Hearts and roses and clubs, and diamonds

David A. Meyer

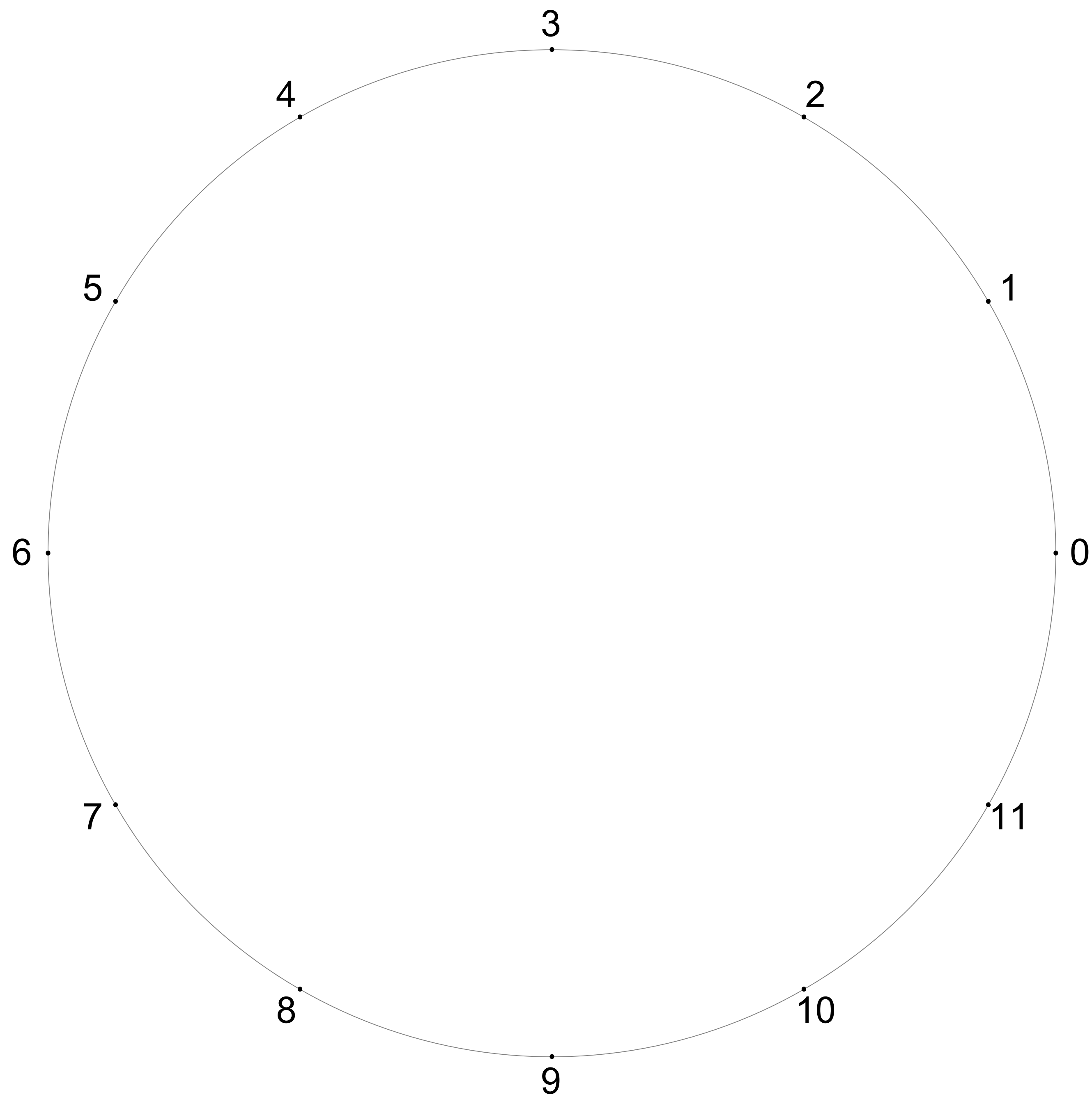
with Grant Allen and Eleanor Meyer

Mathematics Department, UC San Diego

dmeyer@math.ucsd.edu

@dajmeyer

Student Colloquium
University of California, San Diego
La Jolla, CA, the day before Valentine's Day, 2018



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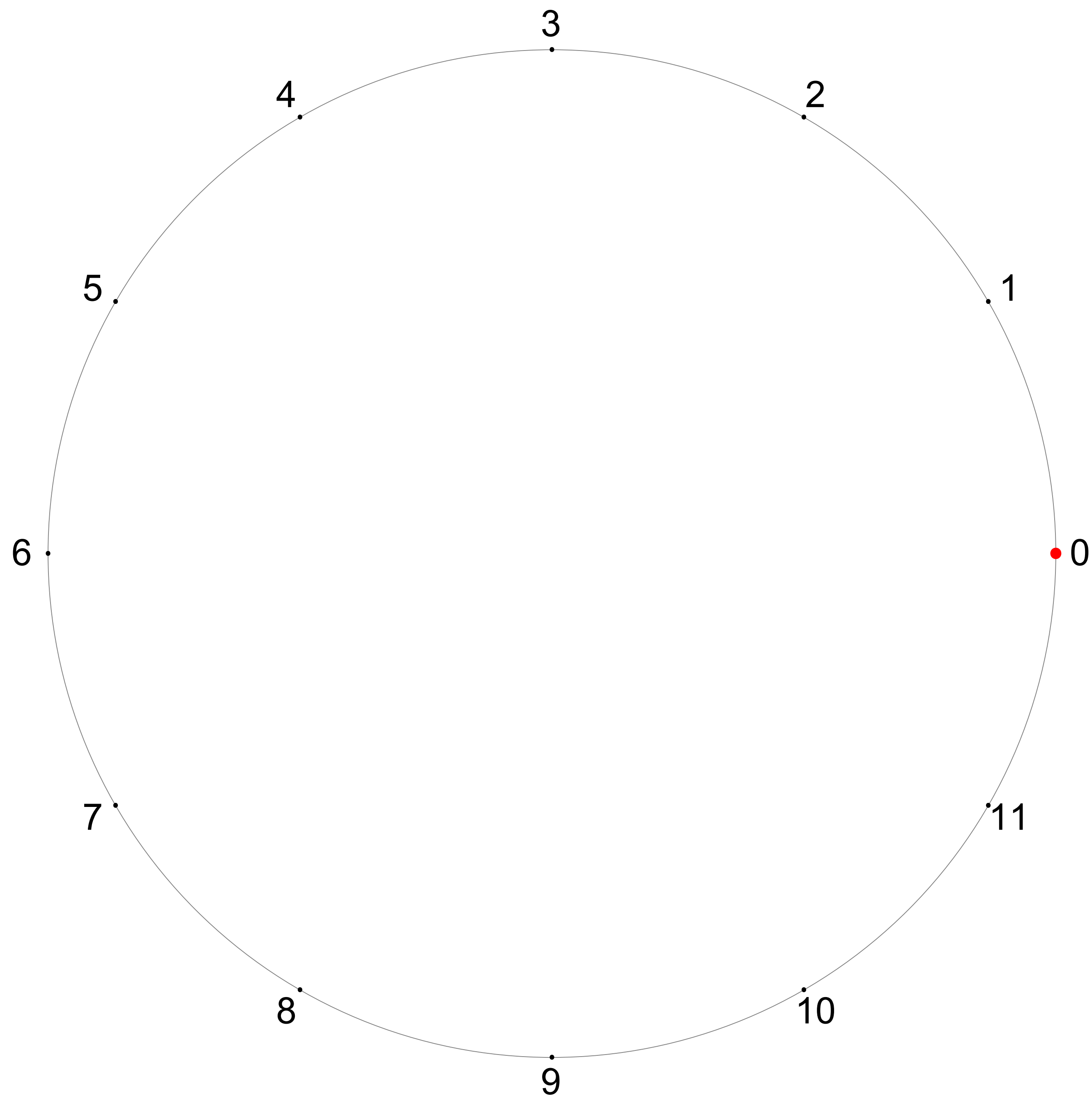

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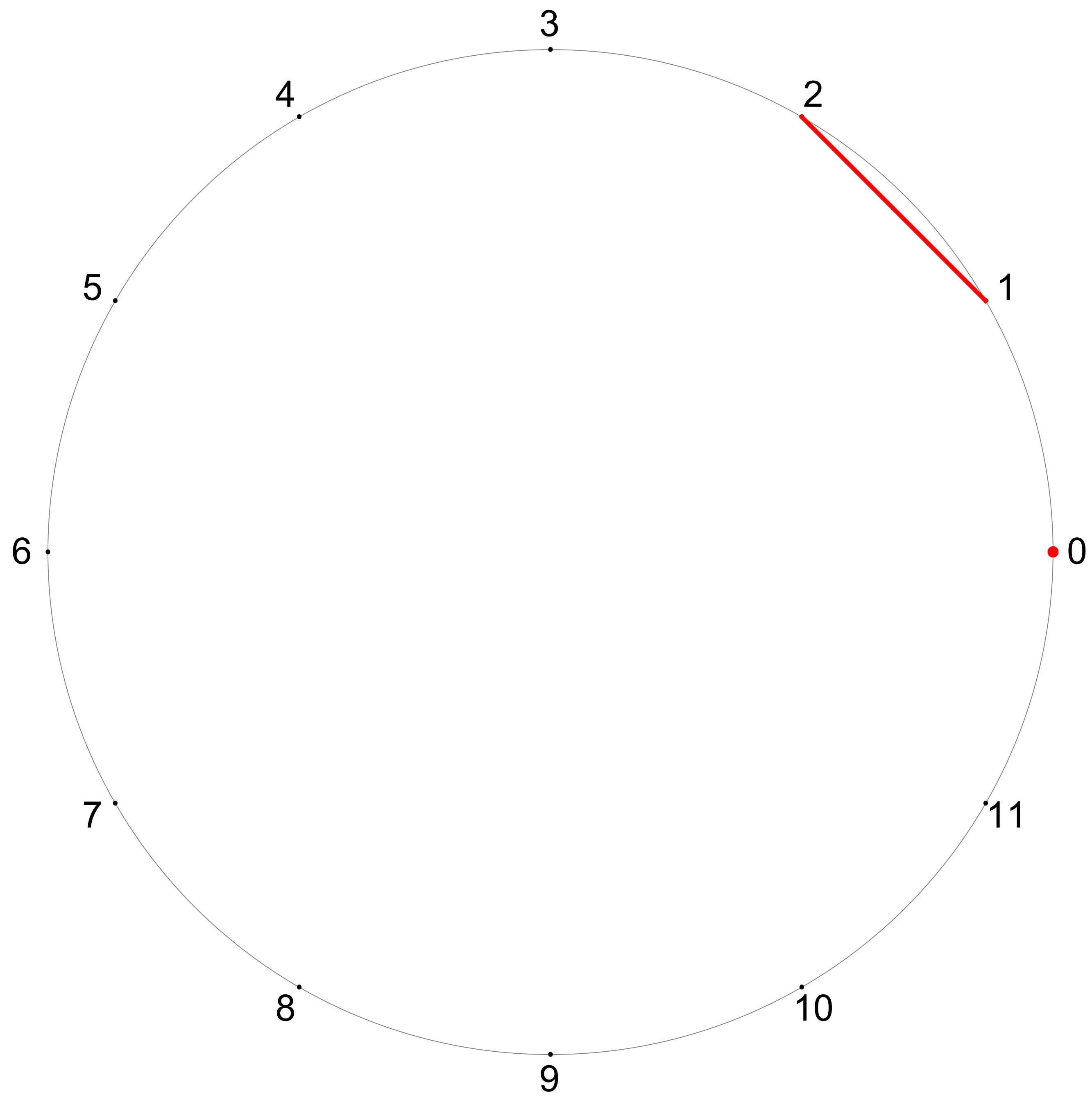

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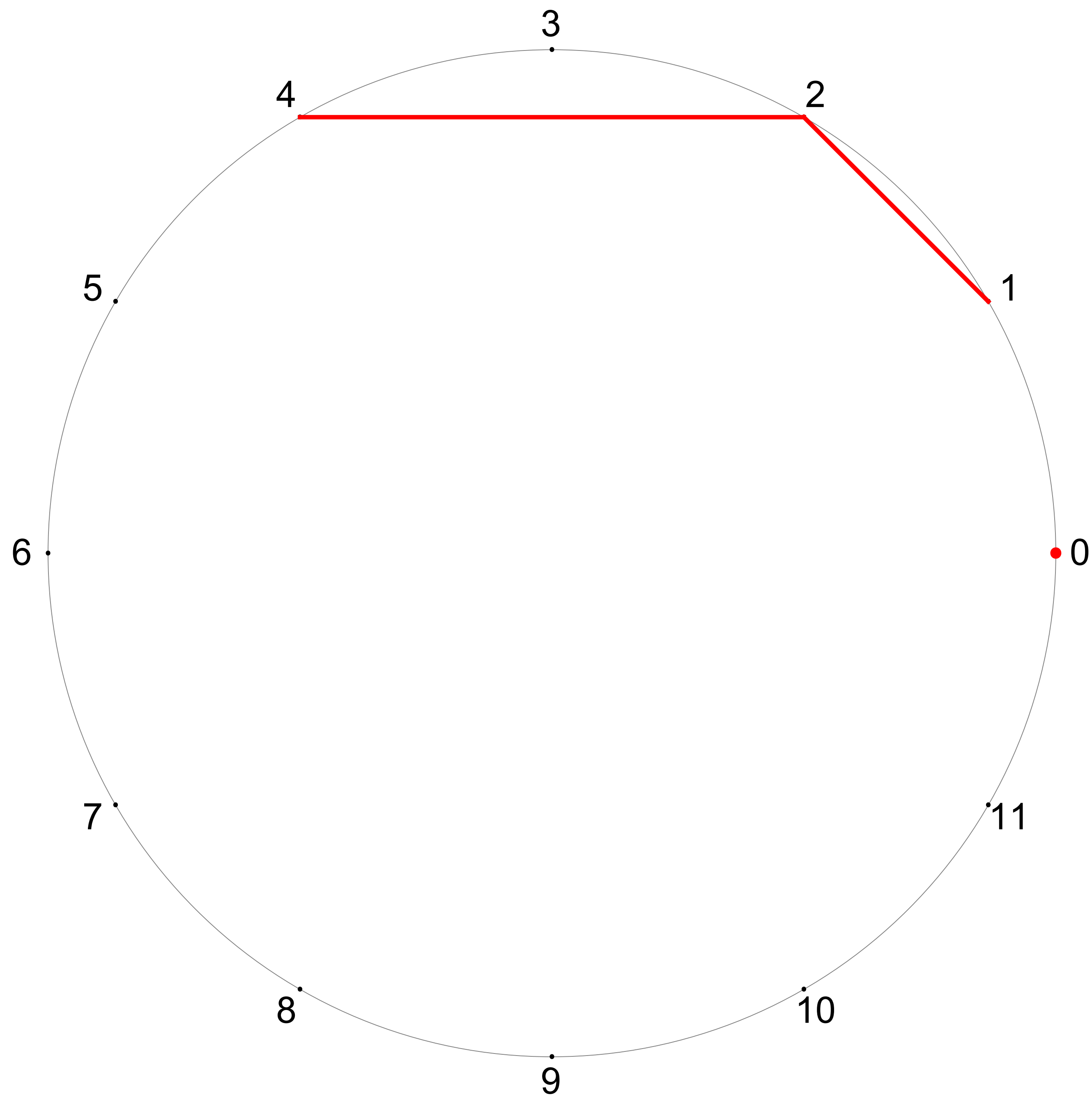
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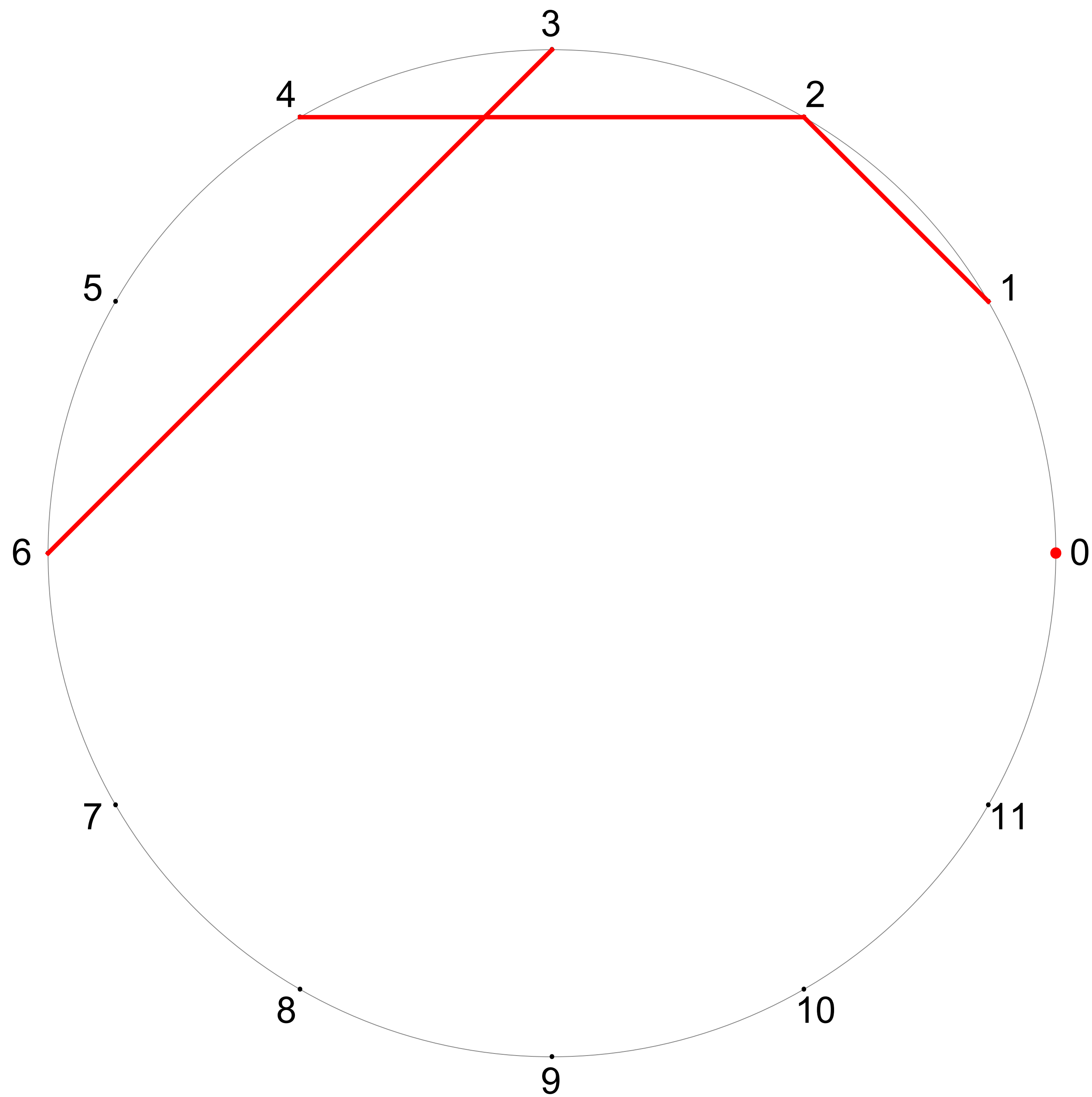

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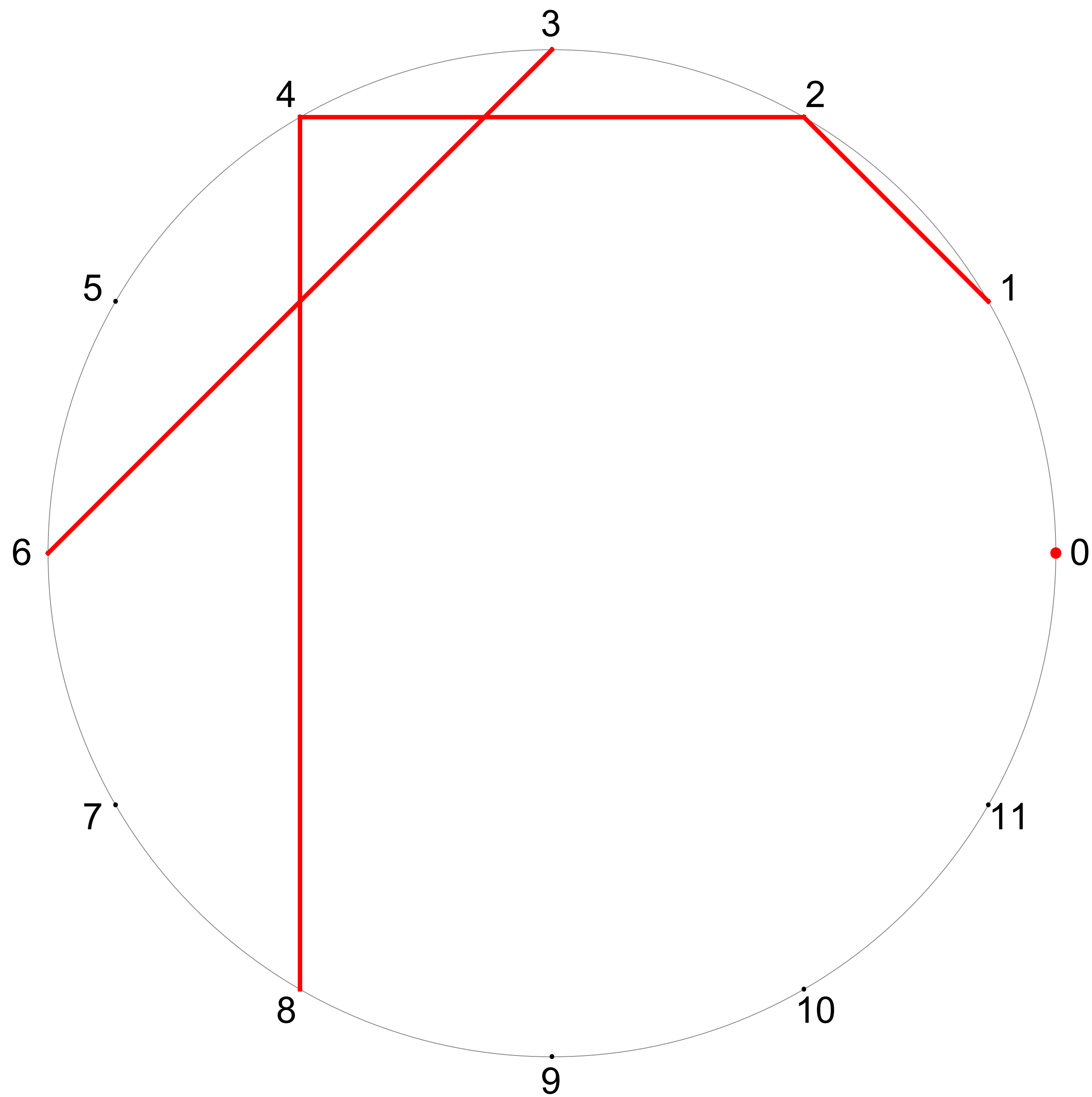

