the way that Cauchy moved in various directions in complex analysis as well.

Joseph Fourier, in his *Analytical Theory of Heat* of 1822, operated like an eighteenth-century mathematician when he modeled how heat flows with a differential equation: its solution was an infinite series of sine and cosine functions. Fourier said that all functions could be written as such a series and said also that such a series converges everywhere to the function it represents. Much later important work in analysis arose from figuring out when his statements are true and when they are not.

Beginning around 1790, Adrien-Marie Legendre studied elliptic integrals, another topic in real analysis with roots in applied mathematics. He might have wanted to evaluate them exactly, but found that he could not, though he successfully reduced them to three canonical forms. Later, Abel and Jacobi, each inverting Legendre-style elliptic integrals by treating the upper endpoints as independent complex variables, discovered elliptic functions. The early theory of elliptic functions remained formal; only later were they brought into a general theory of complex functions, notably by Joseph Liouville's theorem on doubly periodic functions and Cauchy's proof of it. So complex analysis eventually made sense out of elliptic functions and elliptic integrals. In this episode, as in others throughout the book, Gray emphasizes how the real and the complex interacted in unexpected and unpredictable ways.

Gray emphasizes how the real and the complex interacted in unexpected and unpredictable ways.

In the 1820s, Cauchy began to treat real-variable calculus with a new level of rigor. He discredited Lagrange's foundations by exhibiting a function that did not equal its Taylor series. He insisted that formulas hold "only under certain conditions and for certain values of the quantities they contain." He made the Newtonian concept of limit precise and reasoned with epsilons (borrowing many inequality techniques from Lagrange) in key proofs about derivatives and in his systematic treatment of the convergence of series. Instead of taking the existence of the integral for granted, Cauchy defined the definite integral as the limit of sums and gave a proof, assuming the function to be continuous (and, implicitly, uniformly continuous), that the limit existed. All this began the rigorization of the calculus, though much remained to be done. For instance, Cauchy thought he had proved that an infinite series of continuous functions was itself continuous. Abel found an "exception" (a Fourier series), but exactly what went wrong was not worked out until much later.

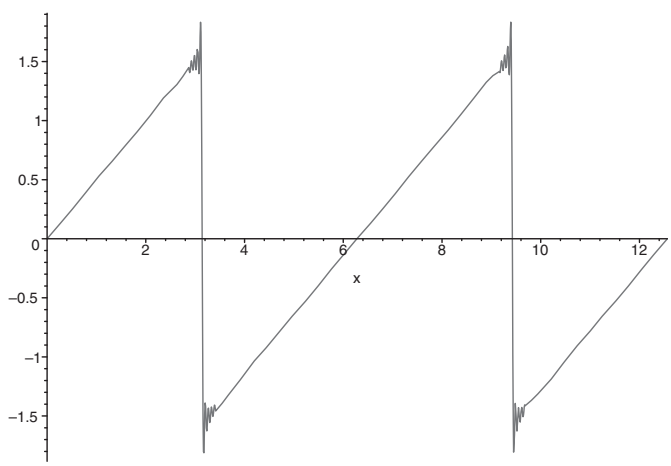


Figure 1. This graph was generated by the computer algebra system Maple and shows the first hundred terms of Abel's function *sin x – (1/2) sin 2x + (1/3) sin 3x –*… . Gray remarks on "how unhappy Maple gets at the points where the Abel function will be discontinuous." This is one of a number of such instructive Maple graphs of functions with unusual properties in Gray's book.

Cauchy also recognized that complex functions could be a separate object of study. Gray lets us follow the way Cauchy moved in various directions, finding the Cauchy integral theorem early in the century, though not yet realizing its fundamental importance. But by the 1840s and 1850s, after many fits and starts, Cauchy had worked out many key properties of complex functions, including what are now called the Cauchy–Riemann equations and their relationship to differentiability, contour integrals, Taylor-series expansions of complex functions, poles, and Laurent series.

Bernhard Riemann (1826–1866) emphasized a conceptual rather than computational approach and transformed the theory of real and complex functions.

At this point in the narrative, Gray has demonstrated that much had changed in analysis. He reflects on what these changes teach us in a brilliant review chapter, his Chapter 12. These reflections enable readers to share in the experiences of leading mathematicians struggling with problems and choices, occasional false trails, as well as unexpected successes, never sure about where their approaches would lead. Gray wants to make absolutely clear that this is what life is like when you do mathematics. If you read nothing else in Gray's book, do read this chapter,[2] which makes this point using historical examples as well as I have ever seen it done. Similarly, the concluding sections of many of his detail-packed chapters successfully convey the essence of what the chapter covered.

