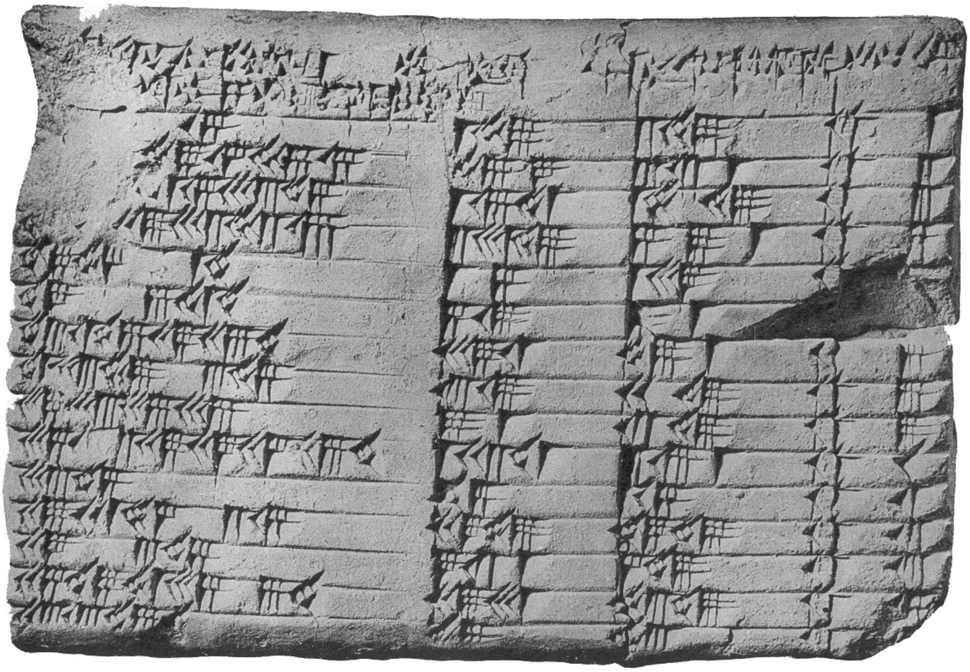
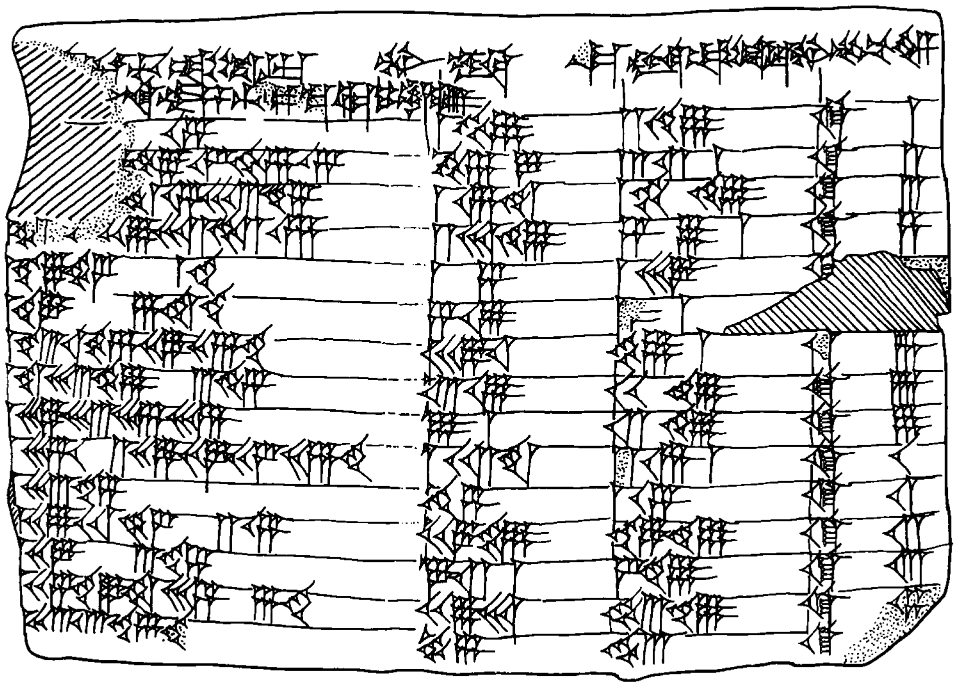




Plimpton 322. *(Photo from* Mathematical Cuneiform Tablets*, Otto Neugebauer and Abraham Sachs, 1945, courtesy of the American Oriental Society; drawing by Eleanor Robson.)*

Rewriting History

Most mathematicians focus on making new mathematical discoveries and finding new applications of mathematics. But some scholars look backward, at work done long ago. Even there, new discoveries are possible. Recent historical research by Eleanor Robson, an Oriental scholar at the University of Oxford, has shed new light on an ancient work of algebra—a cuneiform tablet known as Plimpton 322.