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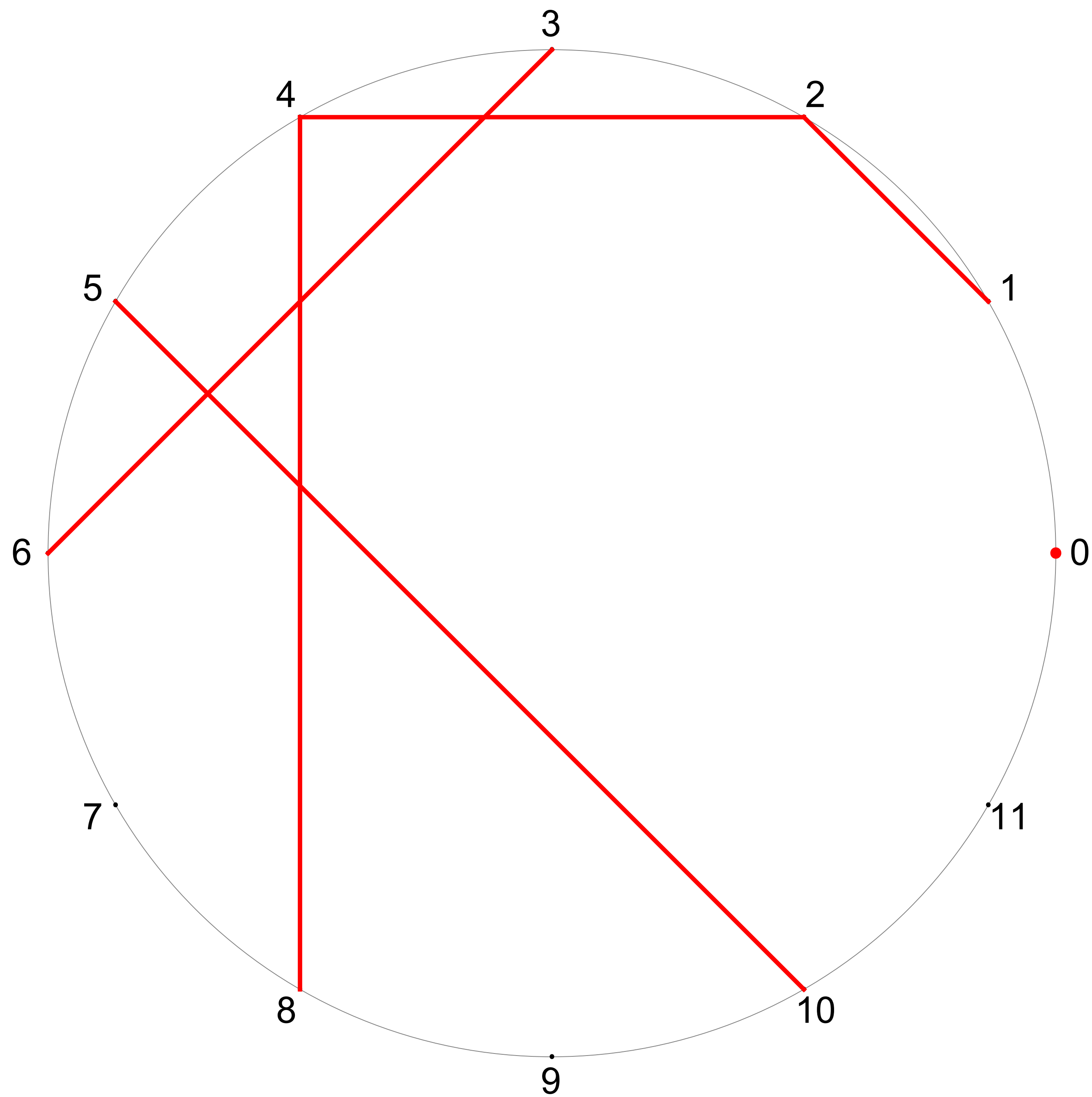

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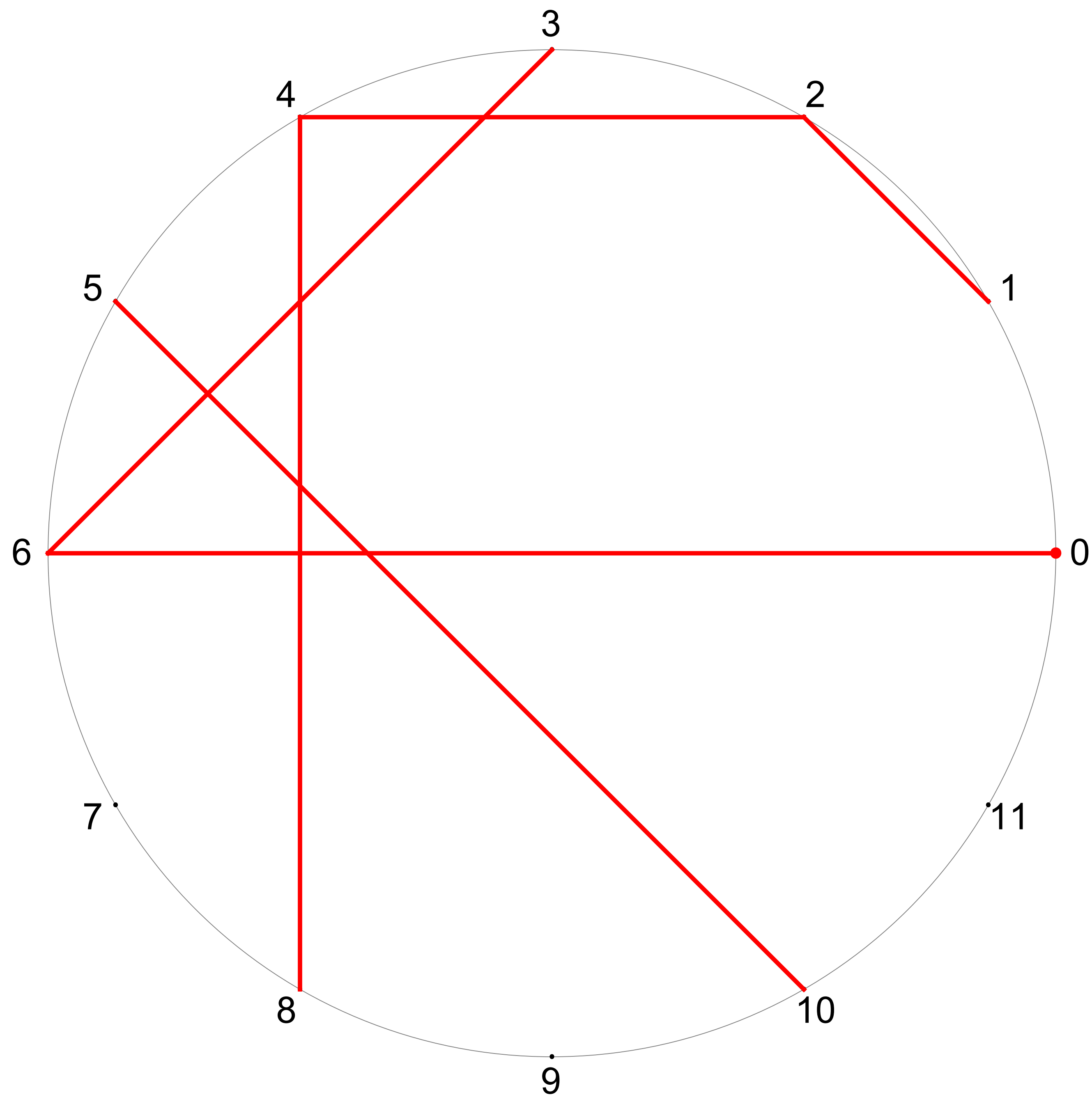
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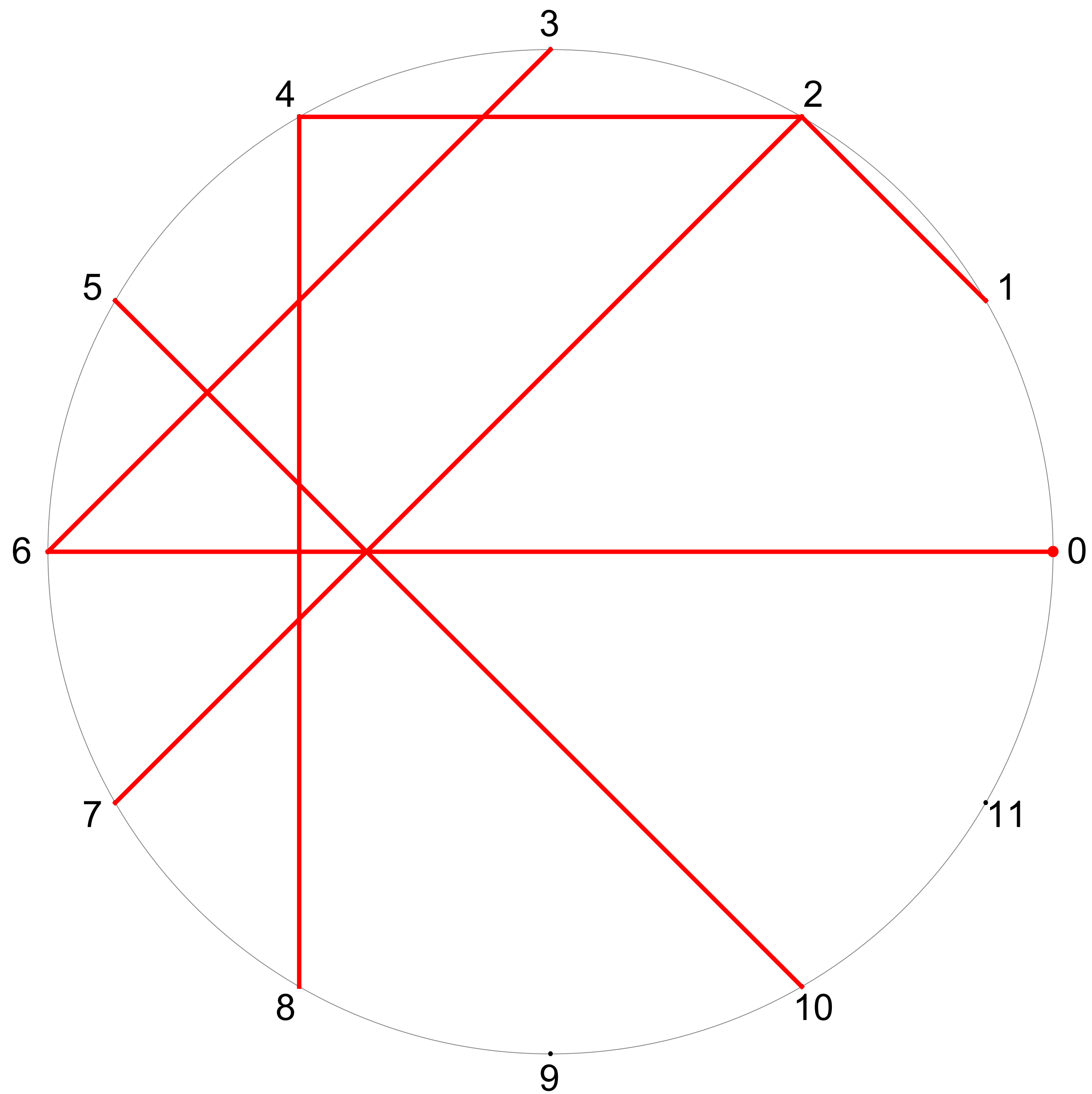

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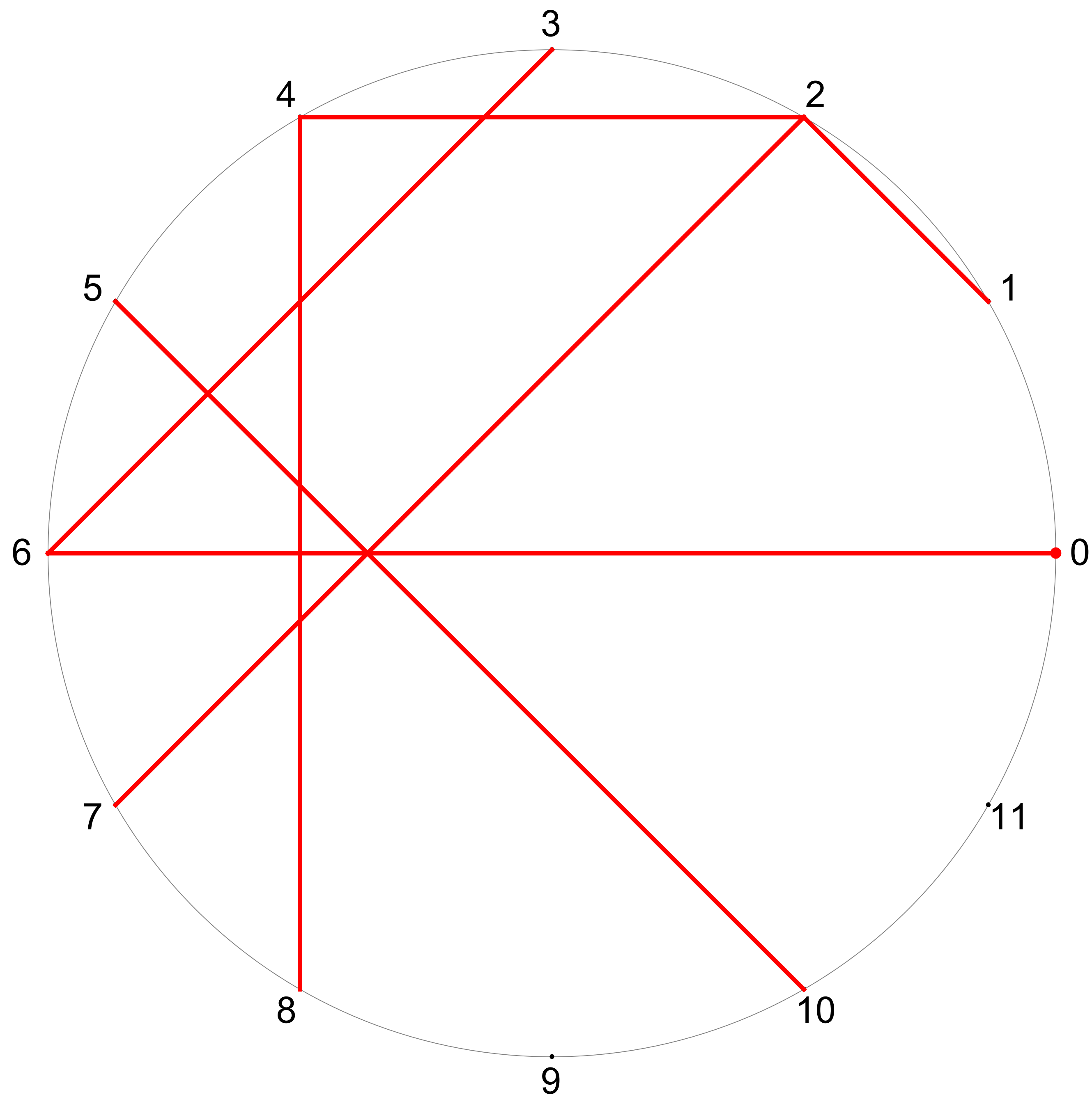
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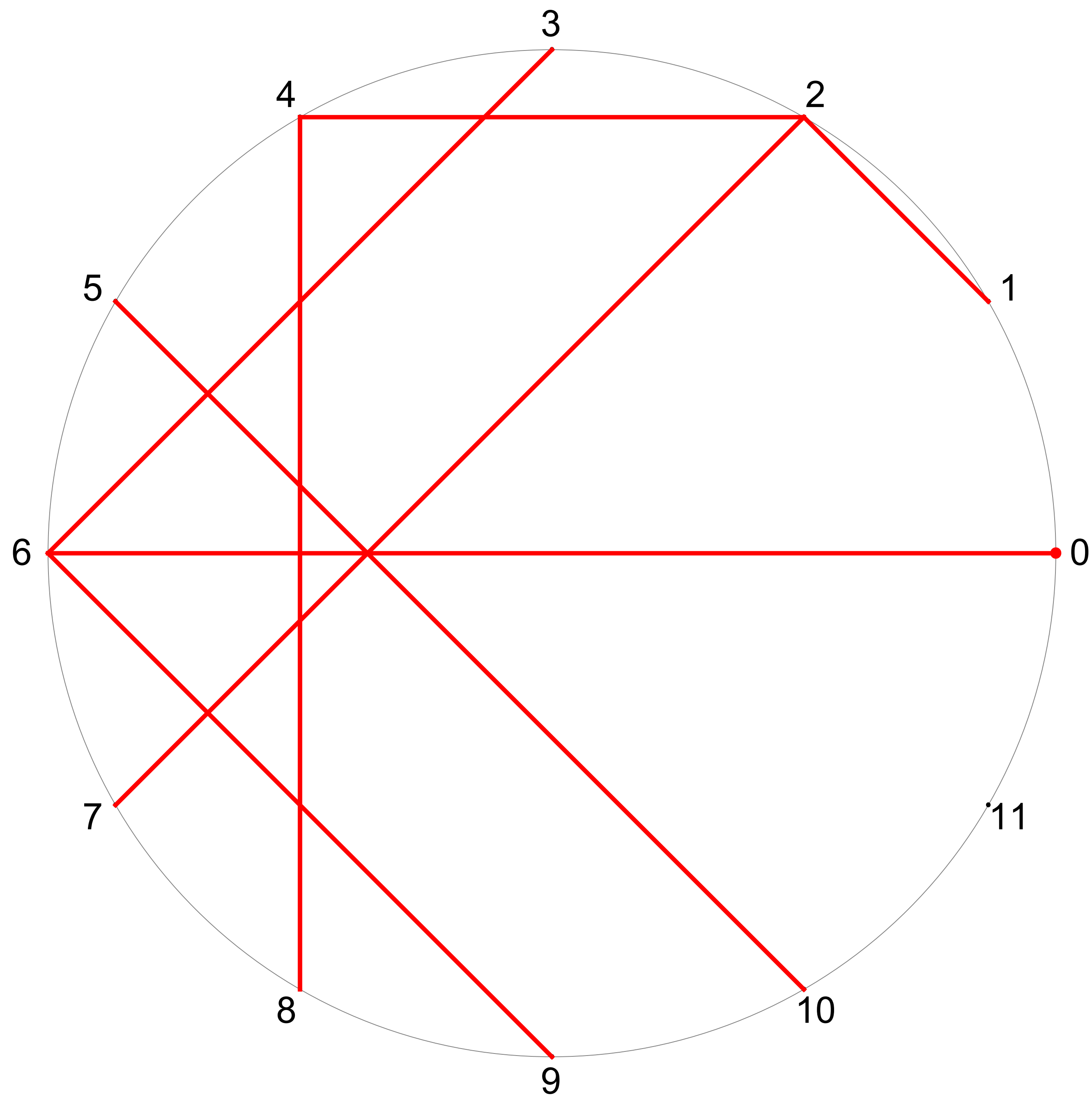

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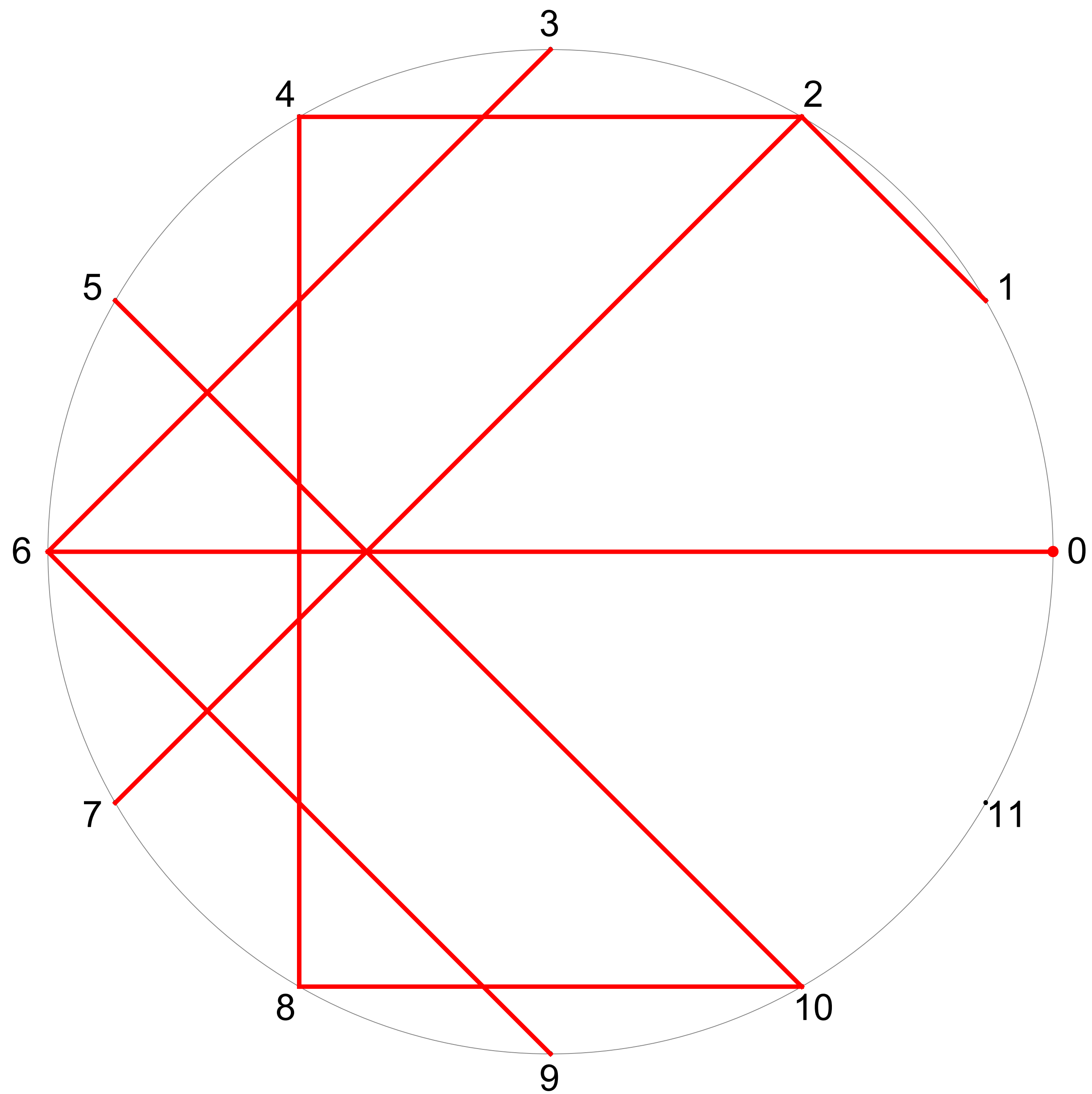

