The Trigonometry of Escher's Woodcut *Circle Limit III*

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Preface. During the 1954 International Congress of Mathematicians in Amsterdam, my wife Hendrina introduced me to M. C. Escher, with whom we became close friends. On one of the occasions when he was visiting his son George in Nova Scotia, he gave an illustrated lecture in the Art Gallery of Ontario and spent a few nights with us in our Toronto house. He gave us four original prints; including one of *Circle Limit III*, which inspired this article, as well as an earlier one. In contrast to the other three *Circle Limits* (*I*, *II* and *IV*), this employs four colours in addition to black and white, and features arcs which are not orthogonal to the peripheral circle. In an earlier article [1], I used hyperbolic trigonometry for my analysis, but several years later I took up the challenge of using Euclidean trigonometry instead. My former student J. Chris Fisher kindly helped by reducing my expressions for the measurements to calculated numbers that could be compared with the actual print on my staircases wall. At first one of the six measurements seemed to be wrong by a few millimetres. (The diameter of the peripheral circle is 41 cm.) Rather than blame Escher, I asked Chris to check his computation again. When he admitted that the mistake was his, I realized that Escher's intuition was completely justified. I still find it almost incredible that he, with no knowledge of algebra or trigonometry obtained accurately the centres and radii (r_1, r_2, r_3) of the three different circles to which the three different axes belong.

The Euclidean material was accepted by Chandler Davis for *The Mathematical Intelligencer* [2]. But neither he nor I was intelligent enough to notice the simplifying relation $r_1 r_2 = 2$. When this and its neat consequences were pointed out to me by Jan van de Craats of Breda, I rearranged the material into a "streamlined" version which was accepted by Koji Miyazaki for publication in his mainly Japanese journal *Hyper-Space* [3]. It is that streamlined version that is republished here.

In M. C. Escher's circular woodcuts, replicas of a fish (or cross, or angel, or devil), diminishing in size as they recede from the centre, fit together so as to fill and cover a disc. *Circle Limits I*, *II*, and *IV* (pages 286, 287) are based on Poincaré's circular model of the hyperbolic plane, whose lines appear as arcs of circles orthogonal to the circular boundary (representing the points at infinity). Suitable sets of such arcs decompose the disc into a theoretically infinite number of similar "triangles," representing congruent triangles filling the hyperbolic plane. Escher replaced these triangles by recognizable shapes. *Circle Limit III* (Fig. 1 and color plate 4) is likewise based on circular arcs, but in this case, instead of being orthogonal to the boundary circle, they meet it at equal angles of almost precisely 80°. (Instead of a straight line of the hyperbolic plane, each arc represents one of the two branches of an "equidistant curve.") Consequently, his

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construction required an even more impressive display of his intuitive feeling for geometric perfection. The present article analyzes the structure, using the elements of trigonometry and the arithmetic of the biquadratic field $\mathbb{Q}(\sqrt{2}+\sqrt{3})$: subjects of which he steadfastly claimed to be entirely ignorant.

Concerning his four Circle Limit woodcuts, M. C. Escher wrote:

> *Circle Limit I, being a first attempt, displays all sorts of shortcomings ... and leaves much to be desired. ... There is no continuity, no "traffic flow" nor unity of colour in each row.... In the coloured woodcut Circle Limit III, the shortcomings of Circle Limit I are largely eliminated. We now have none but "through traffic" series, and all the fish belonging to one series have the same colour and swim after each other head to tail along a circular route from edge to edge. ... Four colours are needed so that each row can be in complete contrast to its surroundings. As all these strings of fish shoot up like rockets ... from the boundary and fall back again whence they came, not a single component reaches the edge. For beyond that there is "absolute nothingness." And yet this round world cannot exist without the emptiness around it ... because it is out there in the "nothingness" that the centre points of the arcs that go to build up the framework are fixed with such geometric exactitude.* [4], p. 109

The purpose of the present article is to demonstrate this "geometric exactitude" (see Fig. 2) by finding the radii and centres of the first three sets of four congruent circles that trace the backs of the "strings of fish." I naturally assume that the relevant arcs of these circles cross one another at equal angles of 60°, decompose the interior of the "boundary" into alternate triangular and quad-