Plimpton 322 is one of thousands of clay tablets dating back to the Old Babylonian period in Mesopotamia nearly 4000 years ago. It resides in the George Arthur Plimpton collection at Columbia University. The tablet was purchased by Plimpton in 1923, from Edgar J. Banks, who said it came from a location near the ancient city of Larsa (modern Tell Senkereh) in Iraq. Robson estimates that Plimpton 322 was created sometime in the six decades before Larsa fell to Hammurabi of Babylon in 1762 BCE.

At 12.7 × 8.8 cm, Plimpton 322 is about the size of a cheap pocket calculator. (It was originally larger: There is a clean break along the left edge. Remnants of glue suggest the missing piece might still be in a drawer somewhere, but Robson suspects that Banks simply removed an unrelated fragment of tablet that a less scrupulous antiquities dealer had glued on to make the piece look complete.) Both sides are ruled like notebook paper. The back side is otherwise blank, but the front side has 15 lines of numbers arranged in four columns, with some labeling across the top.

The rightmost column, labeled "its name," contains only the numbers 1 through 15. It's the other three columns that make the tablet fascinating to mathematicians.

Plimpton 322 was originally assumed to be just another Babylonian ledgerbook, a kind of Sumerian spreadsheet. (The Babylonians were avid record keepers. Many of their tablets list acreages of wheat and quantities of livestock. Others are tax forms that would have done the IRS proud.) But in the early 1940s, Otto Neugebauer, an historian of ancient science at Brown University, and his assistant Abraham Sachs found otherwise. They recognized the entries as, in effect, Pythagorean triples: integer solutions of the equation $a^2+b^2=c^2$.

Pythagorean triples are most closely associated with right triangles. They also make for nice solutions to "reciprocal pair"

Edgar J. Banks. *(Photo from* Bismaya, or the Lost City of Adab, *by Edgar J. Banks (1908), provided courtesy of The University of Chicago Library's Electronic Open Stacks, www.lib.uchicago.edu/eos.)*

(1).9834	119	169	1
(1).9416	3367	11521	2*
(1).9188	4601	6649	3
(1).8862	12709	18541	4
(1).8150	65	97	5
(1).7852	319	481	6
(1).7200	2291	3541	7
(1).6928	799	1249	8*
(1).6427	541	769	9*
(1).5861	4961	8161	10
(1).5625	45	75	11
(1).4894	1679	2929	12
(1).4500	25921	289	13*
(1).4302	1771	3229	14
(1).3872	56	53	15*
(1).3692	175	337	16

Figure 1. *Decimal conversion of P322. The first column is rounded to 4 decimal places. Row 16 does not appear on the tablet. The sharp-eyed reader may notice mistakes (made by the author of Plimpton 322) in the rows marked with asterisks.*

equations of the form $x - 1/x = 2b/a$ and $x + 1/x = 2c/a$ (In each case, $x = (b + c)/a$ is a solution.) Old Babylonian scribes were trained on both kinds of problem, although of course they didn't use modern, algebraic notation—or refer to Pythagoras, who wouldn't appear for another thousand years!

The numbers in the middle two columns of Plimpton 322 record the short side b (with $a > b$) and hypotenuse c for a list of right triangles. The leftmost column contains the ratios $(c/a)^2$ (or possibly $(b/a)^2$, depending on whether the entries there do or don't start with a "1"—the break along the left edge makes it ambiguous). This column also provides the organizing principle: The ratios decrease from top to bottom, beginning (in decimal equivalent) at (1).9834 and ending at (1).3872 (see Figure 1).

Ever since Neugebauer's discovery, mathematicians have had a field day interpreting Plimpton 322 and speculating on how it was composed. The most popular explanation, advanced by Neugebauer, is that the scribe responsible for Plimpton 322 knew that setting $a = 2pq$, $b = p^2 - q^2$, and $c = p^2 + q^2$, with integers p and q (with $p > q$) would produce Pythagorean triples. Armed with this knowledge, he had, in effect, plugged in all the small values of p and q for which a has an exact sexagesimal (base 60) reciprocal—i.e., values with prime divisors 2, 3, and 5. Doing so makes the numbers in the first column exact. (In our modern decimal system, the only fractions that can be written as exact decimals are those with 2's and 5's in the denominator.) The tablet contains the results corresponding to right triangles with base angle ranging from 45º down to around 30º (omitting, however, the pair $p = 16$, $q = 9$—see Figure 2).