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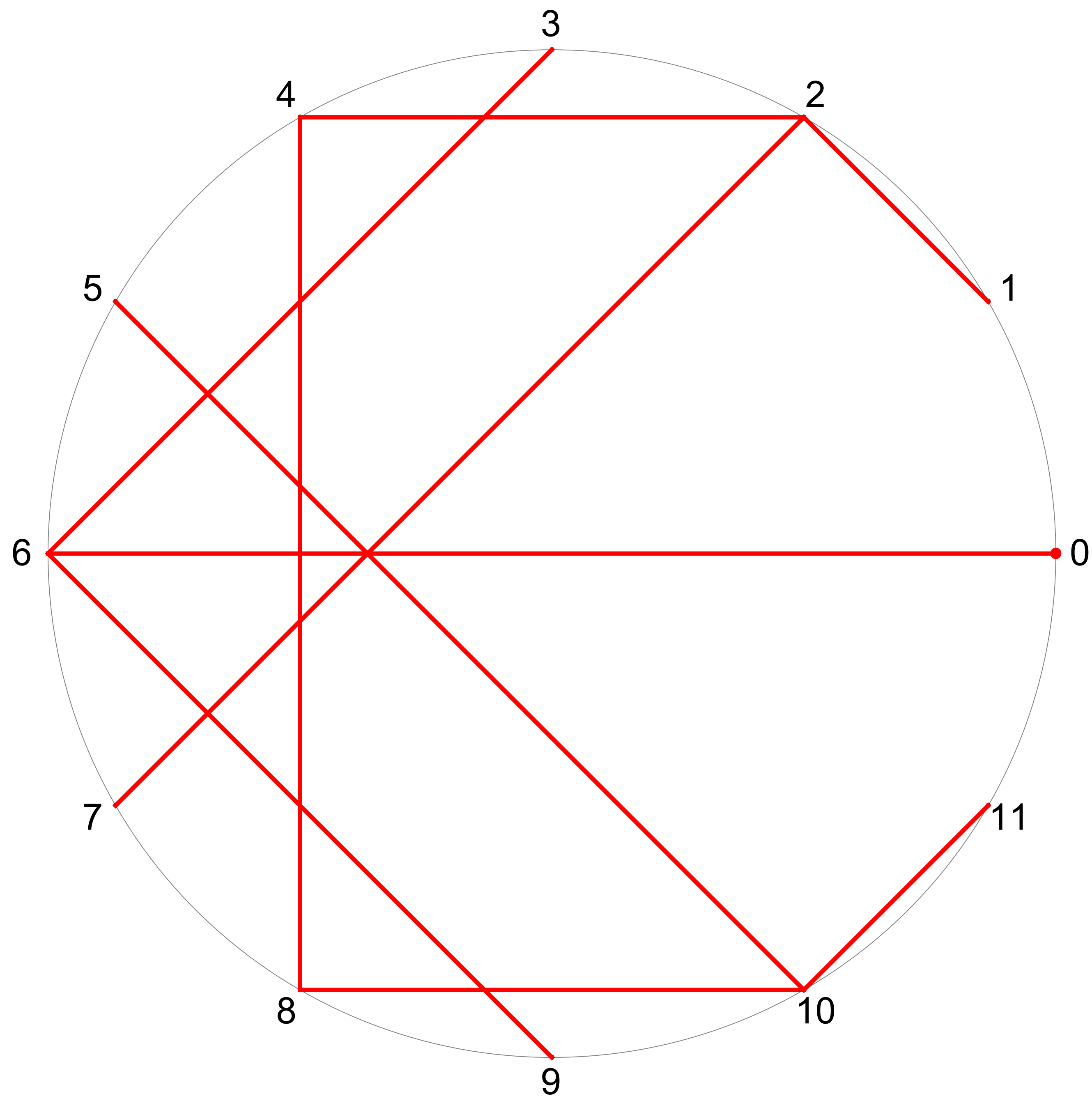
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$$
\begin{aligned}
n &\longmapsto 2n \\
12 - n &\longmapsto 2(12 - n) \equiv 12 - 2n
\end{aligned}
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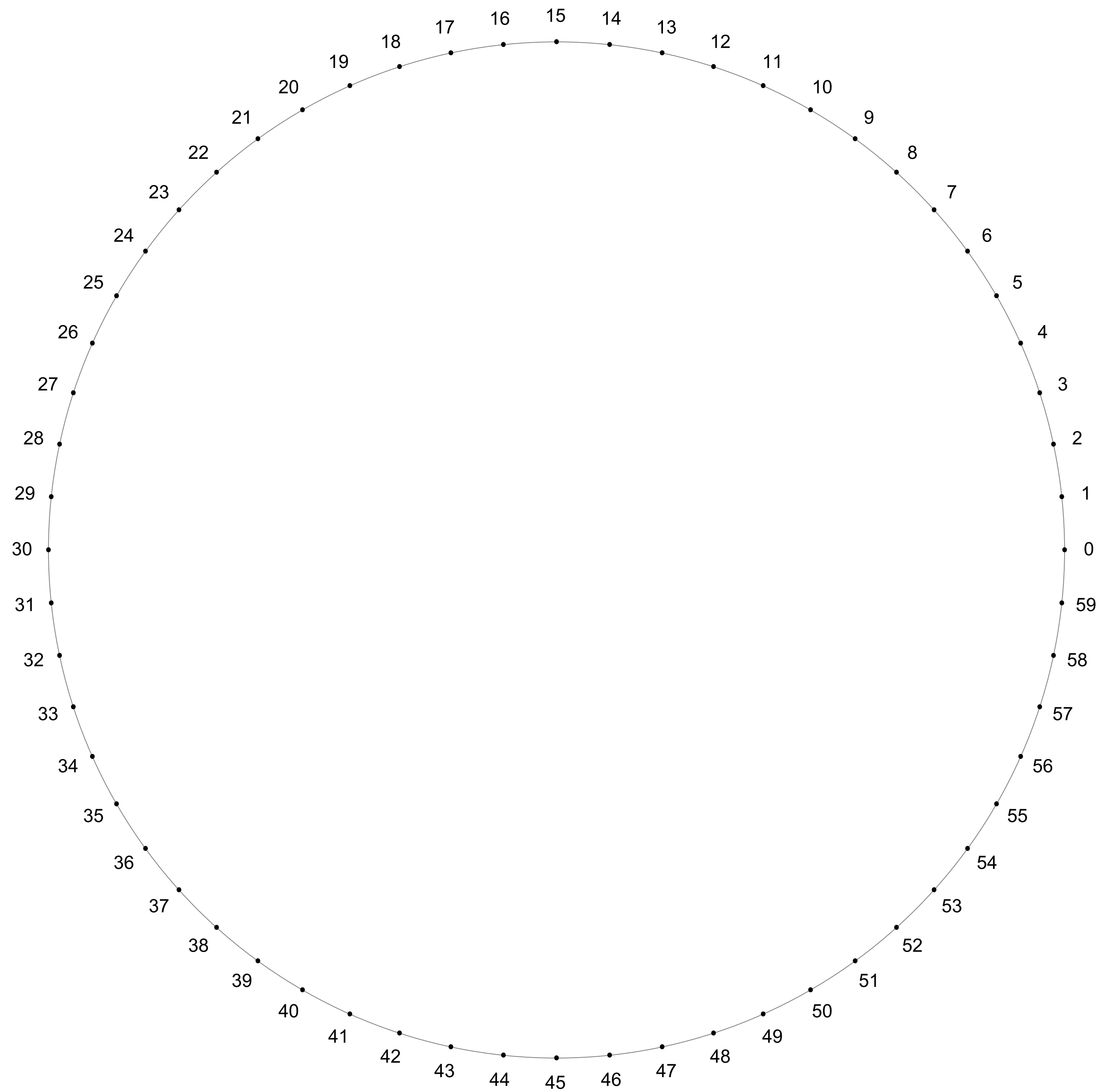

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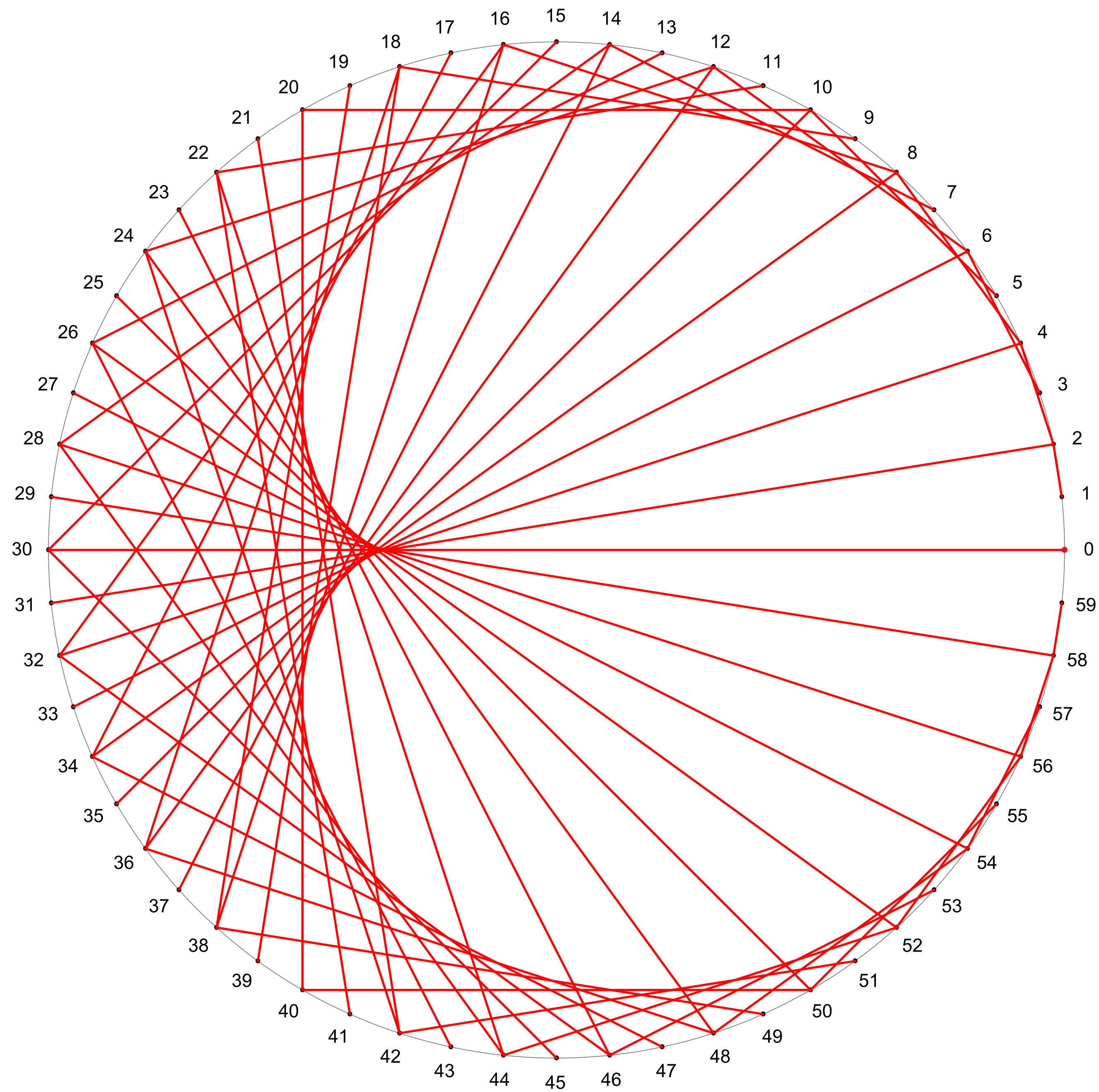