After a chapter on potential functions and Green's theorem, Gray turns to the work of Dirichlet, who brought a new standard of rigor to much earlier work, in particular about the convergence of Fourier series. Rather than assume that every function has a convergent Fourier series that equals it, Dirichlet asked under what conditions this was true. He found integral formulas for partial sums of the series and conditions for which they do converge. Gray illustrates Dirichlet's reputation for rigor by quoting a letter from Jacobi to Alexander von Humboldt: "If Gauss says he has proved something, it seems very probable to me; if Cauchy says so, it is about as likely as not; if Dirichlet says so, it is certain."

Riemann, especially by emphasizing a conceptual rather than computational approach, transformed the theory of real and complex functions. As Gray puts it, "Riemann offered an approach to the study of functions that did not depend on any particular expression for it, and accordingly played down the role of long complicated manipulation of such expressions." His introduction of trigonometric series, even if the coefficients are not coefficients of a Fourier series, made the study of nondifferentiable functions possible. Defining analytic functions independently of the particular analytic expression for them by using the Cauchy–Riemann equations let Riemann relate them to harmonic functions. Riemann outlined a proof that two simply connected planar regions can be mapped conformally onto one another by complex analytic maps, now known as the Riemann mapping theorem. He showed how the geometry of complex curves illuminated the study of elliptic functions. And he defined the Riemann integral and enlarged the set of functions that can be integrated.

Weierstrass had a different, more algebraic, approach to complex function theory, choosing to emphasize the representation of complex analytic functions by convergent power series rather than the equivalent property of complex differentiability.[3] The power-series approach let Weierstrass develop methods that apply to functions of one or several variables. He treated hyperelliptic and Abelian functions; made the concept of analytic continuation, previously known to Riemann, central to his own theory; and distinguished between poles and essential singularities. His lectures at the University of Berlin widely disseminated his conclusions about continuity, convergence, uniform convergence, and term-by-term differentiability for both real and complex functions. In particular, he showed that there are continuous real functions that are nowhere differentiable.

Gray next provides a chapter describing nineteenth-century work on uniform convergence, culminating in Weierstrass's definitive and detailed account. Then, in later chapters, Gray discusses topics for which some questions are answered in the nineteenth century while related questions are answered only in the twentieth. Dirichlet and Riemann had called attention to functions that were not integrable or were defined by series that do not converge uniformly or that exhibit unusual relation-

[2] *He names this chapter "Revision," using the British term for review. Similarly, he describes the book's appendix reviewing potential theory as "a revision of the theory."*

[3] *Recall that Lagrange had tried to base his theory of real functions on a detailed study of their real power series in his* Théorie des fonctions analytiques *of 1797.*

ships between their continuity and differentiability. Heine, Cantor, Hankel, Du Bois-Reymond, H. J. S. Smith, Darboux, and, especially, Weierstrass treated a variety of examples. No longer could it be assumed that functions are continuous and differentiable, except perhaps at isolated points, or that differentiable functions had to have integrable derivatives, and Bolzano, Riemann, Weierstrass, Hankel, Schwarz, and Dini began to consider functions in general and to ask which functions have the important properties of integrability and continuity. Gray describes how Weierstrass related implicit functions to holomorphic functions and how Dini made clear the distinction between the real and complex cases and also treated the several-variable case. Again, there were further developments in the twentieth century. Gray also addresses the completeness of the real numbers and the constructions of the reals from the rationals given by Weierstrass, Dedekind, Heine, and Cantor. Weierstrass realized that the Riemann integral was not sufficient, though he did not know how to make it so, and Gray later shows how such considerations led to Lebesgue's theory of integration.

Gray's last three chapters explicitly address how problems posed by nineteenth-century approaches moved toward the more modern and abstract ideas characteristic of the twentieth century, a process that resembles the way analysis had become both deeper and broader between the eighteenth century and the nineteenth. One chapter describes Lebesgue's theory of integration. Another treats Cantor's set theory and the foundations of mathematics. The final chapter is about topology. The reader may be eager for more forays into the twentieth century, but Gray is true to his goal to let us see nineteenth-century analysis in its own terms and not as a prelude to the more abstract ideas of the modern period. And he concludes with the statement that "the mathematical analysis of the last century also needs its historians." Would that they do as good a job for the twentieth century as Gray has done for the nineteenth.