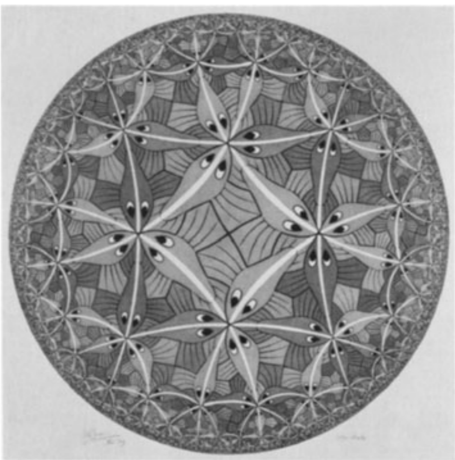

Fig. 1. M. C. Escher, *Circle Limit III*, 1959. Woodcut

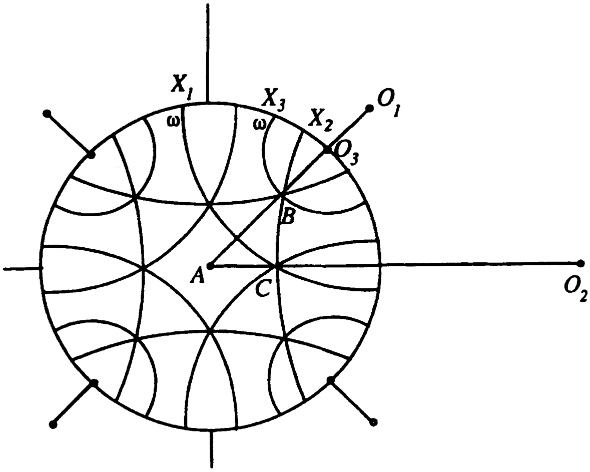

Fig. 2. Escher's "framework"

rangular regions, and all cut the boundary at the same pair of supplementary angles

$$\omega, \quad \pi - \omega .$$

The acute angle ω appears on the side of each arc where the regions are quadrangular.

An earlier article ([1], p. 24) used hyperbolic trigonometry to prove that

$$\begin{aligned} \cos\omega &= \sinh(\frac{1}{4}\log 2) \\ &\approx \sinh 0.1732868 \approx 0.1741553 . \end{aligned}$$

Since $\cos(79°58') \approx 0.17424$, ω scarcely differs from the value 80° which can easily be measured in Escher's woodcut. Here I obtain this expression for ω by a more elementary procedure.

The Angle ω at the Boundary

Figure 2 is a sketch of the middle part of Escher's "framework," showing the centres O_ν, at distances

$$d_\nu = AO_\nu$$

from the centre A of the bounding circle, of radius 1, and showing the radii

$$r_\nu = O_\nu X_\nu .$$

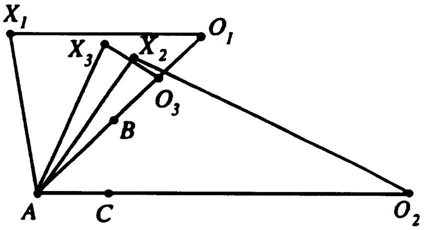

Fig. 3. Triangles with angles ω at X_1 and X_3, $\pi - \omega$ at X_2

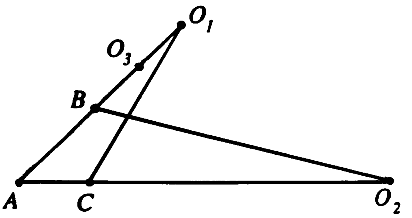

Fig. 4. The similar triangles O_1AC and O_2AB

From the triangle X_1AO_1, whose angle ω at X_1 is opposite to the side $AO_1 = d_1$, as in Fig. 3, we have

$$d_1^2 = 1 + r_1^2 - xr_1 , \tag{1}$$

where

$$x = 2\cos\omega . \tag{2}$$

Similarly, the triangle X_2AO_2, whose angle $\pi - \omega$ at X_2 is opposite to d_2, yields

$$d_2^2 = 1 + r_2^2 + xr_2 . \tag{3}$$

Because the angle between two intersecting circles equals the angle between their radii to a common point, the triangle O_1AC has angles $2\pi/3$, $\pi/4$, and $\pi/12$ opposite to sides

$$AO_1 = d_1 , \quad CO_1 = r_1 , \quad CA = d_2 - r_2 ,$$

respectively, as in Fig. 4. Hence we have

$$\frac{d_1}{\sin(2\pi/3)} = \frac{r_1}{\sin(\pi/4)} = \frac{d_2 - r_2}{\sin(\pi/12)} ,$$

that is,

$$\frac{d_1}{\sqrt{3}} = \frac{r_1}{\sqrt{2}} = \frac{d_2 - r_2}{(\sqrt{3}-1)/\sqrt{2}} . \tag{4}$$