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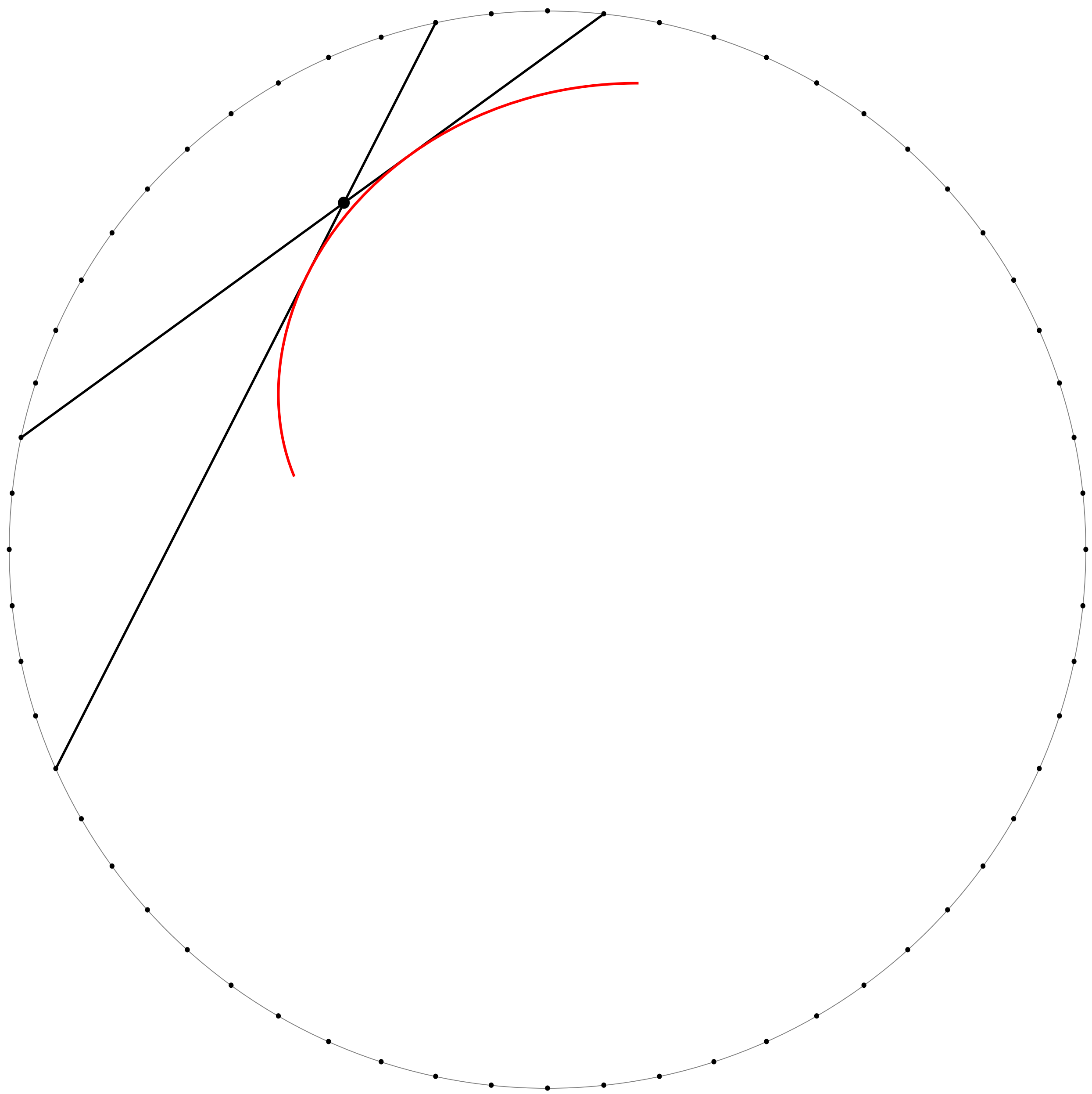
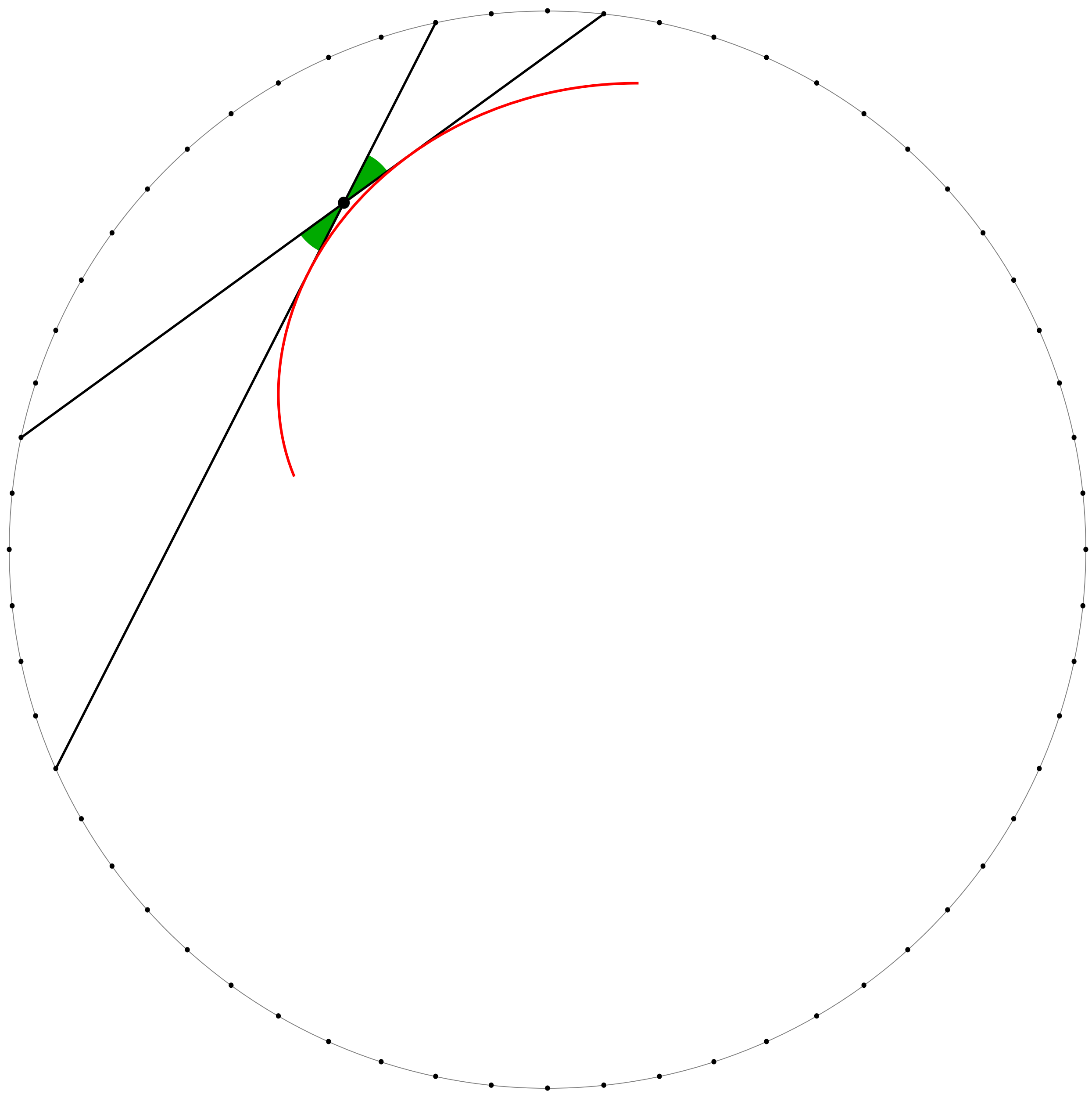
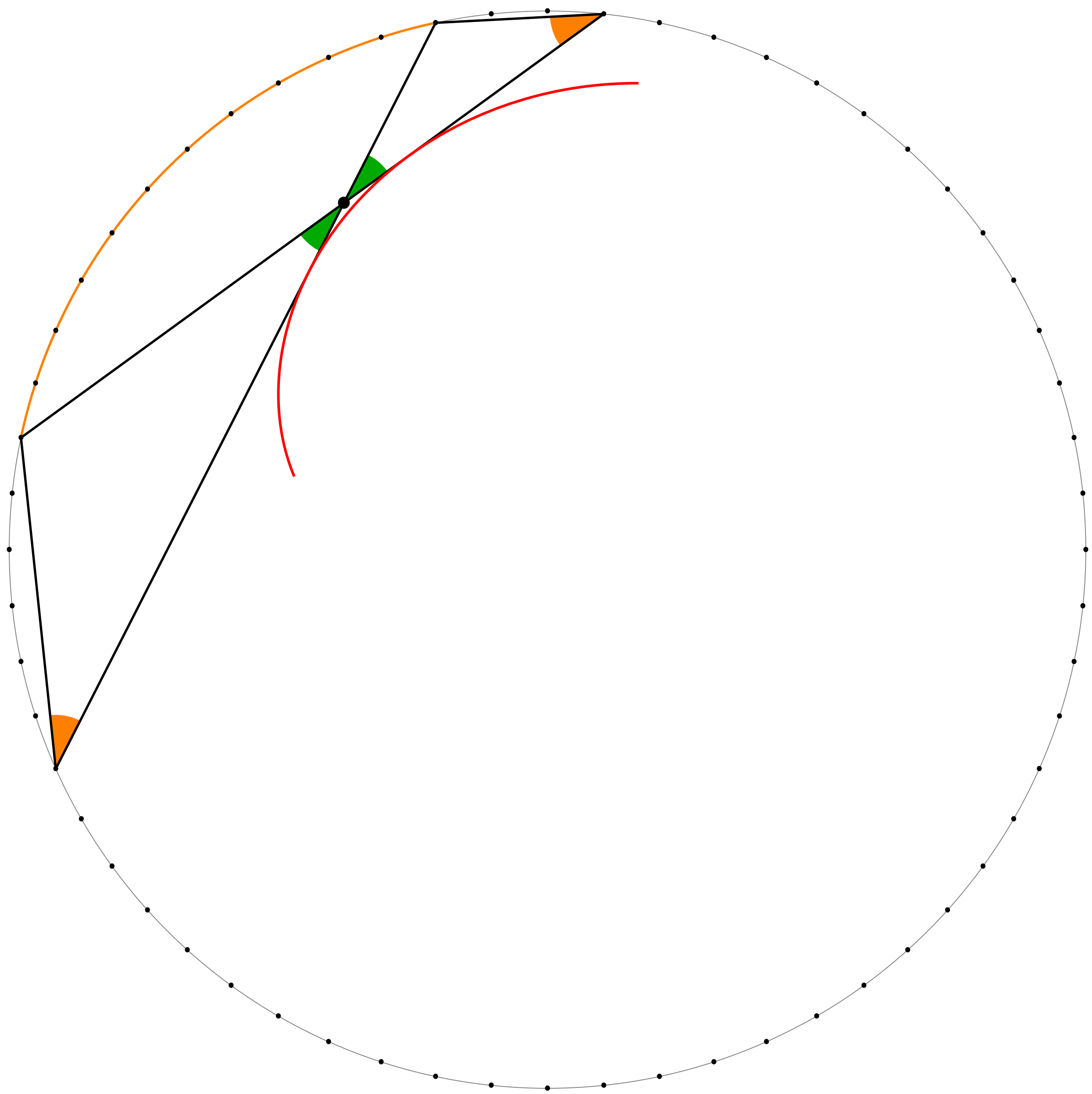
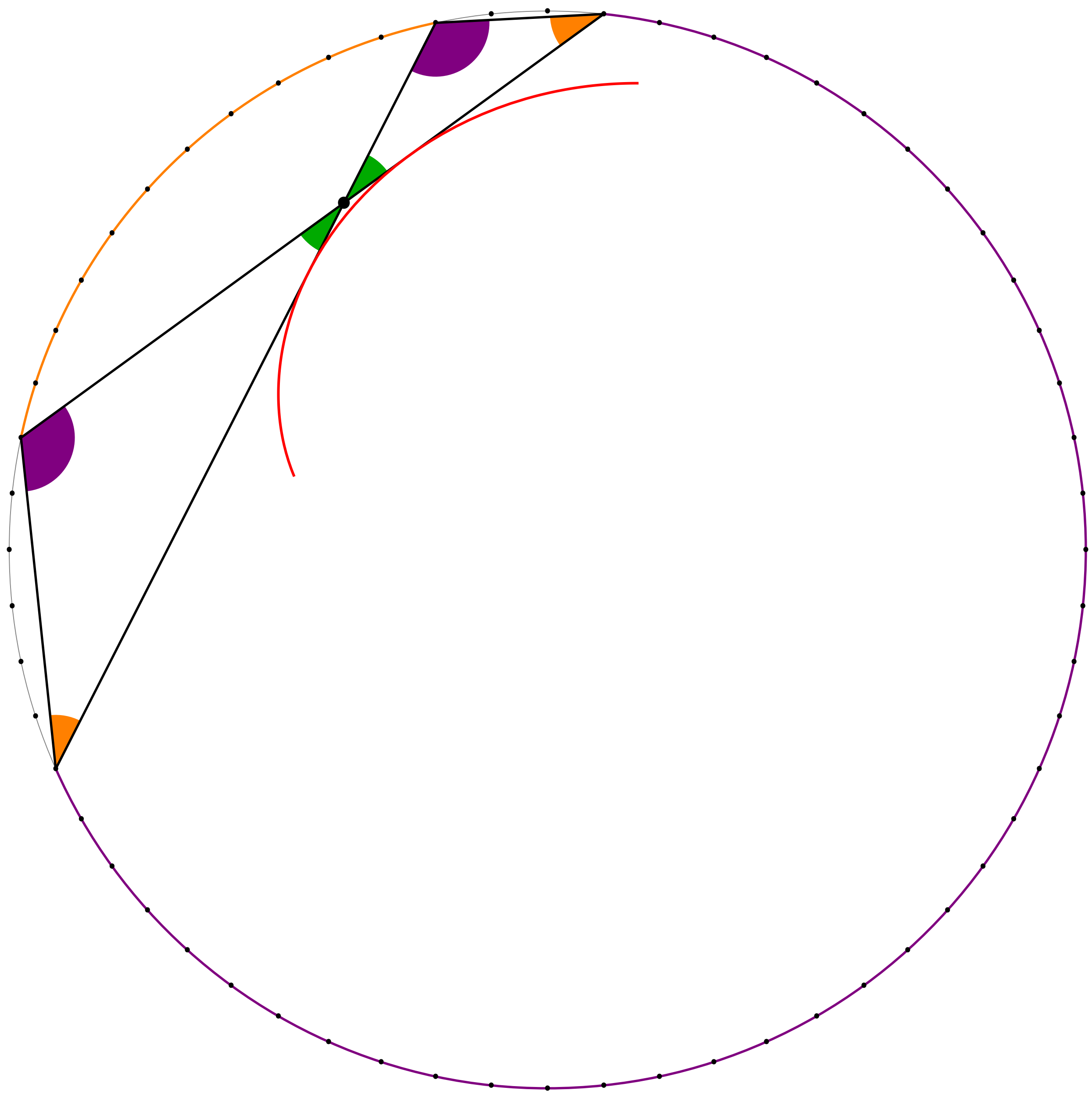
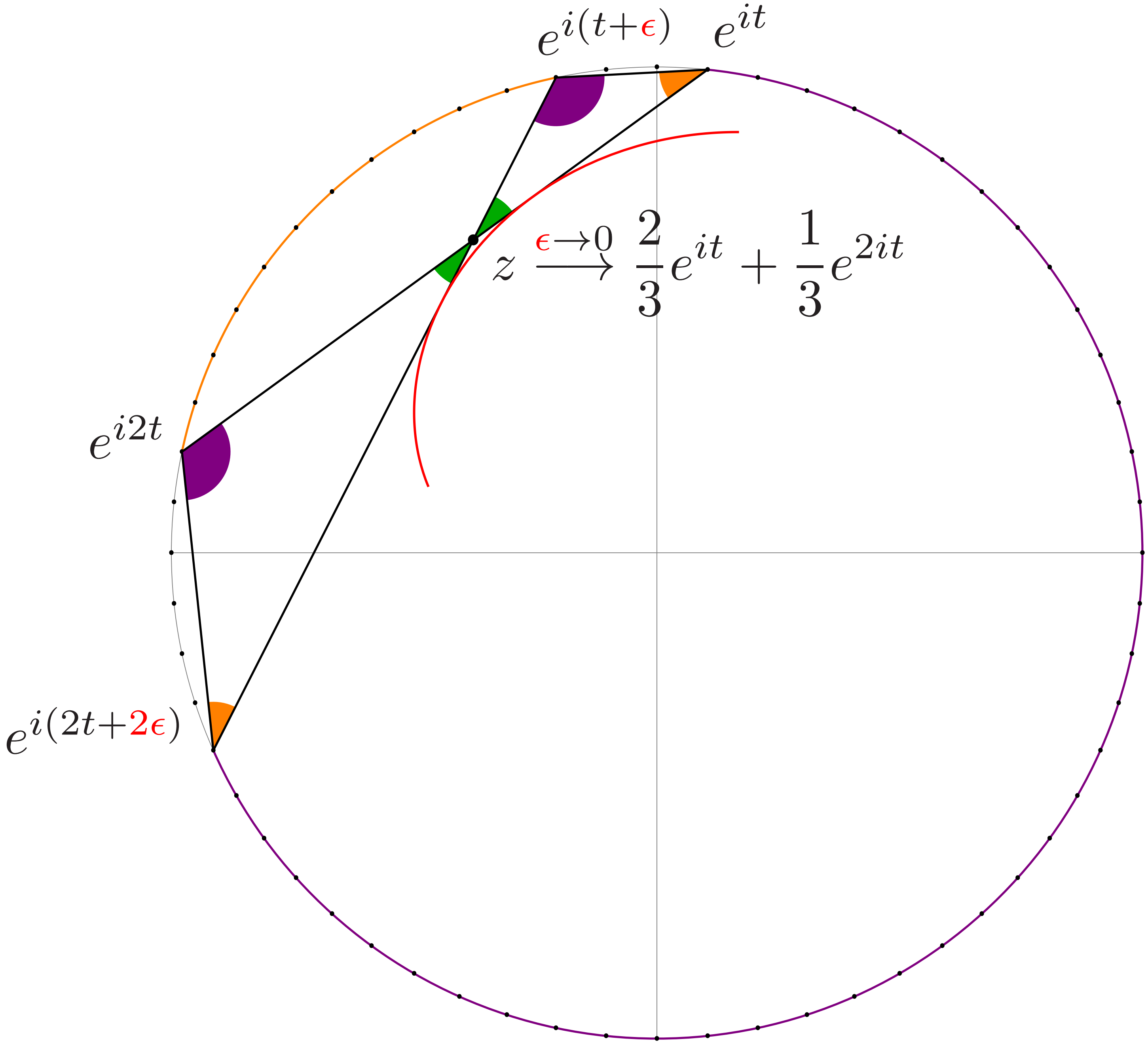


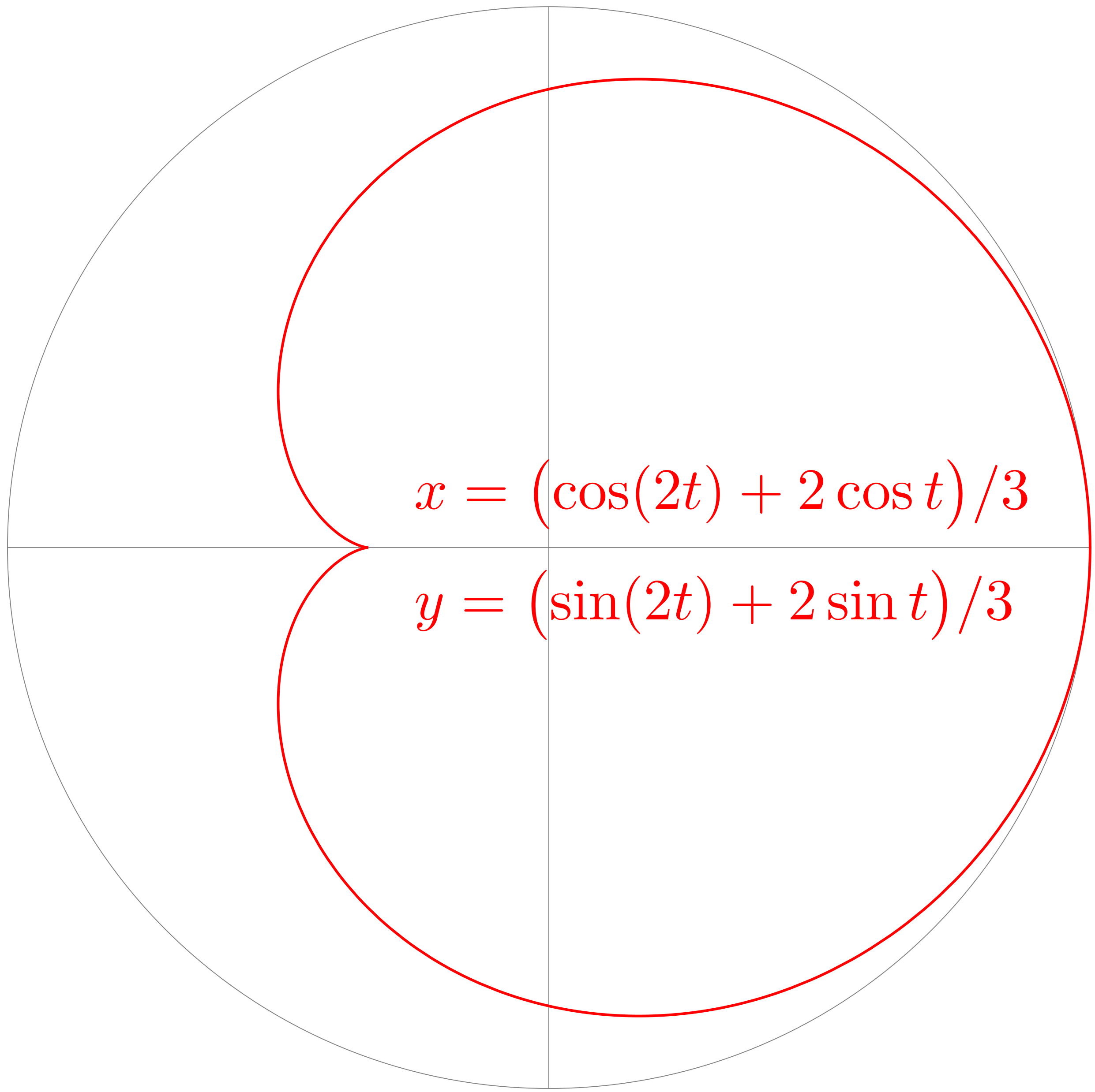


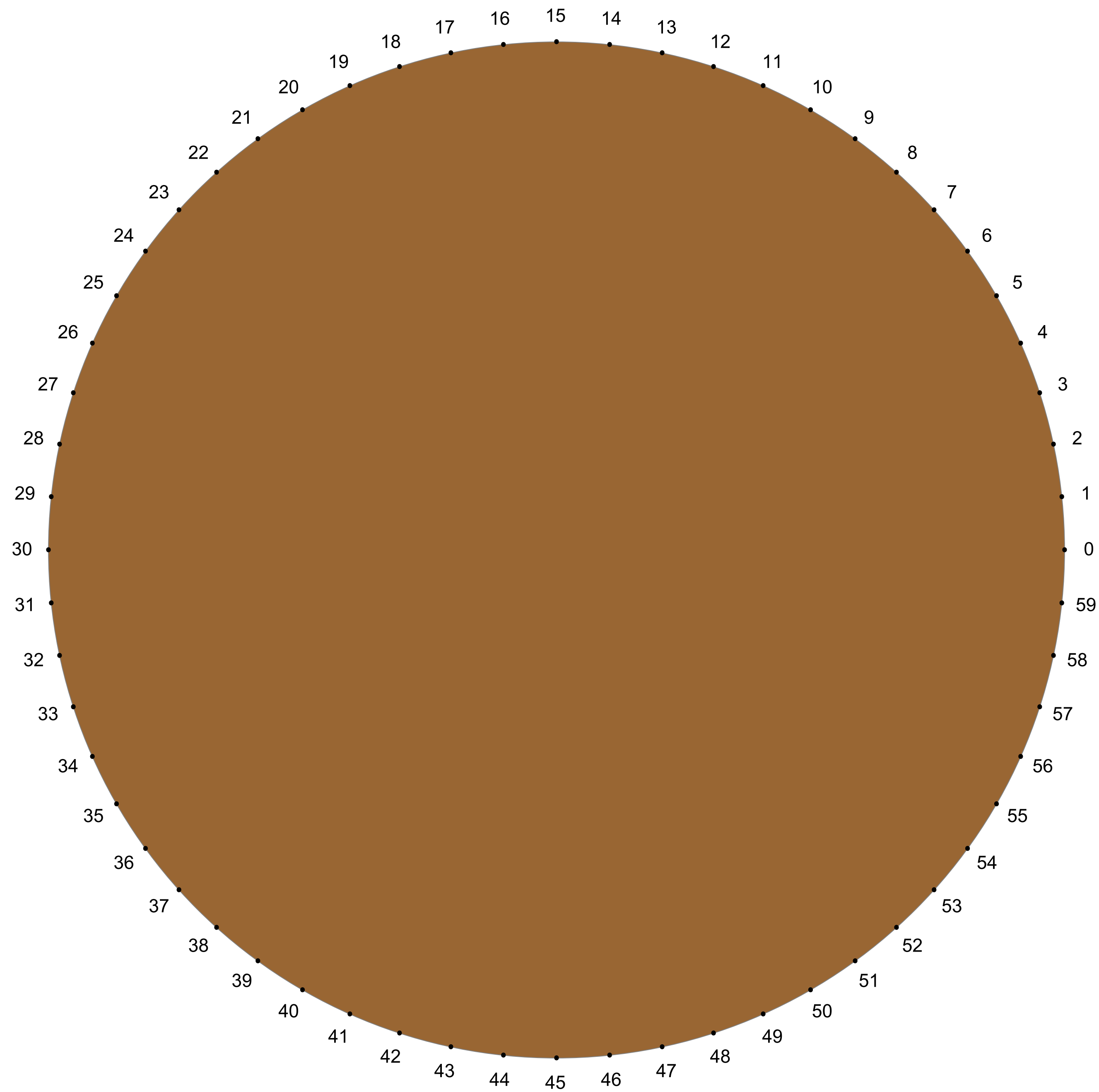






$e^{i(t+\epsilon)}$
e^{it}
$z \xrightarrow{\epsilon\to 0} \frac{2}{3}e^{it} + \frac{1}{3}e^{2it}$
e^{i2t}
$e^{i(2t+2\epsilon)}$