Gray shows, rather than tells, providing a great deal of close-up mathematical detail.

As this brief overview may indicate, Gray's book is not light reading. But it is a very good book indeed. Any mathematician who would like to know how analysis developed should find it interesting. Those who teach analysis can gain much insight from seeing the blind alleys as well as the great ideas, since watching how the mathematics unfolded helps one appreciate what can be really hard and what types of confusion can occur even to the most able student. Seeing past contrasting approaches to the same topic, like those of Riemann and Weierstrass on complex functions, both humanizes the subject and enhances understanding. And Gray really knows his material, both the original sources and the best of the current literature. He scrupulously gives credit to historians whose research helped inform his own. His conclusions about controversial questions are generally well supported and judicious. He shows, rather than tells, as much as he can, providing a great deal of close-up mathematical detail. The writing style is clear and unpretentious, as well as marked by immense learning.

Karl Weierstrass (1815–1897) took a more algebraic approach to complex function theory than did Riemann. His lectures at the University of Berlin widely disseminated much of his work.

There are many ways one could write the history of nineteenth-century analysis. Gray emphasizes the internal development of the mathematics, though he does include some biographical information about the lives of the mathematicians and their mutual relationships, especially when he deems it crucial to explain how the mathematics developed. Other historians might have said much more about how nineteenth-century society and culture affected mathematicians' careers, choice of problems, the teaching of mathematics, and the needs of the sciences, or might have discussed in general the nineteenth-century professionalization of mathematics, increase in specialization, and journals and societies dedicated solely to mathematics. It would have been worth mentioning the work of Sofya Kovalevskaya and noting the institutional barriers that prevented talented women from entering higher mathematics. Of course both approaches mentioned here are valuable, and Gray's choice is designed for his intended audience.

Both my paperback review copy and the corresponding eBook version have more than the typical number of typographical errors. For instance, Cauchy's function that does not converge to its Taylor series is not $e^{-1/x}$; the exponent should be $-1/x^2$. But most are simply punctuation marks or words that are left out—in one case the word "not"—and a second printing could correct these; a careful reader can usually navigate around them. Other matters worth noting: Cauchy did not, in defining the definite integral in terms of sums, restrict the mode of division of the interval of integration to equal subdivisions. And, especially since Gray makes a point of recommending accessible and instructive English-language sources, the Bradley–Sandifer translation of Cauchy's *Cours d'analyse* ought to be added to the 12-page multilingual and comprehensive bibliography.

I strongly recommend *The Real and the Complex* to readers of the *Notices*. As a result of Gray's vision and scholarship, any mathematically informed reader can now watch the giants of the nineteenth century, most notably Legendre, Fourier, Cauchy, Gauss, Dirichlet, Riemann, Weierstrass, Dedekind, and Cantor, developing much of the mathematics we use today—not flying straight as an arrow to their target, not possessing all our modern concepts, but doing the best they could so that, as Gray says, they "found their ways, imperfectly, from what they knew to what they wanted to know."

ACKNOWLEDGMENT. I dedicate this review to the memory of Dr. Uta C. Merzbach (1933–2017), inspiring mentor, valued colleague, and beloved friend. Her long-anticipated study of Dirichlet, with the editorial assistance of Judy Green and Jeanne LaDuke, is forthcoming from Birkhäuser.

Image Credits

Illustration of Augustin Cauchy courtesy of Library of Congress Prints and Photographs Division. From an illustration in: *Das Neunzehnte Jahrhundert in Bildnissen* / Karl Werckmeister, ed. Berlin : Kunstverlag der photographische Gesellschaft, 1901, vol. V, no. 581

Figure 1 by Jeremy Gray.

Photo of Bernhard Riemann courtesy of Smithsonian Libraries Scientific Identity website.

Photo of Karl Weierstrass courtesy of Smithsonian Libraries Scientific Identity website.

Photo of Judith V. Grabiner courtesy of Pitzer College.

Judith V. Grabiner

ABOUT THE REVIEWER

Judith V. Grabiner, author of *The Origins of Cauchy's Rigorous Calculus* and "The role of mathematics in liberal arts education" (in M. Matthews, ed., *International Handbook of Research in History, Philosophy and Science Teaching*), received the Beckenbach Prize from the Mathematical Association of America for her book *A Historian Looks Back: The Calculus As Algebra and Selected Writings*.