The similar triangle O_2AB, with angles $2\pi/3$ and $\pi/4$ opposite to sides

$$AO_2 = d_2 , \quad \text{and} \quad BO_2 = r_2 ,$$

respectively, yields

$$\frac{d_2}{\sqrt{3}} = \frac{r_2}{\sqrt{2}} = \frac{AB}{(\sqrt{3}-1)/\sqrt{2}} . \tag{5}$$

Thus,

$$d_\nu^2 = \frac{3}{2} r_\nu^2 \quad (\nu = 1 \text{ or } 2) \tag{6}$$

and expressions (1) and (3) for d_ν^2 yield quadratic equations for r_ν:

$$r_1^2 + 2xr_1 - 2 = 0\,, \quad r_2^2 - 2xr_2 - 2 = 0\,.$$

Solving these equations for the positive numbers r_ν, we find

$$r_1 = -x + \sqrt{x^2+2}\,, \quad r_2 = x + \sqrt{x^2+2}\,. \tag{7}$$

From (4) we have

$$(\sqrt{3}-1)r_1 = 2(d_2 - r_2) = (\sqrt{6}-2)r_2\,,$$

and from (7),

$$r_1 r_2 = 2\,.$$

It follows that

$$\begin{aligned} r_1^2 &= (\sqrt{6}-2)(\sqrt{3}+1)\,, \\ r_2^2 &= (\sqrt{6}+2)(\sqrt{3}-1)\,, \\ 4x^2 &= (r_2 - r_1)^2 = r_1^2 + r_2^2 - 2r_1r_2 = 2\sqrt{2}(\sqrt{2}-1)^2 \end{aligned} \tag{8}$$

and

$$x = 2^{-1/4}(2^{1/2}-1) = 2^{1/4} - 2^{-1/4} = 2\sinh(\tfrac{1}{4}\log 2)\,.$$

The First Two Circles

Since $\sqrt{x^2+2} = \sqrt{2^{1/2}+2^{-1/2}} = 2^{-1/4}\sqrt{3}$, (7) yields

$$\begin{aligned} r_1 &= 2^{-1/4}(1-\sqrt{2}+\sqrt{3}) \approx 1.1081646\,, \\ r_2 &= 2^{-1/4}(\sqrt{2}-1+\sqrt{3}) \approx 1.8047860\,, \end{aligned} \tag{9}$$

and, from (6),

$$\begin{aligned} d_1 &= 2^{-3/4}(\sqrt{3}-\sqrt{6}+3) \approx 1.3572189\,, \\ d_2 &= 2^{-3/4}(\sqrt{6}-\sqrt{3}+3) \approx 2.2104024\,. \end{aligned}$$

From (5) we have

$$\begin{aligned} AB &= \tfrac{1}{2}(\sqrt{3}-1)r_2 \\ &= 2^{-3/4}(-1+2\sqrt{2}+\sqrt{3}-\sqrt{6}) \approx 0.6605975\,. \end{aligned}$$

The Biquadratic field $\mathbb{Q}(\sqrt{2}+\sqrt{3})$

The numbers $(a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6})/q$, where a, b, c and d are integers and q is a positive integer, are easily seen to constitute a *field* ([5], p. 230). This field is called $\mathbb{Q}(\sqrt{2}+\sqrt{3})$ because it can be expressed as the set of all rational functions of the special number $\theta = \sqrt{2}+\sqrt{3}$ in terms of which

$$\sqrt{2} = \tfrac{1}{2}(\theta-\theta^{-1}), \quad \sqrt{3} = \tfrac{1}{2}(\theta+\theta^{-1}), \quad \sqrt{6} = \tfrac{1}{2}(\theta^2-5).$$

In this field, θ is called an *integer* because it satisfies a monic equation, namely

$$\theta^4 - 10\theta^2 + 1 = 0.$$

When we assert that "factorization is unique," we disregard, as factors, the *units*, which are divisors of 1; for if $st = 1$, any number

$$n = nst$$

has the trivial factorization $ns \times t$.