$x = \big(\cos(2t) + 2\cos t\big)/3$
$y = \big(\sin(2t) + 2\sin t\big)/3$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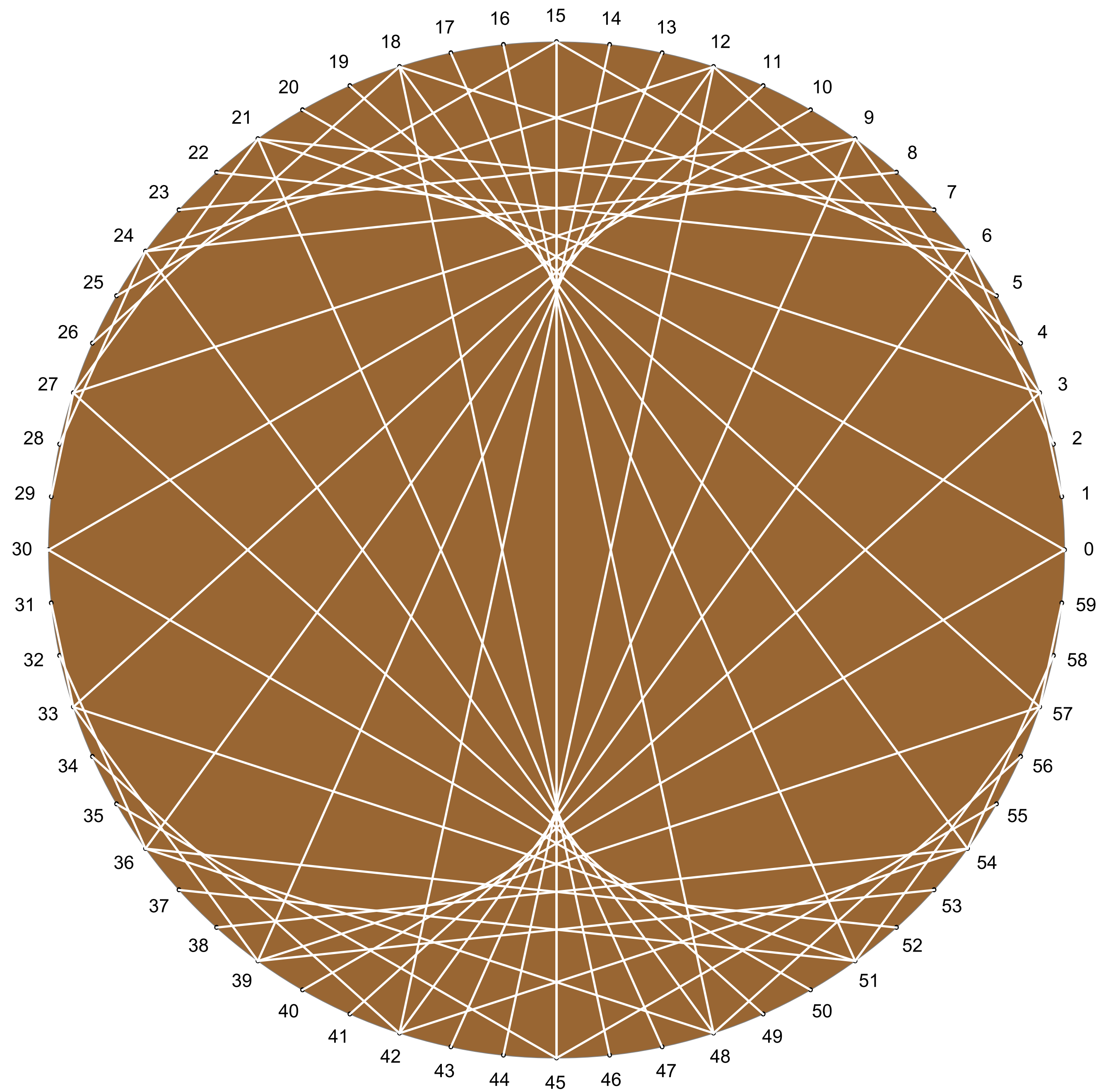
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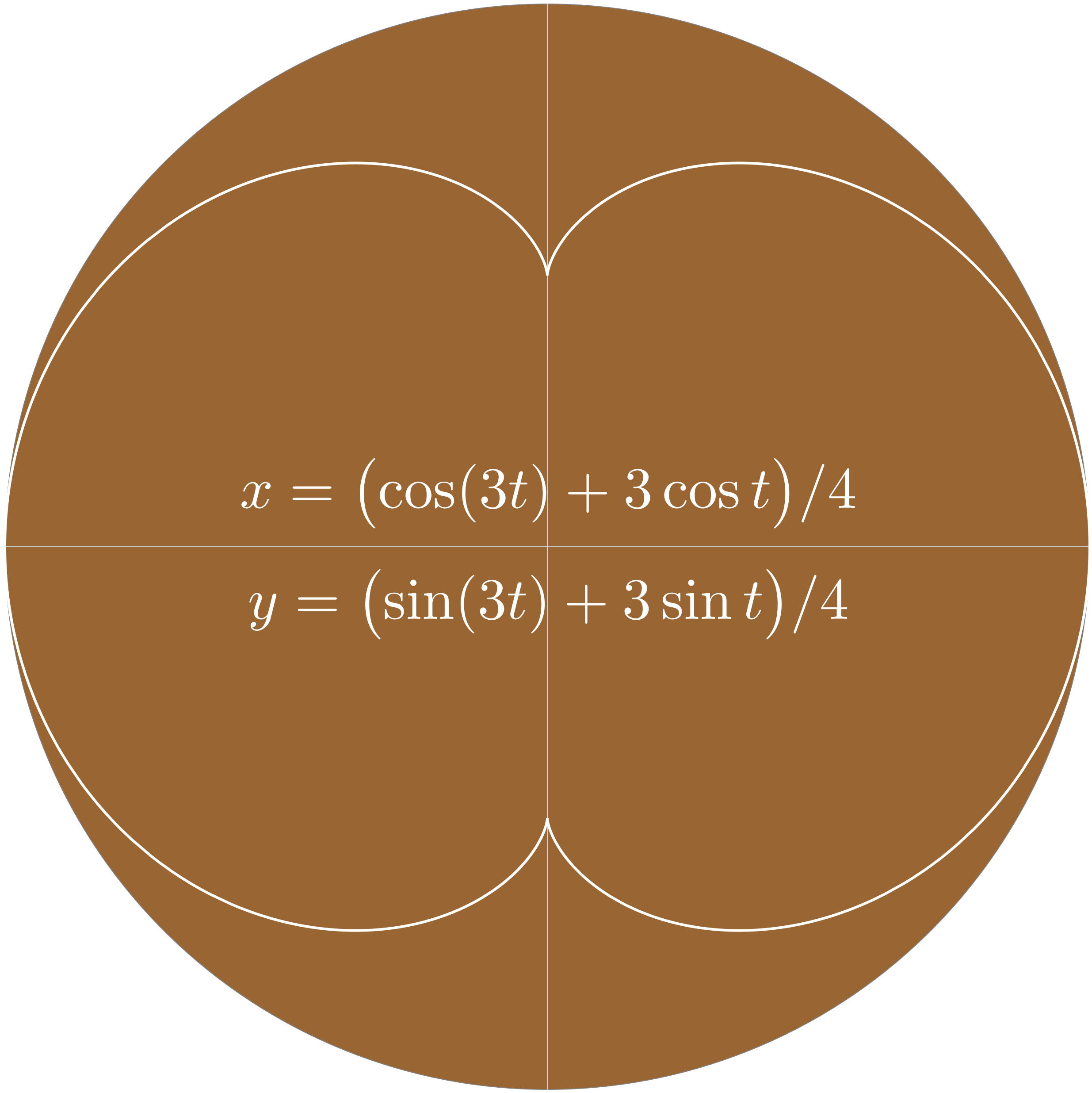
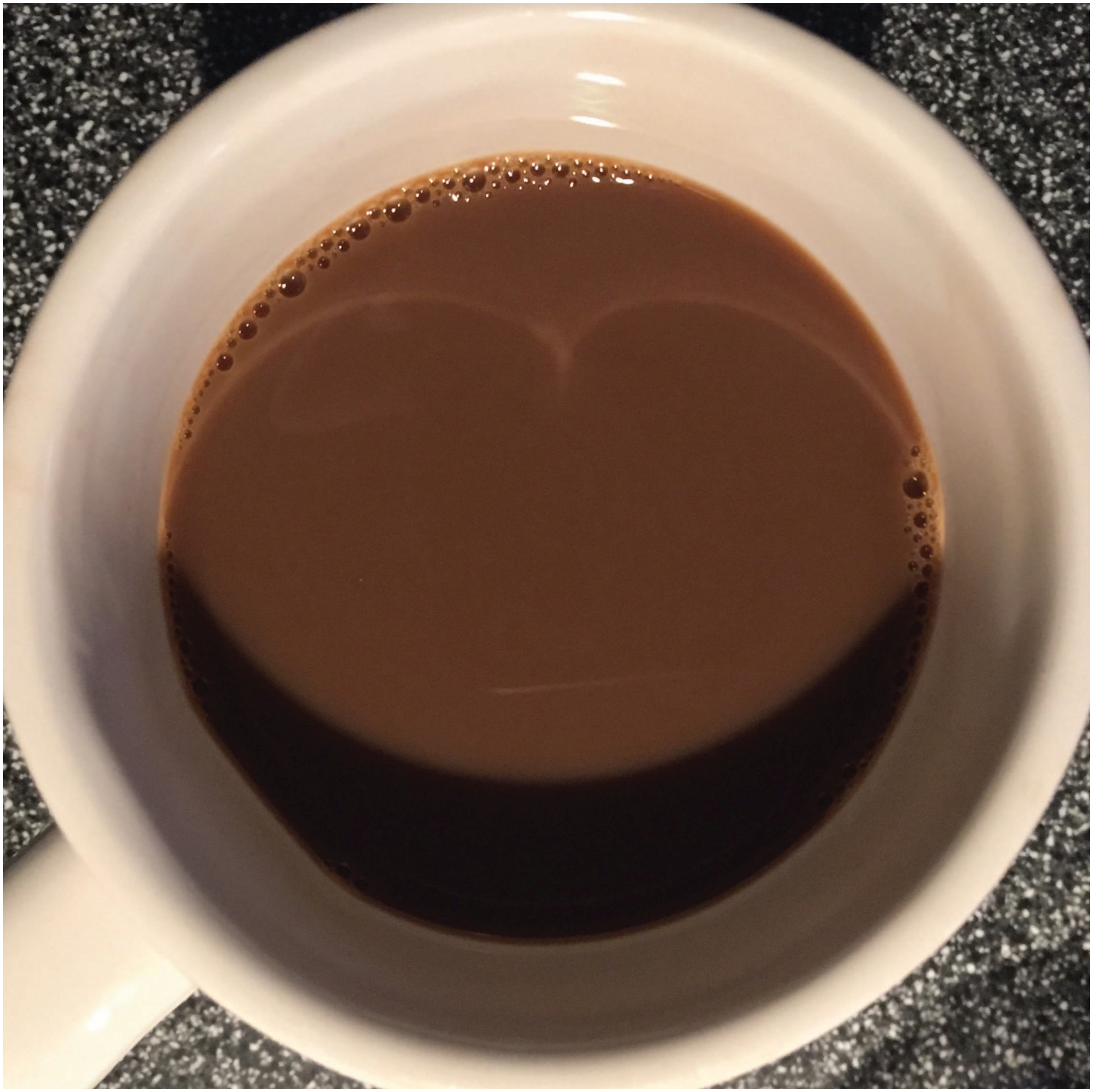
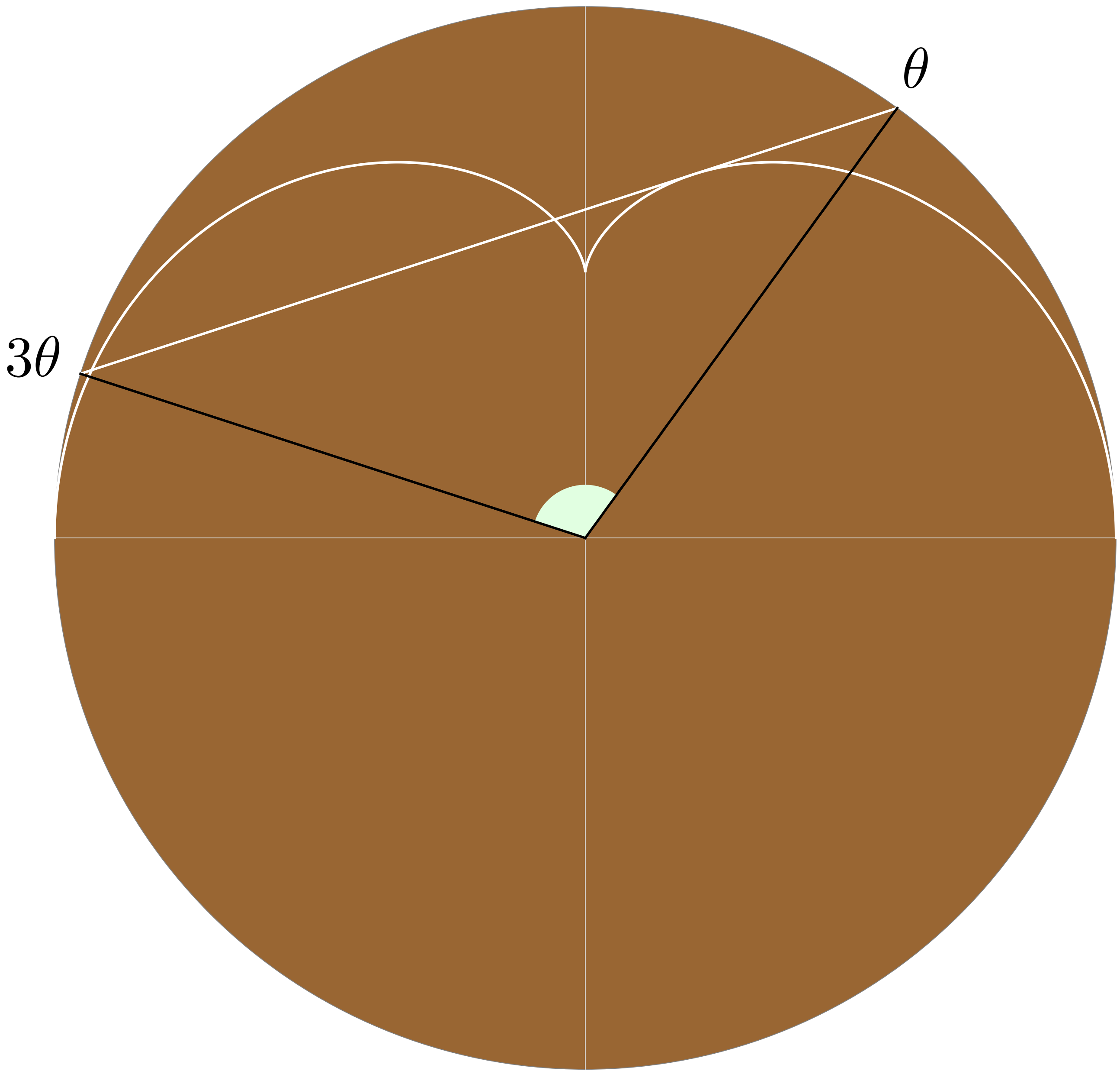


$x = \big(\cos(3t) + 3\cos t\big)/4$
$y = \big(\sin(3t) + 3\sin t\big)/4$





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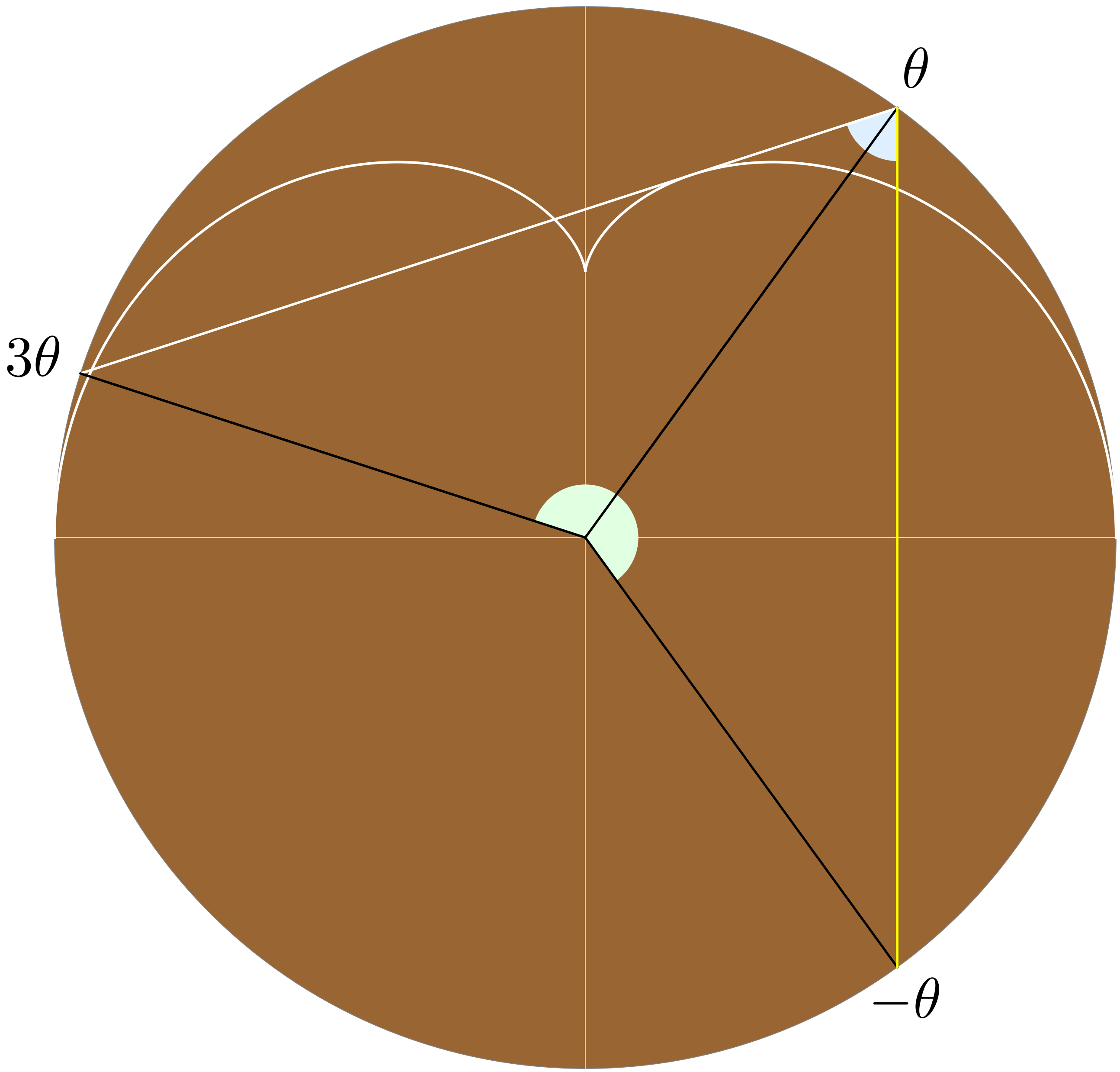

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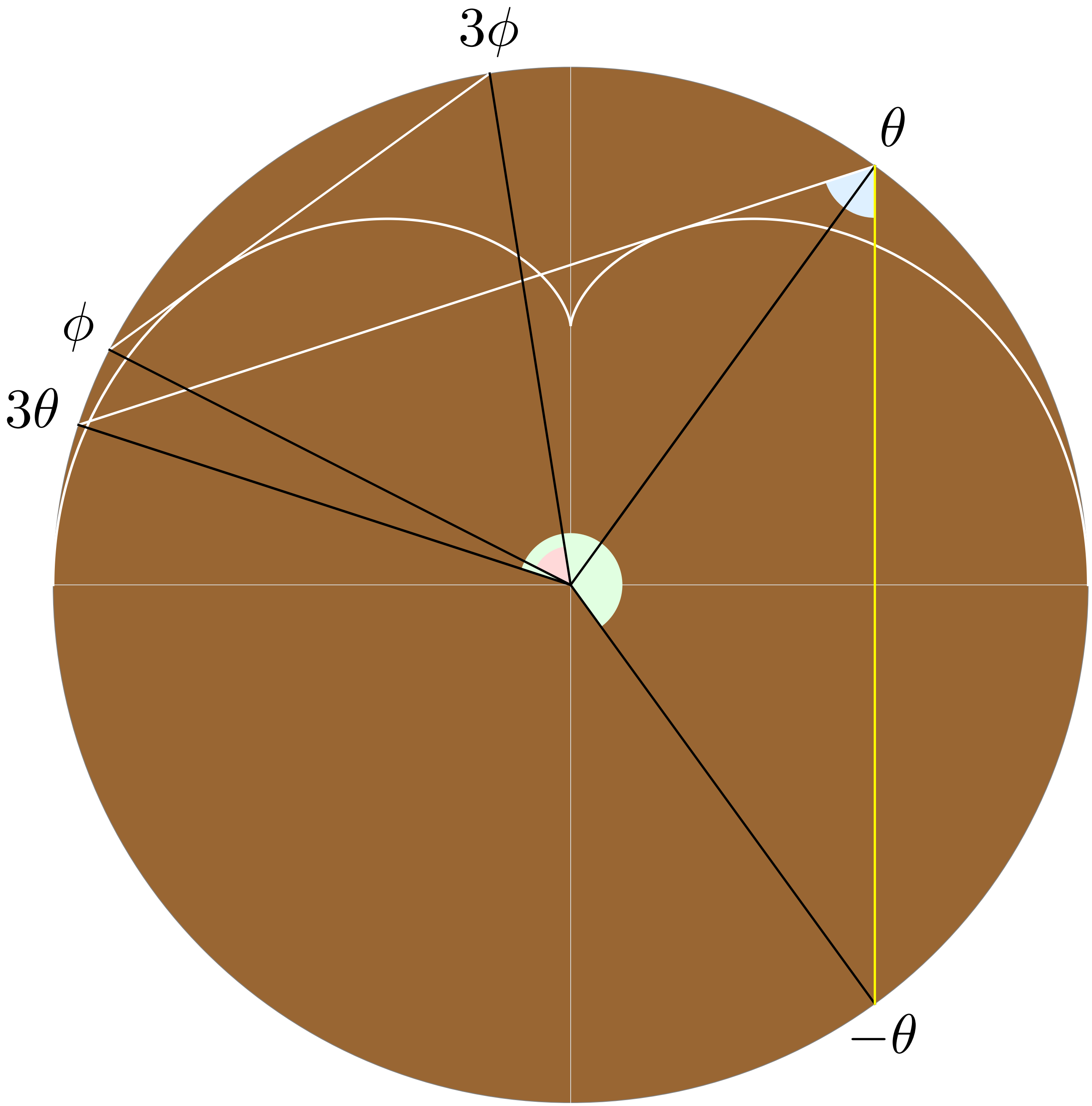
$-\theta$



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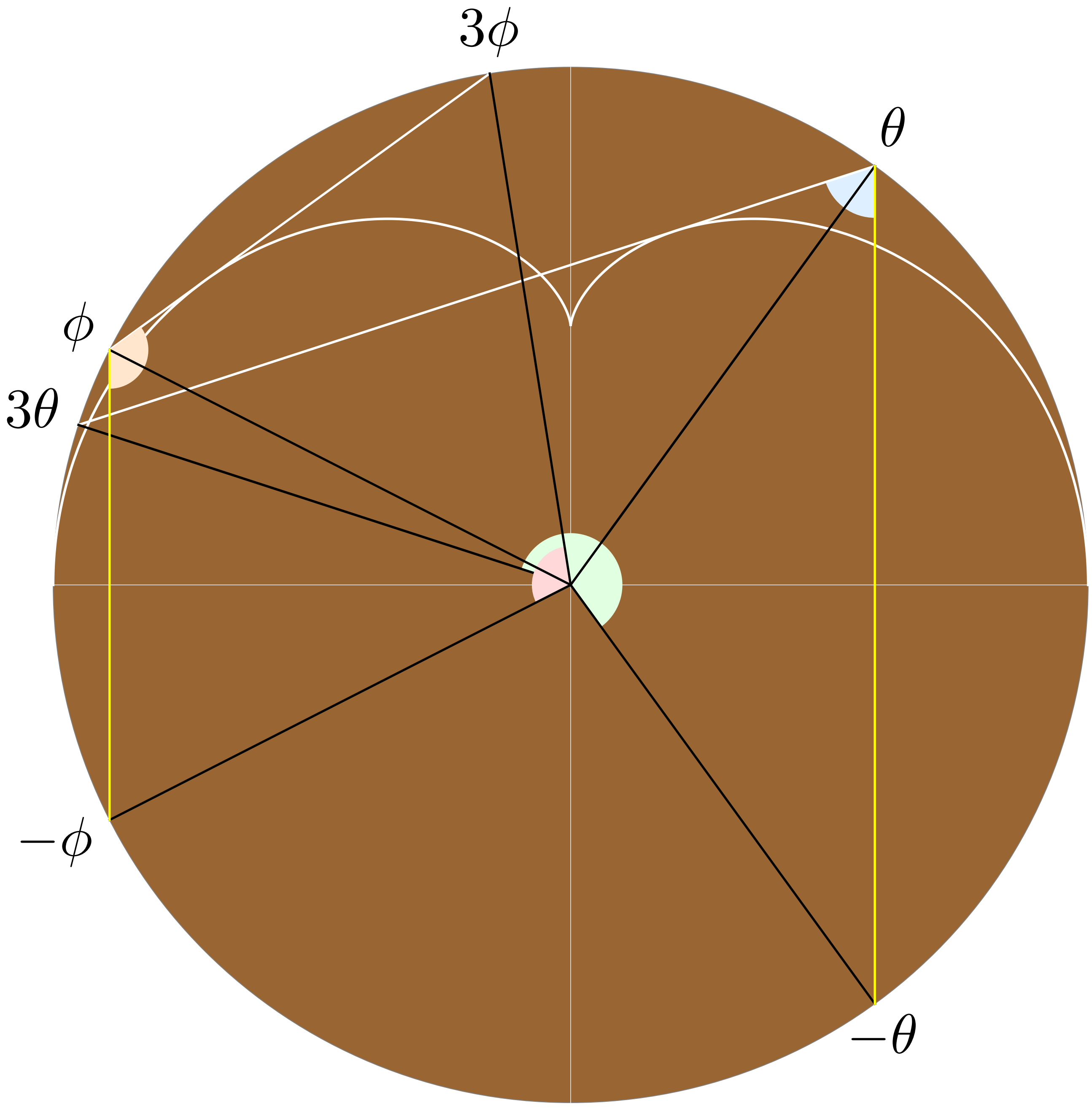