Eleanor Robson. *(Photo courtesy of Eleanor Robson.)*

That ingenious

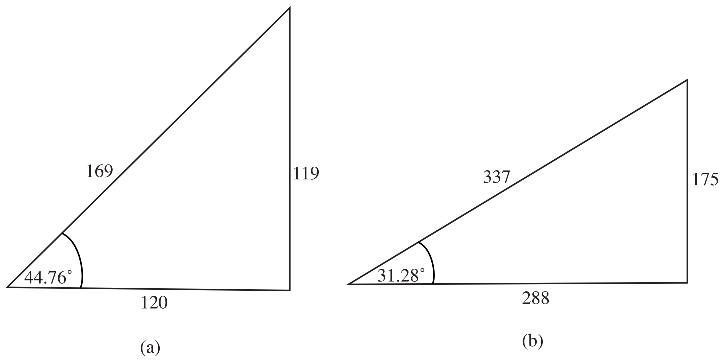


Figure 2. *Right triangles corresponding to the first and "last" entry of P322. The latter, which corresponds to* $p = 16$, $q = 9$, *is not actually on the tablet. The angle for the actual last line is* 31.89°.

Mathematicians have had a field day interpreting Plimpton 322.

explanation implies a flattering view of the creator of Plimpton 322: a mathematical prodigy, doing original research in number theory and trigonometry. Robson favors a more mundane alternative. The tablet, she believes, was created as a teacher's aid, designed for generating problems involving right triangles and reciprocal pairs. Robson has marshalled her arguments in a recent article in *Historia Mathematica*.

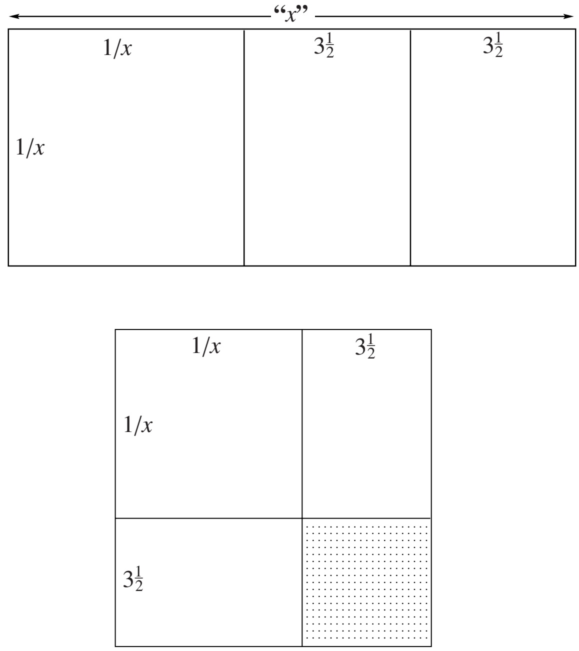
A reciprocal pair are two numbers whose product is a power of 60. (Lacking a symbol for zero, the Old Babylonian scribes would write any such power simply as "1.") There are several examples of cuneiform tablets with school problems like "A number and its reciprocal differ by 7. What is the number?" The solution proceeds by a kind of "cut and paste" geometry (see Figure 3, next page): Divide 7 by 2, getting 3;30 (i.e., 3 + 30/60,), square 3;30 to get 12;15, "add" 1 to get 1;12;15, and find the square root of that, which is 8;30 (You can check this by computing

$$(8;\,30)^2 = (8 \times 60 + 30)^2 = 510^2 = 260100$$
$$= 1 \times 60^3 + 12 \times 60^2 + 15 \times 60 = 1;\,12;\,15.$$

The Babylonians were keenly interested in squares and square roots.) Finally, add and subtract 3;30 to 8;30, getting 12 and 5, respectively. These numbers are reciprocals, and they differ by 7.

In modern terms, if x and y are a reciprocal pair (that is, $xy = 1$), then $1 + ((x - y)/2)^2 = ((x + y)/2)^2$. This implies that $a = 1$, $b = (x - y)/2$, and $c = (x + y)/2$ are a Pythagorean triple,

The *p/q* theory fails to account for many of the features of the tablet.