Comparing (8) and (9), we obtain the apparently surprising identity

$$(1-\sqrt{2}+\sqrt{3})^2 = 2(\sqrt{3}-\sqrt{2})(\sqrt{3}+1).$$

This "factorization" loses its element of surprise when we face the obvious fact that $\sqrt{3}-\sqrt{2}$ is a unit:

$$(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2}) = 1.$$

The Third Circle

Looking again at Fig. 3, we see that

$$d_3^2 = 1 + r_3^2 - xr_3$$

and, since the third circle passes through B,

$$d_3 - r_3 = AB.$$

Thus

$$d_3 + r_3 = \frac{1 - xr_3}{AB}, \quad 2r_3 = \frac{1 - xr_3}{AB} - AB,$$

$$\begin{aligned}
r_3 &= \frac{(1/AB) - AB}{2 + x/AB} = \frac{1 - AB^2}{2AB + x} \\
&= \frac{1 - 2^{-3/2}(18 - 10\sqrt{2} - 10\sqrt{3} + 6\sqrt{6})}{2^{1/4}(-1 + 2\sqrt{2} + \sqrt{3} - \sqrt{6}) + 2^{-1/4}(\sqrt{2} - 1)} \\
&= \frac{-9 + 6\sqrt{2} + 5\sqrt{3} - 3\sqrt{6}}{2^{1/4}(3 - 2\sqrt{3} + \sqrt{6})} \\
&= 2^{-1/4} \frac{5 - 3\sqrt{2} - 3\sqrt{3} + 2\sqrt{6}}{-2 + \sqrt{2} + \sqrt{3}} \\
&= 2^{-1/4} \left(\frac{-1 + 2\sqrt{6}}{-2 + \sqrt{2} + \sqrt{3}} - 3 \right) \\
&= 2^{-1/4} \left(\frac{(-1 + 2\sqrt{6})(-2 + 5\sqrt{2} + 3\sqrt{3} + 4\sqrt{6})}{(-2 + \sqrt{2} + \sqrt{3})(-2 + 5\sqrt{2} + 3\sqrt{3} + 4\sqrt{6})} - 3 \right) \\
&= 2^{-1/4} \left(\frac{50 + 13\sqrt{2} + 17\sqrt{3} - 8\sqrt{6}}{23} - 3 \right) \\
&= 2^{-1/4} \left(\frac{-19 + 13\sqrt{2} + 17\sqrt{3} - 8\sqrt{6}}{23} \right) \\
&\approx 0.3375915
\end{aligned}$$

and

$$d_3 = r_3 + AB = 2^{-3/4} \frac{3 + 27\sqrt{2} + 7\sqrt{3} - 6\sqrt{6}}{23} \approx 0.998189.$$

Since Escher's bounding circle has diameter 41 cm, our results

$$\begin{aligned}
r_1 &\approx 1.10816, & d_1 &\approx 1.3572, \\
r_2 &\approx 1.8048, & d_2 &\approx 2.2104, \\
r_3 &\approx 0.3376, & d_3 &\approx 0.9982,
\end{aligned}$$

should be multiplied by 20.5 to obtain the distances in centimetres:

$$\begin{aligned}
&22.7, \quad 27.8, \\
&37.0, \quad 45.3, \\
&6.92, \quad 20.46.
\end{aligned}$$

These distances agree perfectly with actual measurements in the woodcut itself.

Acknowledgements

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References

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[2] H.S.M. Coxeter: The Trigonometry of Escher's Woodcut 'Circle Limit III,' *The Mathematical Intelligencer* **18**, no. 4, (1996), 42–46.
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[4] B. Ernst: *The Magic Mirror of M.C. Escher,* New York: Random House, 1976.
[5] G.H. Hardy and E.M. Wright: *An Introduction to the Theory of Numbers,* 4th ed., Oxford: Clarendon Press, 1960.