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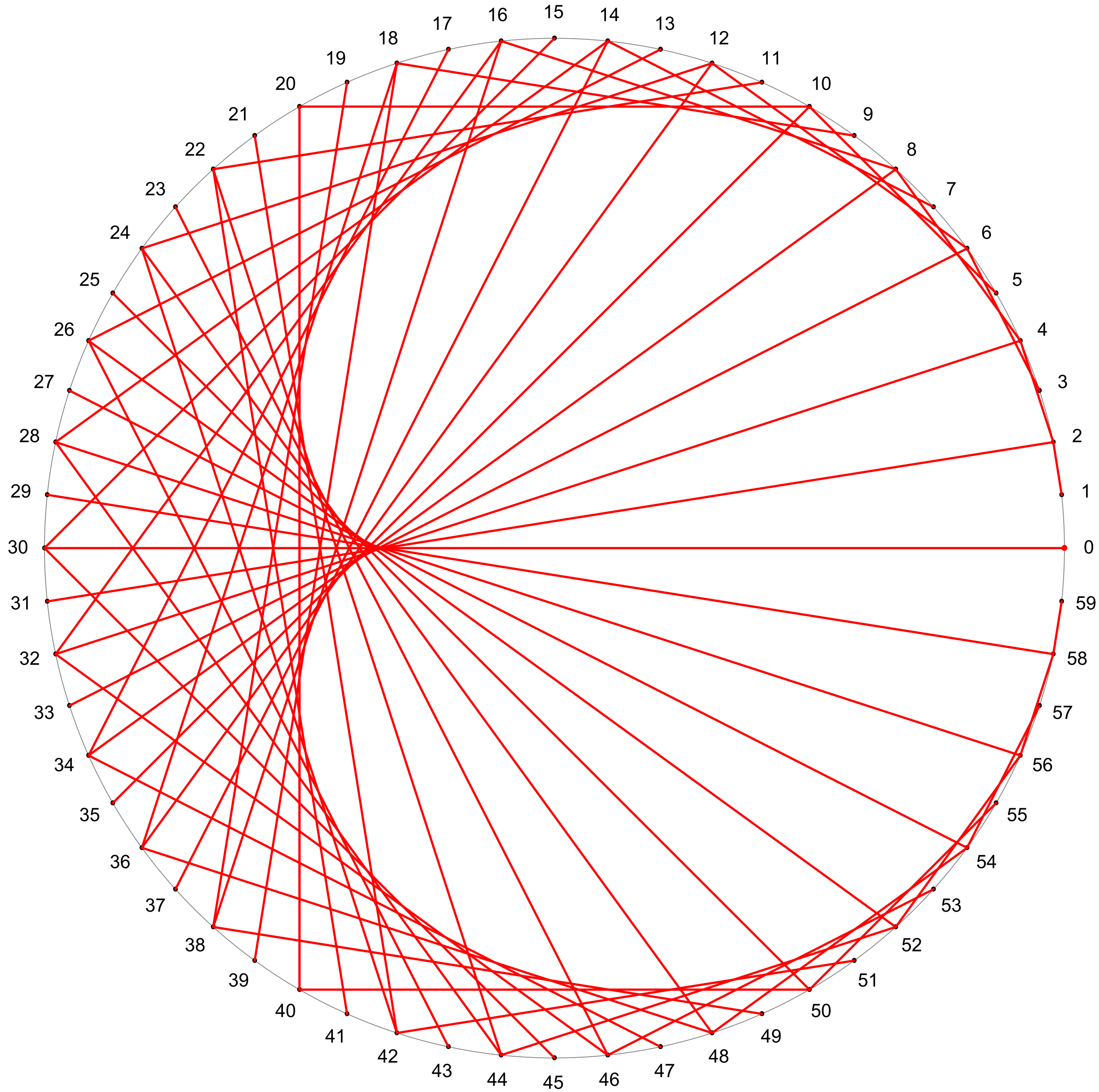
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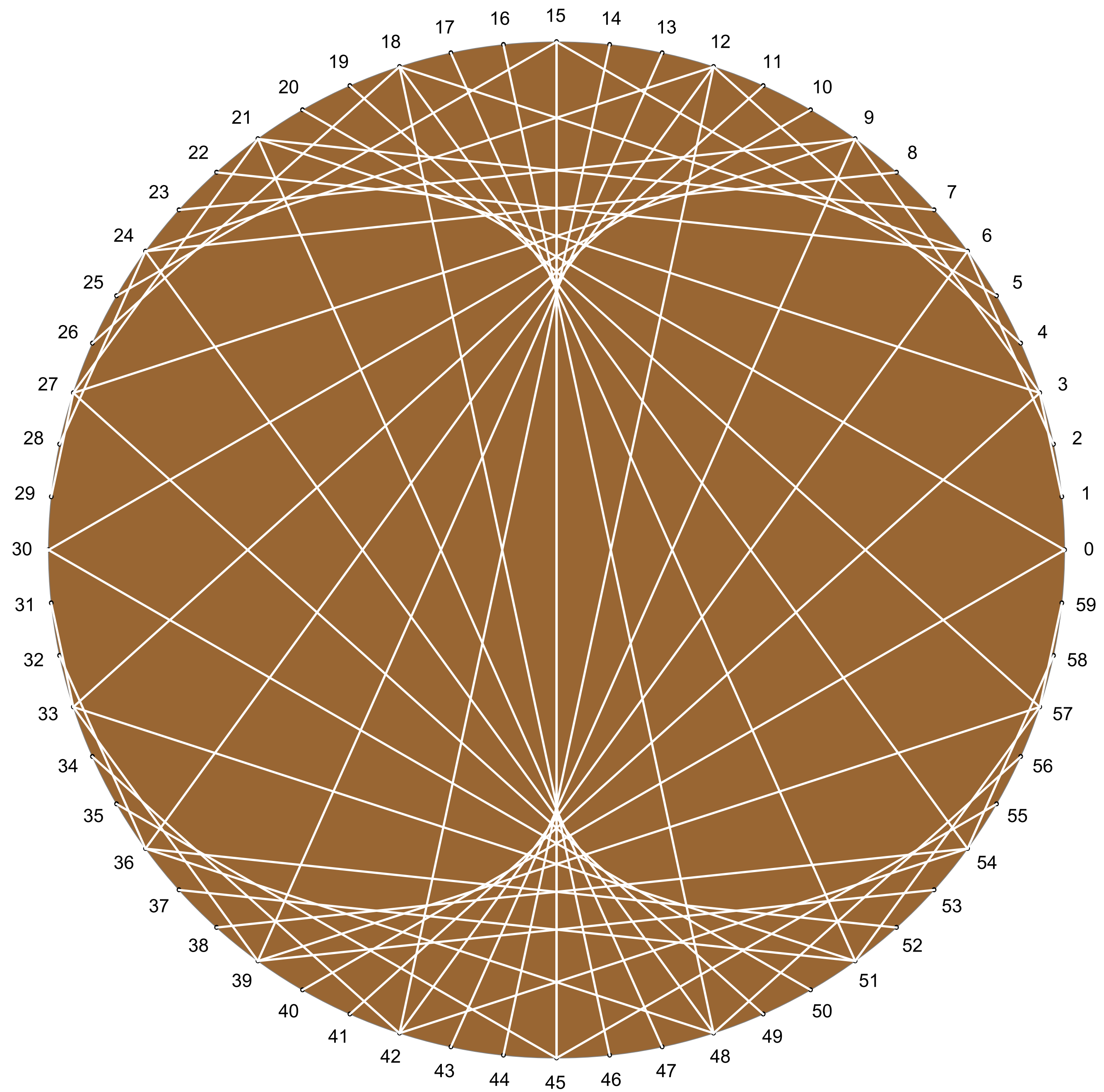
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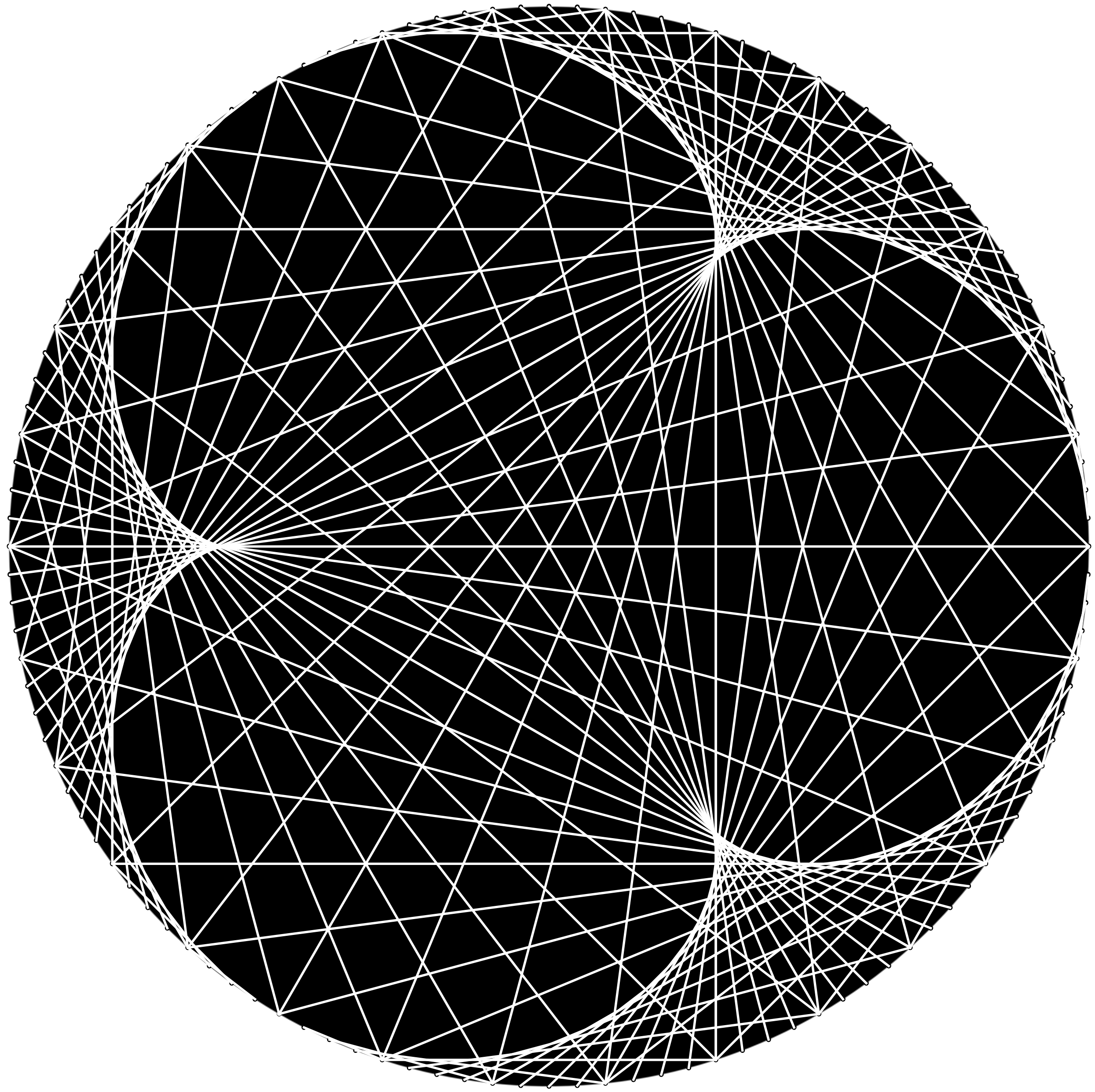
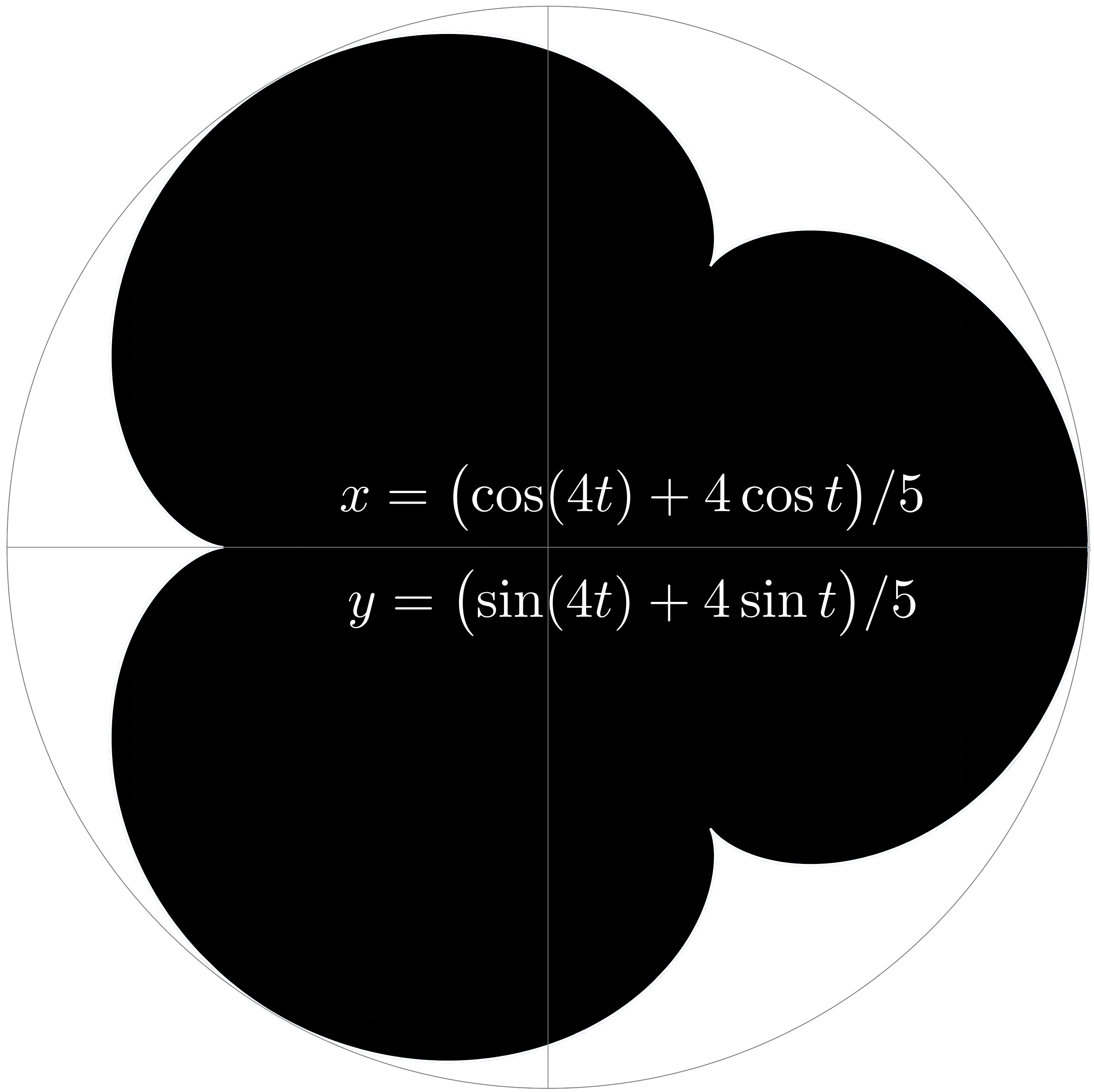
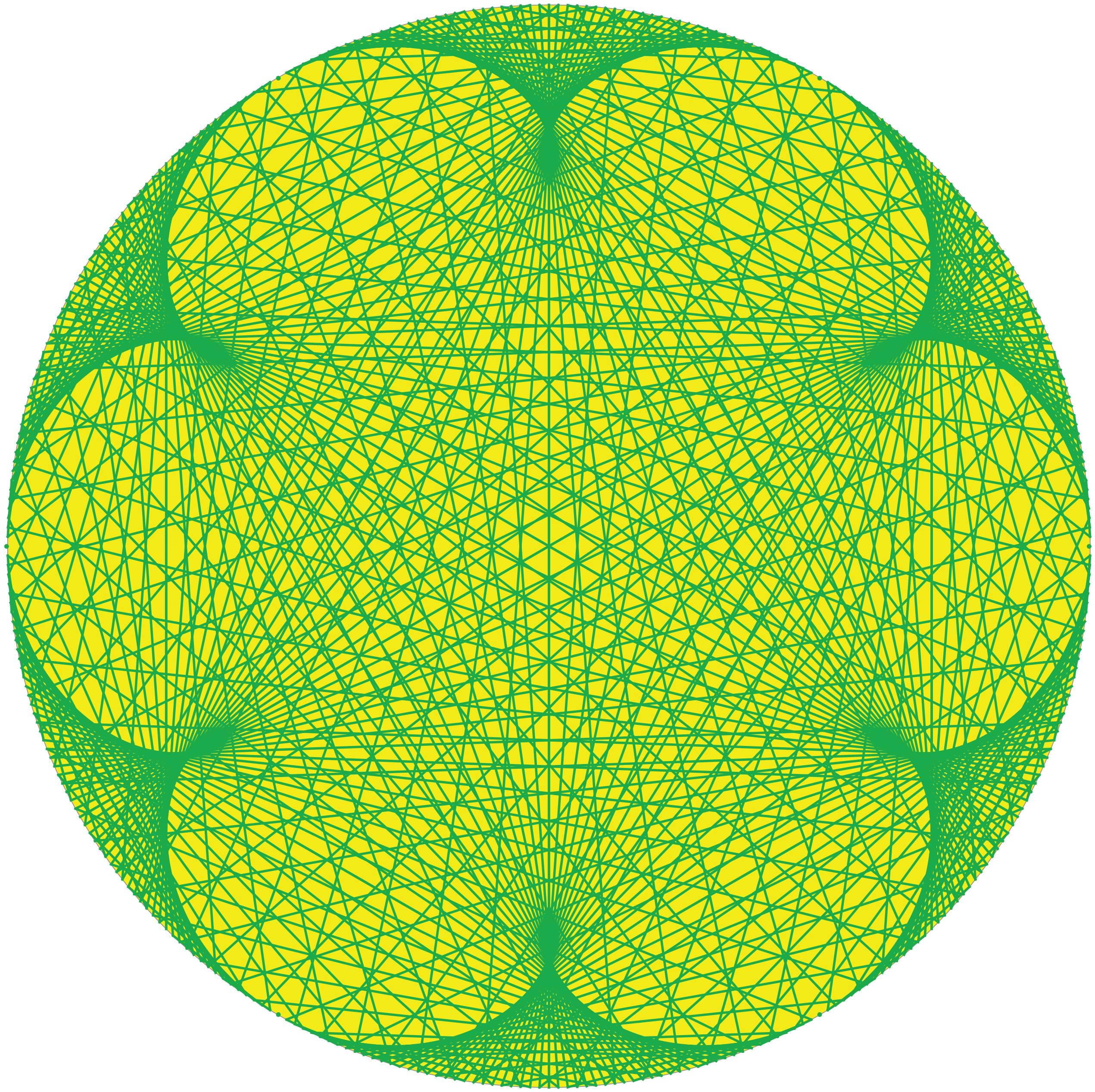
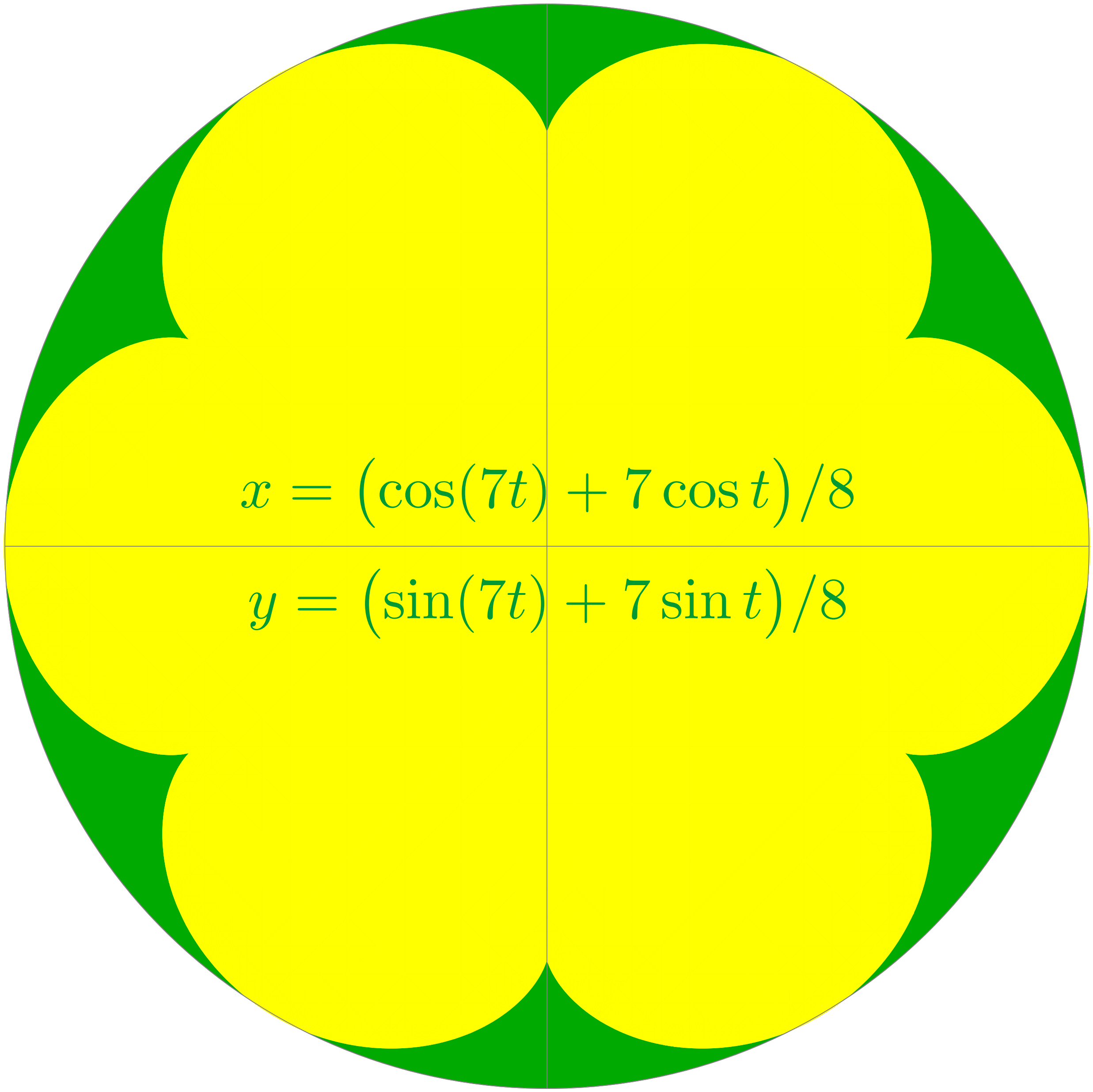



$x = \big(\cos(4t) + 4\cos t\big)/5$
$y = \big(\sin(4t) + 4\sin t\big)/5$




$x = \left(\cos(7t) + 7\cos t\right)/8$
$y = \left(\sin(7t) + 7\sin t\right)/8$

An algebra problem

$$x = \frac{\cos(kt) + k\cos t}{k+1}$$
$$y = \frac{\sin(kt) + k\sin t}{k+1}$$

For each k, we would like to find a real-valued function $g(x, y)$ that is negative exactly on the (compact) region bounded by this curve.