Area = $1 + (3\frac{1}{2})^2 = 72\frac{1}{4} = (8\frac{1}{2})^2$

Figure 3. *Cut and paste mathematics. In working with reciprocal pairs, sometimes “1” means 60.*

once you clear out the denominators in b and c. To be a reciprocal pair in the Old Babylonian sense, x and y must have the form p/q and q/p with 2, 3, and 5 as the only prime divisors of p and q. Thus, clearing denominators leads to $a = 2pq$, $b = p^2 - q^2$, and $c = p^2 + q^2$. In other words, p/q the theory and the reciprocal pairs explanation are mathematically equivalent from a modern viewpoint.

But the author of Plimpton 322 did not have a modern viewpoint. According to Robson, the p/q theory fails to account for many of the features of the tablet, including that fact that it records values of $(c/a)^2$ instead of a. The reciprocal pair explanation, she says, makes more sense in light of what's been learned about Old Babylonian tablets in the last half century.

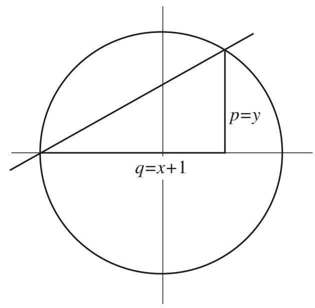
One key is the label for the first column. Neugebauer and Sachs rendered it as “The *takiltum* of the diagonal which has been subtracted such that the width...,” leaving *takiltum* untranslated and the label unfinished, because part of it near the end is unreadable. (“Diagonal” means “hypotenuse,” since right triangles arise by cutting a rectangle diagonally in half. “Width” and “short side” are also synonymous.) Subsequent scholars, observing the use of *takiltum* in other mathematical tablets, determined that it refers to a “helping” or “holding” number. With that meaning and an educated guess for what makes grammatical sense (and also fits physically) in the unreadable and damaged portions, Robson offers a new translation: “The holding-square of the diagonal from which 1 is torn out, so that the short side comes up.” That reading, she says, aligns well with the Old Babylonian approach to solving reciprocal-pair-type problems and with other mathematical tablets of the time. So it seems that the author of Plimpton 322 was no lone genius—but he was probably a very good teacher.

Take Home Assignment

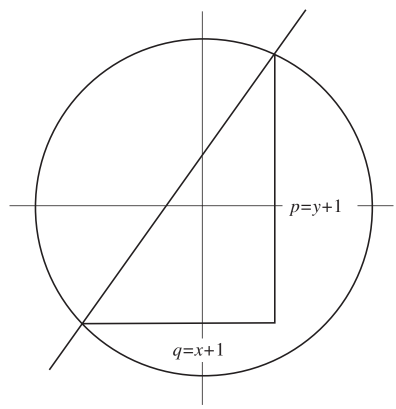
Thanks to mathematicians such as René Descartes and Pierre de Fermat, many of the problems that the ancients struggled with are a virtual snap today. Finding Pythagorean triples, for example, is a simple exercise in algebraic geometry, since $a^2 + b^2 = c^2$ (with $c \neq 0$) if and only if $(a/c,\ b/c)$ is a point on the “unit circle,” which is defined by the equation $x^2 + y^2 = 1$. But a point (x, y) on the unit circle has rational coordinates if and only if the line connecting it to $(-1, 0)$ has rational slope (see Figure 4a). If the slope is taken as p/q, then a little algebra leads to $x = (q^2 - p^2)/(q^2 + p^2)$ and $y = 2pq/(q^2 + p^2)$, giving the now-familiar formula for Pythagorean triples.

A similar approach works for problems like “Find three squares in arithmetic progression”—that is, a^2, b^2, c^2 with $b^2 - a^2 = c^2 - b^2$. Solutions correspond to rational points $(a/b,\ c/b)$ on the circle $x^2 + y^2 = 2$. In this case, a point (x, y) has rational coordinates if and only if the line connecting it to $(-1,-1)$ has rational slope (see Figure 4b). This time the algebra leads to $x = (q^2 + 2pq - p^2)/(q^2 + p^2)$ and $y = (p^2 + 2pq - q^2)/(q^2 + p^2)$, which gives $(a,b,c) = (q^2 + 2pq - p^2,\ q^2 + p^2,\ p^2 + 2pq - q^2)$ for numbers whose squares are in arithmetic progression. The values $(p, q) = (2, 1)$, for example, give $(1, 5, 7)$, corresponding to squares 1, 25, and 49.

All this still works fairly easily if you generalize to Pythagorean “quadruples,” in which three squares sum to a square: $a^2 + b^2 + c^2 = d^2$. But if you go looking for four squares in arithmetic progression, the going gets a little tricky. Try it!



(*a*)



(*b*)

Figure 4. *Rational points on the circle* $x^2 + y^2 = 1$ *(a) and* $x^2 + y^2 = 2$ *(b).*