An algebra problem

$$x = \frac{\cos(kt) + k\cos t}{k+1}$$
$$y = \frac{\sin(kt) + k\sin t}{k+1}$$

For each k, we would like to find a real-valued function $g(x, y)$ that is negative exactly on the (compact) region bounded by this curve.

If $g(x, y)$ is continuous, *e.g.*, polynomial, then it will vanish on the curve.

An algebra problem

$$x = \frac{\cos(kt) + k\cos t}{k+1}$$
$$y = \frac{\sin(kt) + k\sin t}{k+1}$$

For each k, we would like to find a real-valued function $g(x, y)$ that is negative exactly on the (compact) region bounded by this curve.

If $g(x, y)$ is continuous, *e.g.*, polynomial, then it will vanish on the curve.

The simplest case is $k = 1$, *i.e.*, a circle:

$$x = \cos t$$
$$y = \sin t$$

The circle

Let $c = \cos t$. Then

$$x = c$$
$$y = \sin t$$

The circle

Let $c = \cos t$. Then

$$x = c$$

$$y = \sin t$$

$$y^2 = \sin^2 t = 1 - c^2$$

The circle

Let $c = \cos t$. Then

$$x = c$$

$$y = \sin t$$

$$y^2 = \sin^2 t = 1 - c^2$$

$$x^2 = c^2$$

The circle

Let $c = \cos t$. Then

$$\begin{aligned} x &= c \\ y &= \sin t \\ y^2 &= \sin^2 t = 1 - c^2 \\ x^2 &= c^2 \end{aligned}$$

So $g(x, y) = -1 + x^2 + y^2 = 0$ describes the circle, and $g(x, y) < 0$ on the interior of the disk.

The cardioid

$$\begin{aligned} 3x &= \cos(2t) + 2\cos t \\ &= \cos^2 t - \sin^2 t + 2\cos t \\ &= 2\cos^2 t - 1 + 2\cos t \\ &= -1 + 2\cos t + 2\cos^2 t \end{aligned}$$

The cardioid

$$
\begin{aligned}
3x &= \cos(2t) + 2\cos t \\
&= \cos^2 t - \sin^2 t + 2\cos t \\
&= 2\cos^2 t - 1 + 2\cos t \\
&= -1 + 2\cos t + 2\cos^2 t
\end{aligned}
$$

$$
\begin{aligned}
3y &= \sin(2t) + 2\sin t \\
&= 2\cos t \sin t + 2\sin t
\end{aligned}
$$

The cardioid

$$
\begin{aligned}
3x &= \cos(2t) + 2\cos t \\
&= \cos^2 t - \sin^2 t + 2\cos t \\
&= 2\cos^2 t - 1 + 2\cos t \\
&= -1 + 2\cos t + 2\cos^2 t
\end{aligned}
$$

$$
\begin{aligned}
3y &= \sin(2t) + 2\sin t \\
&= 2\cos t \sin t + 2\sin t
\end{aligned}
$$

$$
\begin{aligned}
9y^2 &= 4\cos^2 t \sin^2 t + 8\cos t \sin^2 t + 4\sin^2 t \\
&= 4\cos^2 t(1 - \cos^2 t) + 8\cos t(1 - \cos^2 t) + 4(1 - \cos^2 t) \\
&= 4 + 8\cos t - 8\cos^3 t - 4\cos^4 t
\end{aligned}
$$

The cardioid

	1	c	c^2	c^3	c^4
1	1				
$3x$	-1	2	2		
$9x^2$	1	-4		8	4
$9y^2$	4	8		-8	-4

The cardioid

	1	c	c^2	c^3	c^4	c^5	c^6
1	1						
$3x$	-1	2	2				
$9x^2$	1	-4		8	4		
$9y^2$	4	8		-8	-4		
$27x^3$	-1	6	-6	-16	12	24	8
$27xy^2$	-4		24	24	-12	-24	-8

The cardioid

	1	c	c^2	c^3	c^4	c^5	c^6	c^7	c^8
1	1								
$3x$	-1	2	2						
$9x^2$	1	-4		8	4				
$9y^2$	4	8		-8	-4				
$27x^3$	-1	6	-6	-16	12	24	8		
$27xy^2$	-4		24	24	-12	-24	-8		
$81x^4$	1	-8	16	16	-56	-32	64	64	16
$81x^2y^2$	4	-8	-32	24	108	48	-64	-64	-16
$81y^4$	16	64	64	-64	-160	-64	64	64	16

The cardioid

	1	c	c^2	c^3	c^4	c^5	c^6	c^7	c^8
1	1								
$3x$	-1	2	2						
$9x^2$	1	-4		8	4				
$9y^2$	4	8		-8	-4				
$27x^3$	-1	6	-6	-16	12	24	8		
$27xy^2$	-4		24	24	-12	-24	-8		
$81x^4$	1	-8	16	16	-56	-32	64	64	16
$81x^2y^2$	4	-8	-32	24	108	48	-64	-64	-16
$81y^4$	16	64	64	-64	-160	-64	64	64	16

$$\Rightarrow \; -3 - 8 \cdot 3x - 6 \cdot 9(x^2 + y^2) + 81(x^4 + 2x^2y^2 + y^4) = 0$$

The cardioid

	1	c	c^2	c^3	c^4	c^5	c^6	c^7	c^8
1	1								
$3x$	-1	2	2						
$9x^2$	1	-4		8	4				
$9y^2$	4	8		-8	-4				
$27x^3$	-1	6	-6	-16	12	24	8		
$27xy^2$	-4		24	24	-12	-24	-8		
$81x^4$	1	-8	16	16	-56	-32	64	64	16
$81x^2y^2$	4	-8	-32	24	108	48	-64	-64	-16
$81y^4$	16	64	64	-64	-160	-64	64	64	16

$$\Rightarrow\ -3-8\cdot 3x-6\cdot 9(x^2+y^2)+81(x^4+2x^2y^2+y^4)=0$$

So $g(x,y)=-1-8x-18(x^2+y^2)+27(x^2+y^2)^2=0$ describes the cardioid, and $g(x,y)<0$, its interior.

Counting

We had to eventually find a linear combination of monomials in x and y that summed to 0, because up to degree d there are

$$\begin{aligned} &\left(\frac{d}{2}+1\right)^2 && \text{if } d \text{ even;} \\ &\frac{d+1}{2}\left(\frac{d+1}{2}+1\right) && \text{if } d \text{ odd;} \end{aligned}$$

monomials, but only $2d+1$ constraints on the coefficients, coming from the powers of c.

Fewer linear equations

Let $f_1(c)$ and $f_2(c)$ be polynomials of degrees m and n, respectively. For example,

$$f_1(c) = (1 + 3x) - 2c - 2c^2$$
$$f_2(c) = (4 - 9y^2) + 8c - 8c^3 - 4c^4,$$

in which case $m = 2$ and $n = 4$.

Fewer linear equations

Let $f_1(c)$ and $f_2(c)$ be polynomials of degrees m and n, respectively. For example,

$$f_1(c) = (1 + 3x) - 2c - 2c^2$$
$$f_2(c) = (4 - 9y^2) + 8c - 8c^3 - 4c^4,$$

in which case $m = 2$ and $n = 4$.

$f_1(c) = 0$ and $f_2(c) = 0$ share solutions iff they have a common factor, a polynomial $D(c)$ such that $f_i(c) = Q_i(c)D(c)$, whence

$$\frac{f_1(c)}{Q_1(c)} = \frac{f_2(c)}{Q_2(c)},$$

so $0 = Q_2(c)f_1(c) - Q_1(c)f_2(c)$.

Sylvester's matrix

That is, there are $m+n$ scalars $a_0, \ldots, a_{n-1}$ and $b_0, \ldots, b_{m-1}$ such that:

$$
\begin{aligned}
0 &= (a_0 + a_1 c + \cdots + a_{n-1} c^{n-1}) f_1(c) \\
&\quad + (b_0 + b_1 c + \cdots + b_{m-1} c^{m-1}) f_2(c) \\
&= \begin{bmatrix}
1+3x & & & & 4-9y^2 & \\
-2 & 1+3x & & & 8 & 4-9y^2 \\
-2 & -2 & 1+3x & & 0 & 8 \\
 & -2 & -2 & 1+3x & -8 & 0 \\
 & & -2 & -2 & -4 & -8 \\
 & & & -2 & & -4
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ b_0 \\ b_1 \end{bmatrix}
\end{aligned}
$$

Sylvester's matrix

That is, there are $m+n$ scalars $a_0, \ldots, a_{n-1}$ and $b_0, \ldots, b_{m-1}$ such that:

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-2 & -2 & 1+3x & & 0 & 8 \\
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 & & & -2 & & -4
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ b_0 \\ b_1 \end{bmatrix}
\end{aligned}
$$

In general, this $(m+n) \times (m+n)$ matrix is called the Sylvester matrix of f_1 and f_2, $\mathrm{Syl}(f_1, f_2)$.

Sylvester's matrix

This homogeneous system of $m+n$ linear equations in $m+n$ variables has a nontrivial solution iff Sylvester's matrix is singular, *i.e.*, iff the resultant:

$$\mathrm{Res}(f_1, f_2) = \det \mathrm{Syl}(f_1, f_2) = 0.$$

Sylvester's matrix

This homogeneous system of $m+n$ linear equations in $m+n$ variables has a nontrivial solution iff Sylvester's matrix is singular, *i.e.*, iff the resultant:

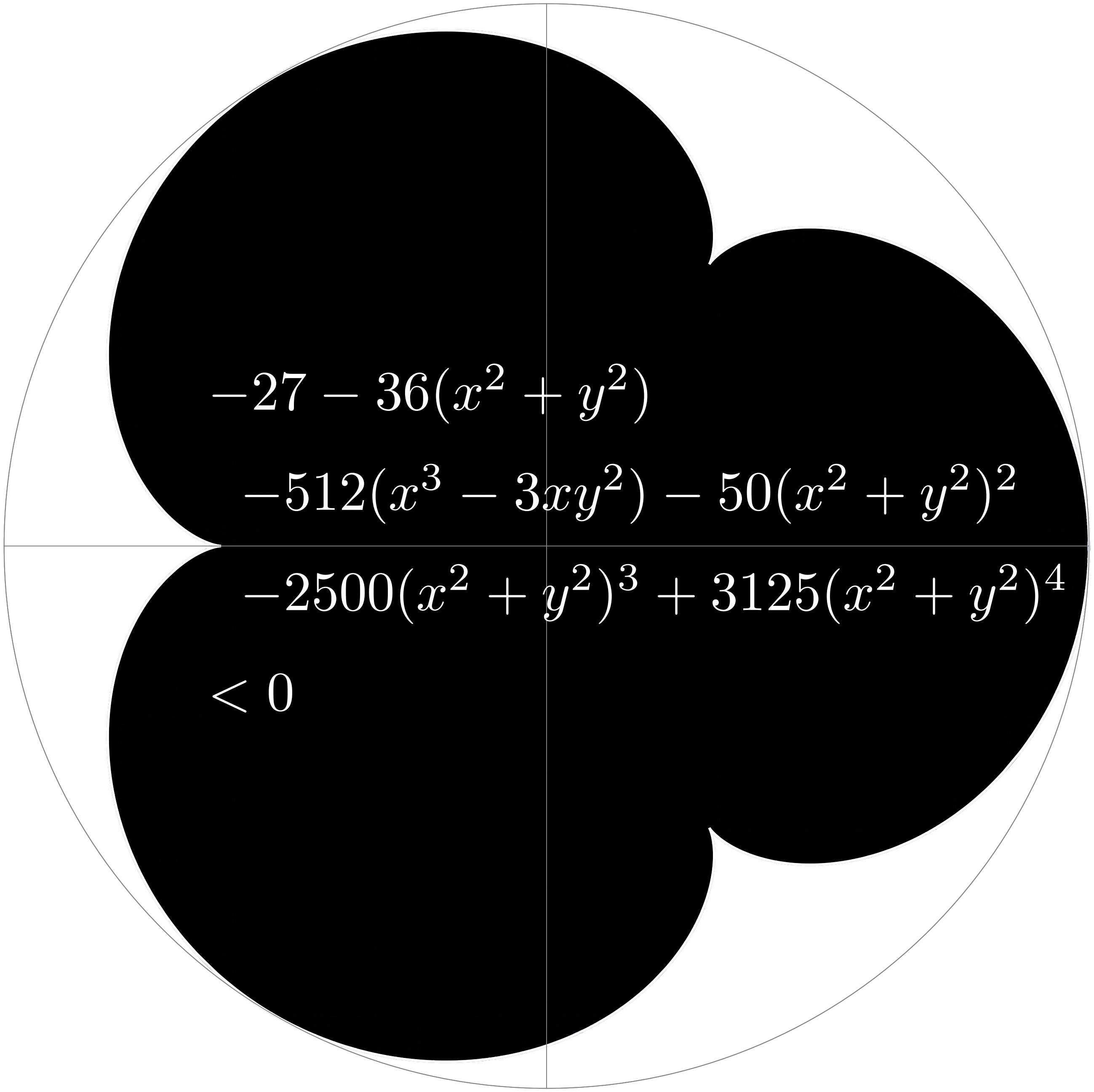
$$\mathrm{Res}(f_1, f_2) = \det \mathrm{Syl}(f_1, f_2) = 0.$$

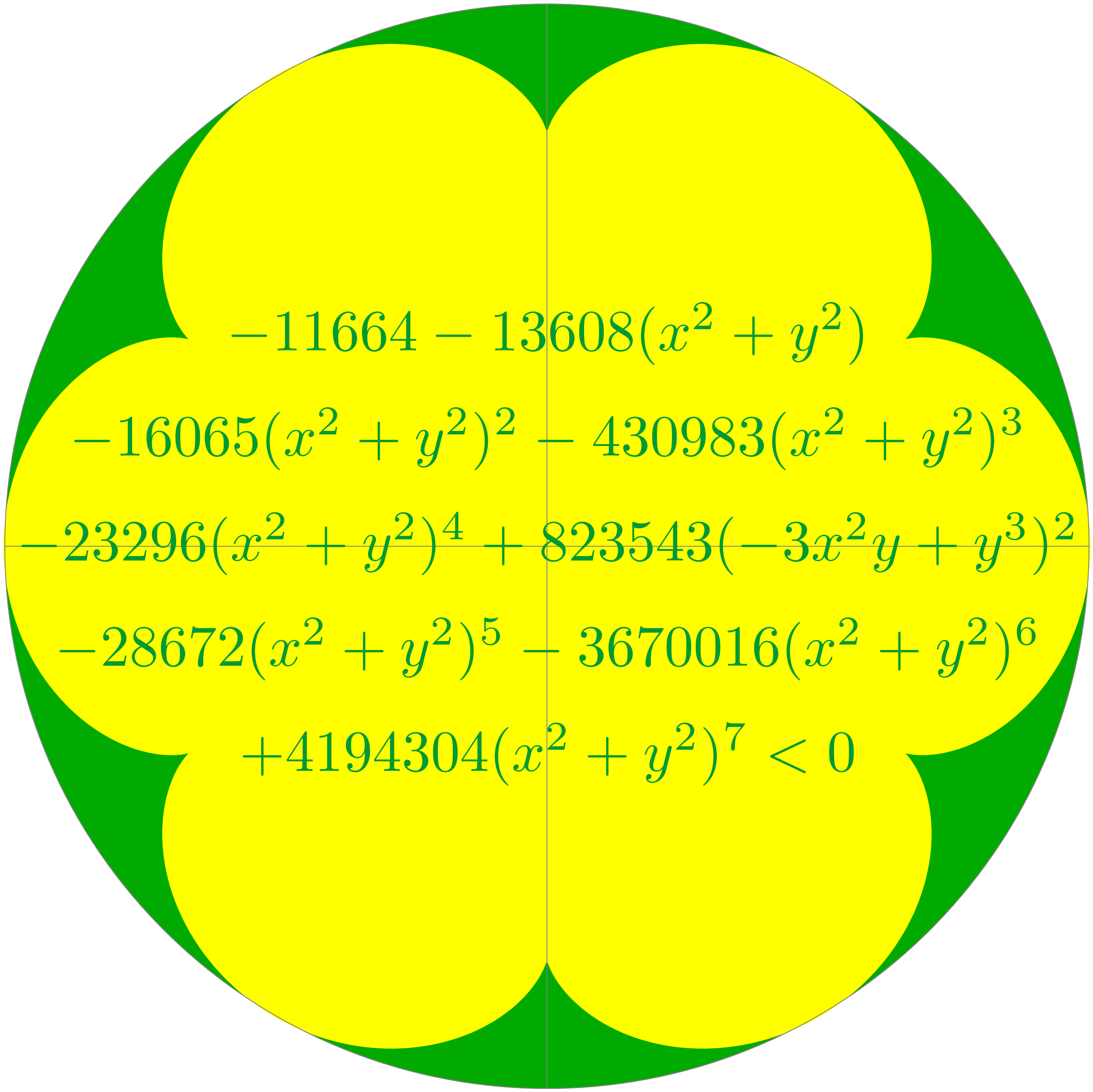
For the f_i coming from the cardioid,

$$0 = \begin{vmatrix} 1+3x & & & & 4-9y^2 & \\ -2 & 1+3x & & & 8 & 4-9y^2 \\ -2 & -2 & 1+3x & & 0 & 8 \\ & -2 & -2 & 1+3x & -8 & 0 \\ & & -2 & -2 & -4 & -8 \\ & & & -2 & & -4 \end{vmatrix}$$

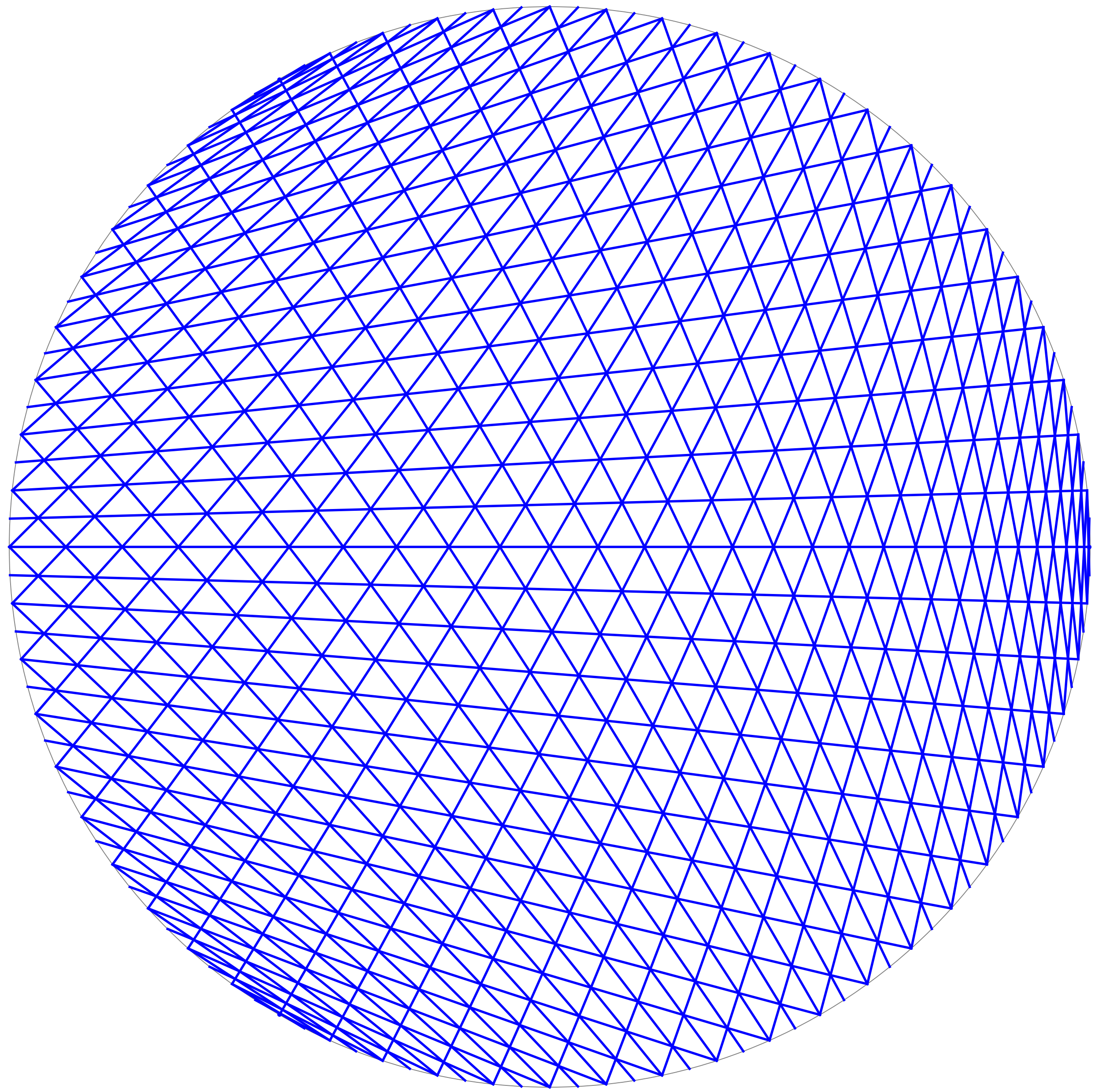
$$= -48 - 384x - 864(x^2+y^2) + 1296(x^4 + 2x^2y^2 + y^4)$$

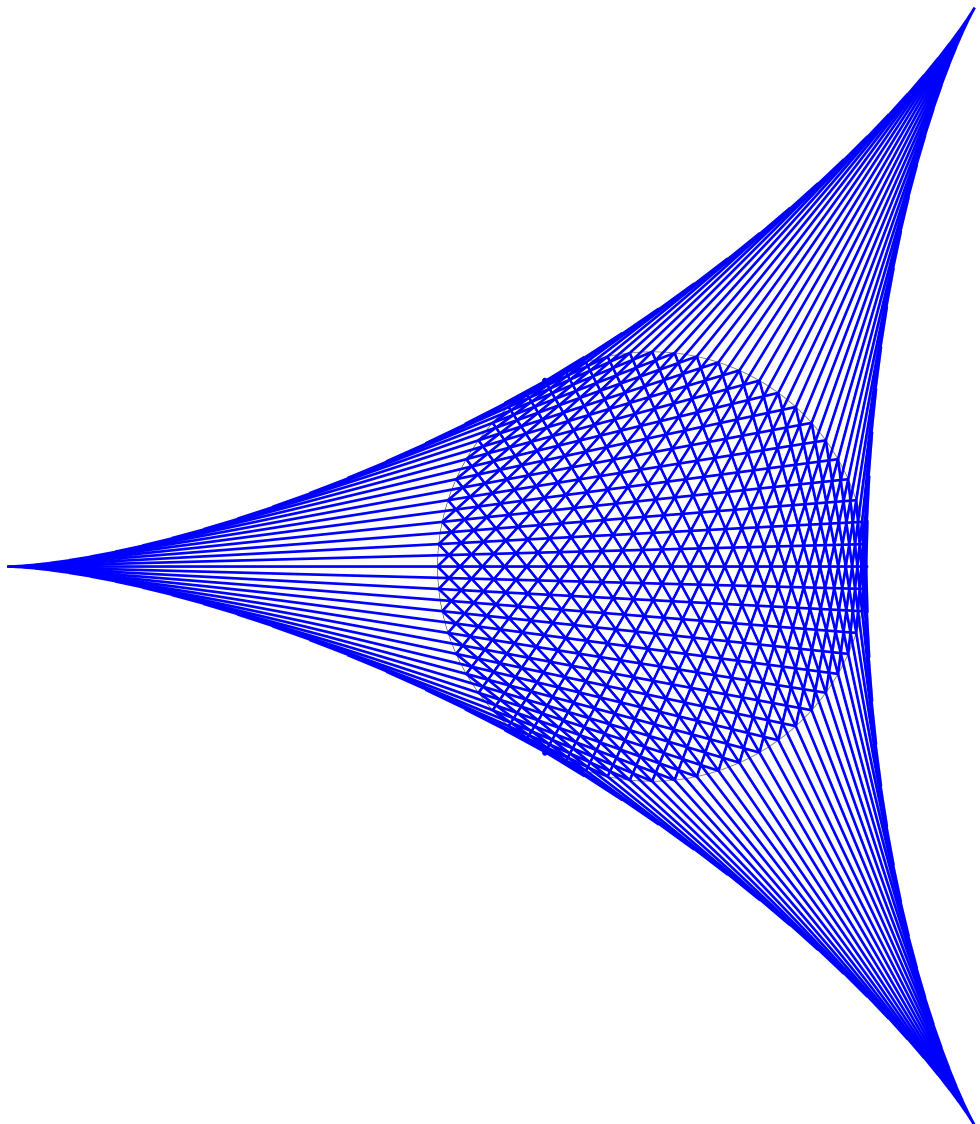
$$= 48\big(-1 - 8x - 18(x^2+y^2) + 27(x^2+y^2)^2\big).$$


$-27 - 36(x^2 + y^2)$
$-512(x^3 - 3xy^2) - 50(x^2 + y^2)^2$
$-2500(x^2 + y^2)^3 + 3125(x^2 + y^2)^4$
< 0



$-11664 - 13608(x^2+y^2)$
$-16065(x^2+y^2)^2 - 430983(x^2+y^2)^3$
$-23296(x^2+y^2)^4 + 823543(-3x^2y+y^3)^2$
$-28672(x^2+y^2)^5 - 3670016(x^2+y^2)^6$
$+4194304(x^2+y^2)^7 < 0$





Tricuspid

When $k = -2$:

$$x = \frac{\cos(-2t) - 2\cos t}{-2+1}$$

$$y = \frac{\sin(-2t) - 2\sin t}{-2+1}$$

Tricuspid

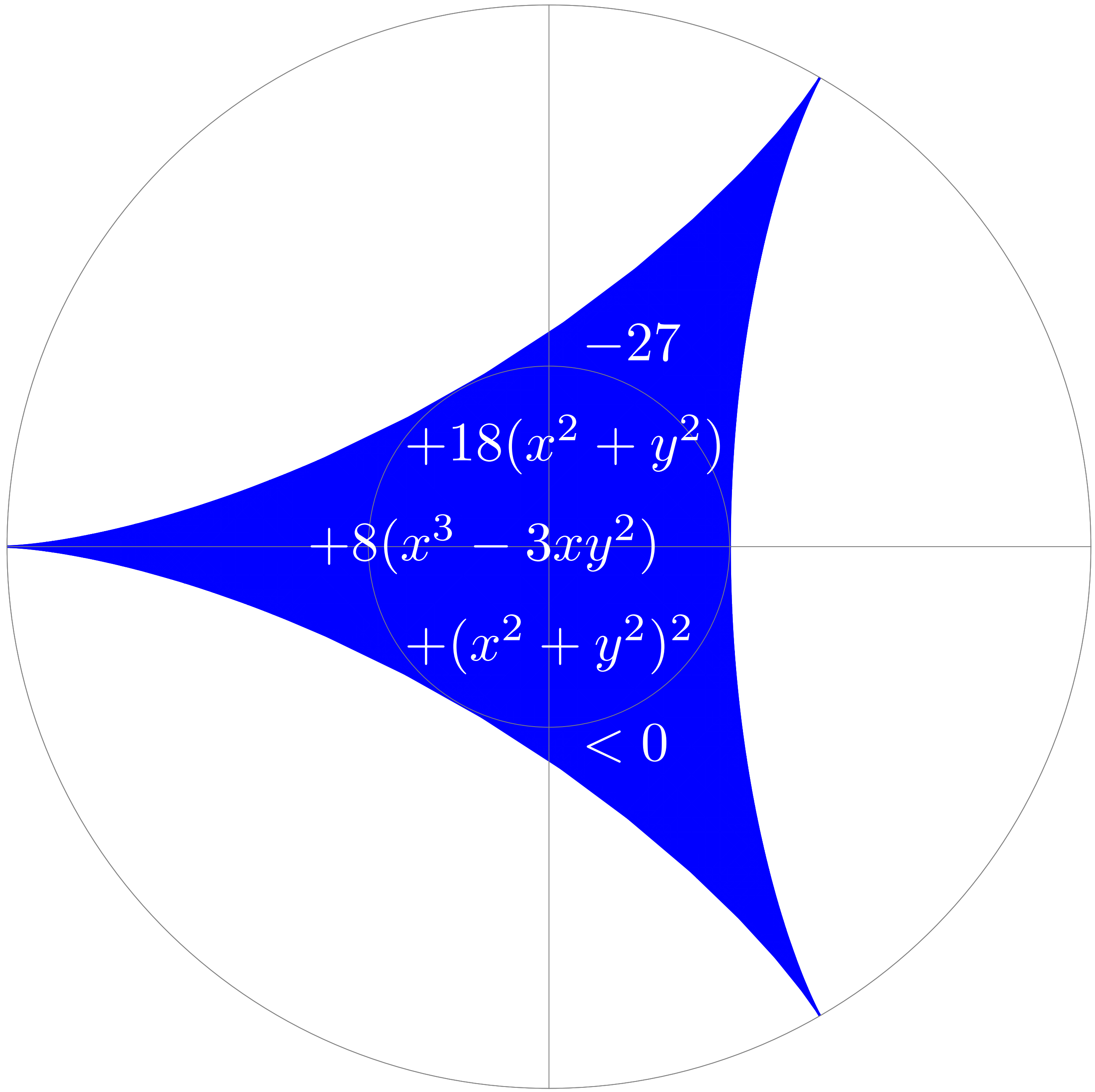
When $k = -2$:

$$x = \frac{\cos(-2t) - 2\cos t}{-2+1}$$

$$y = \frac{\sin(-2t) - 2\sin t}{-2+1}$$

Eliminating t gives the equation for the tricuspid:

$$-27 + 18(x^2 + y^2) + 8(x^3 - 3xy^2) + (x^2 + y^2)^2 = 0.$$



-27
$+18(x^2 + y^2)$
$+8(x^3 - 3xy^2)$
$+(x^2 + y^2)^2$
< 0

Tricuspid

When $k = -2$:

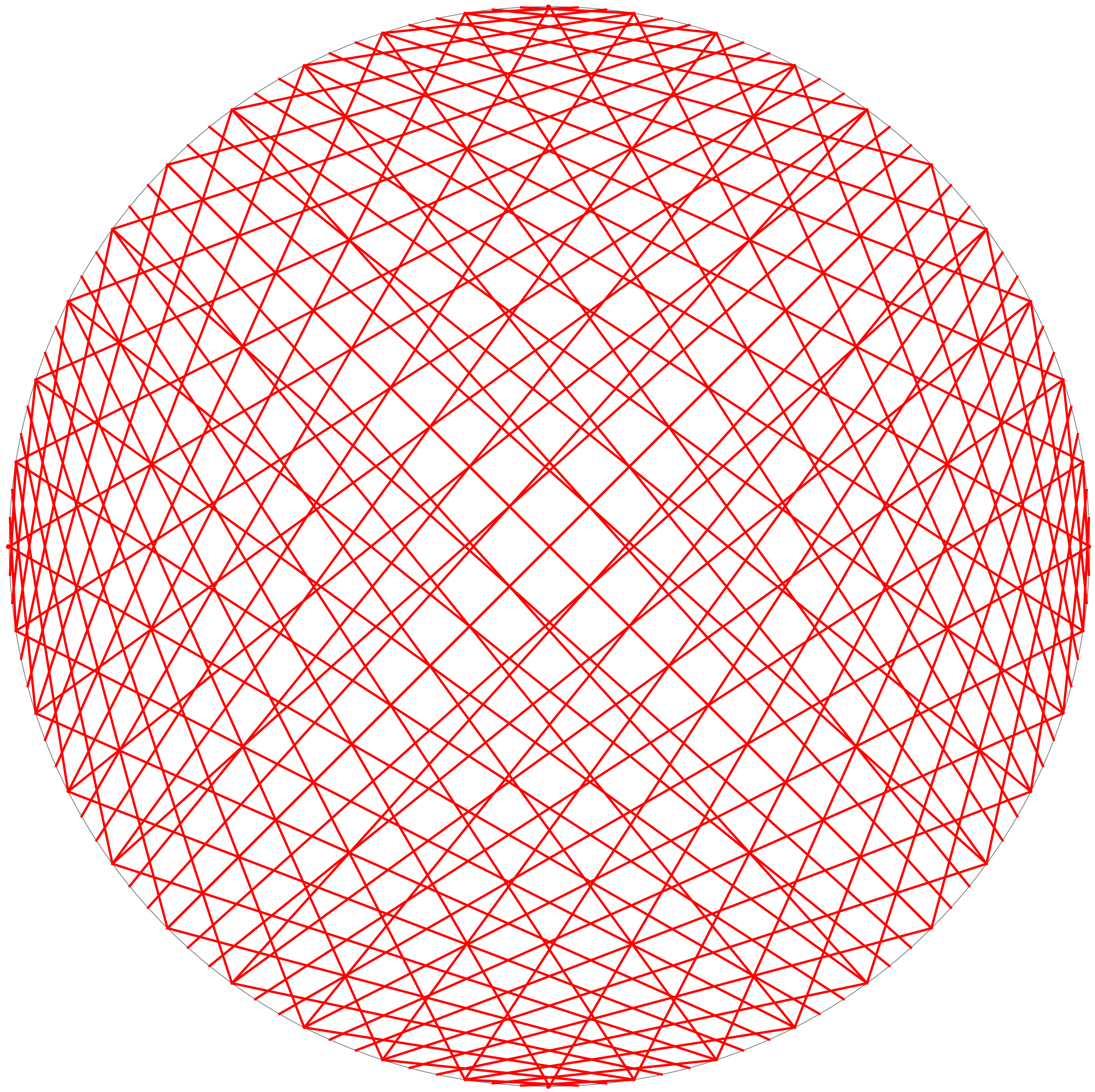
$$x = \frac{\cos(-2t) - 2\cos t}{-2+1}$$
$$y = \frac{\sin(-2t) - 2\sin t}{-2+1}$$

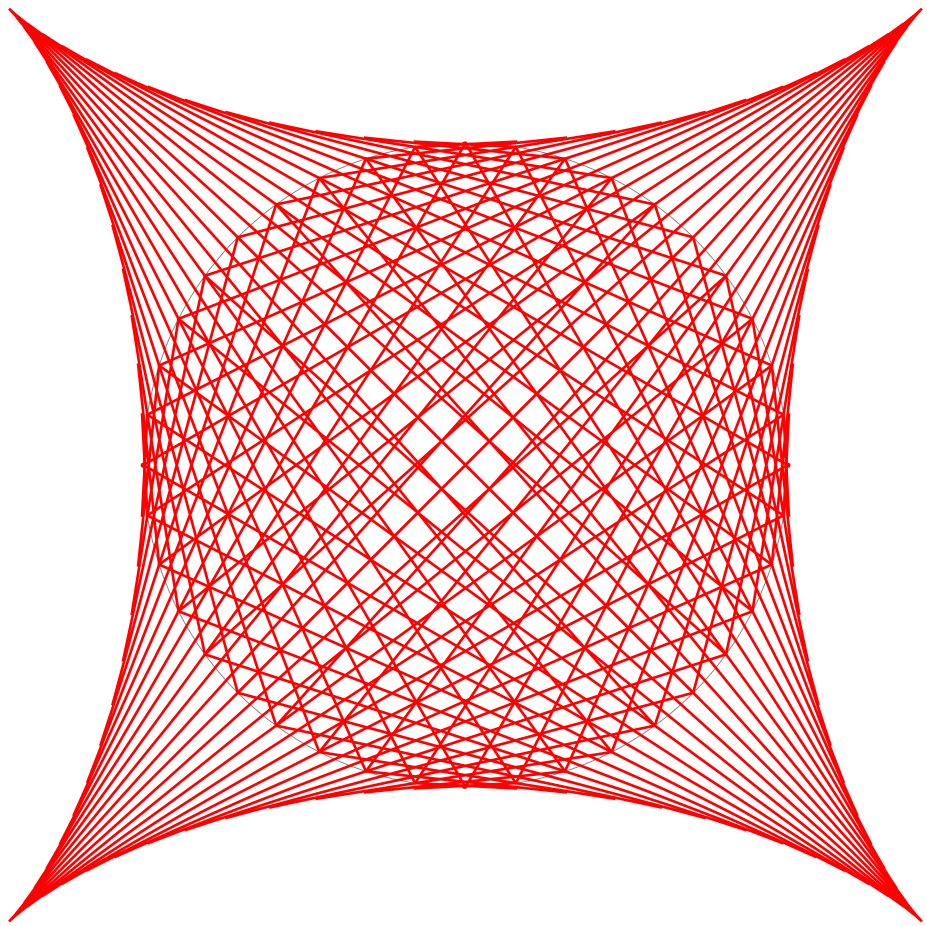
Eliminating t gives the equation for the tricuspid:

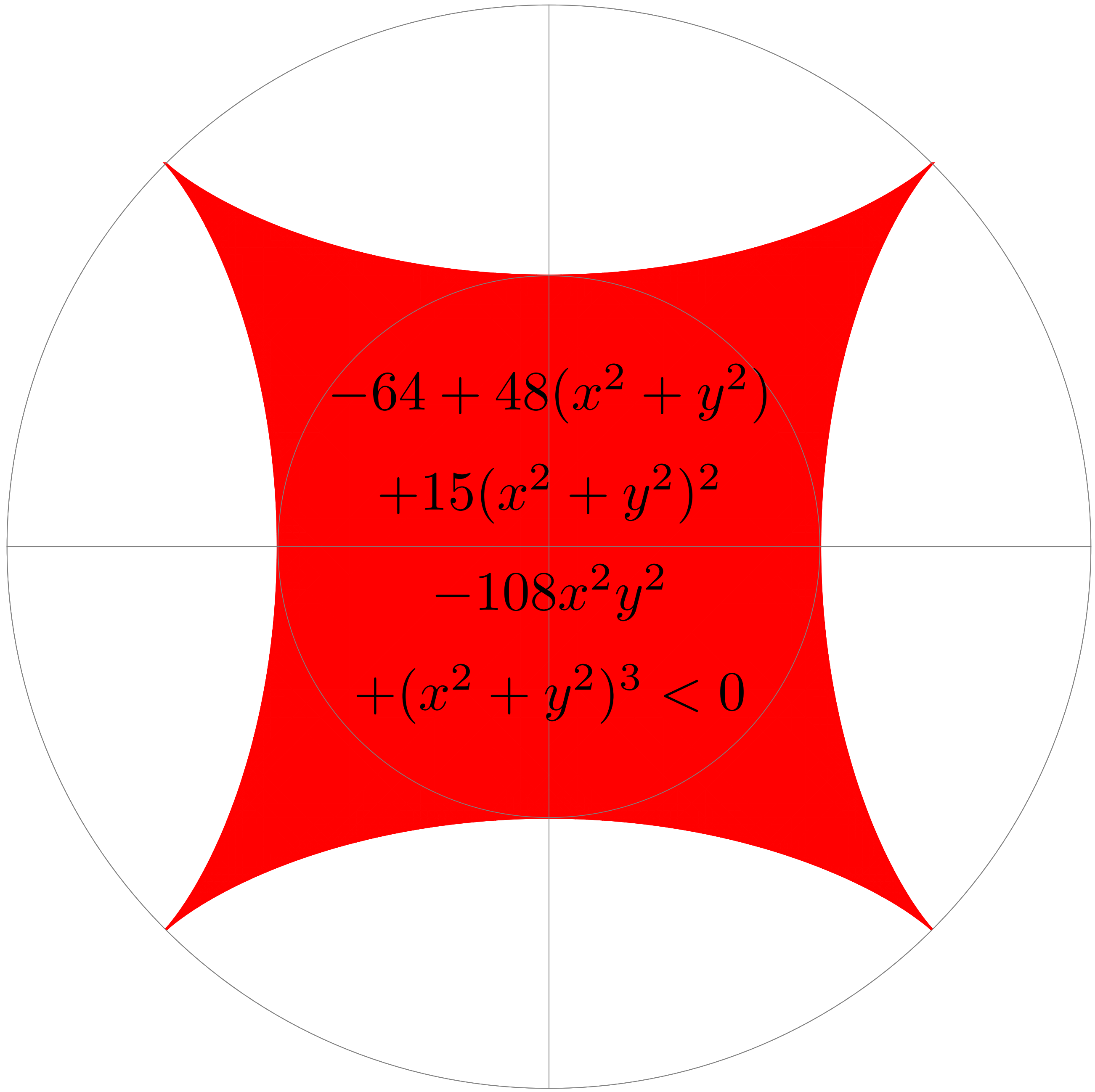
$$-27 + 18(x^2 + y^2) + 8(x^3 - 3xy^2) + (x^2 + y^2)^2 = 0.$$

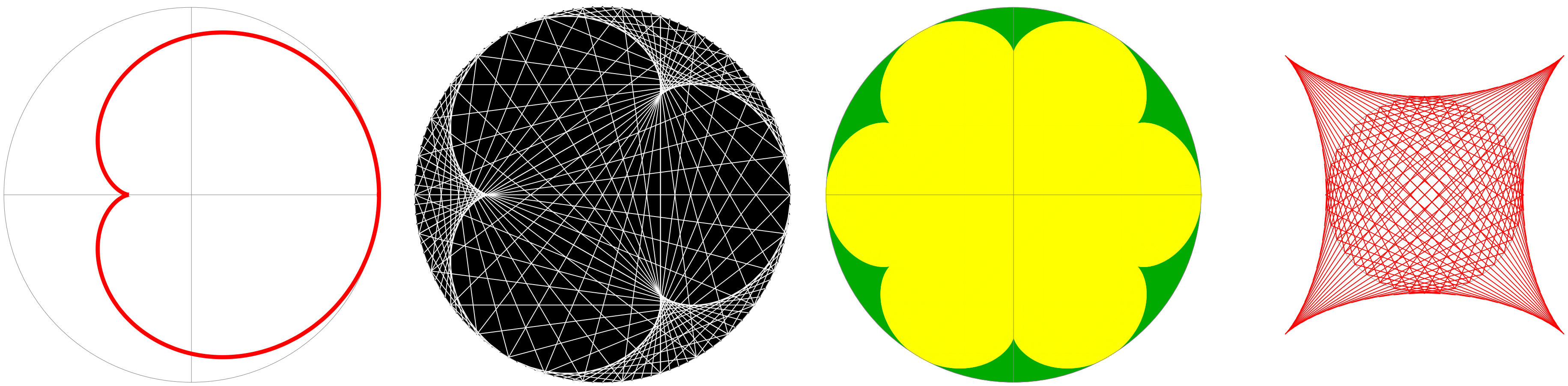
Compare with the equation for the cardioid:

$$-1 - 8x - 18(x^2 + y^2) + 27(x^2 + y^2)^2 = 0.$$






$-64 + 48(x^2 + y^2)$
$+15(x^2 + y^2)^2$
$-108x^2y^2$
$+(x^2 + y^2)^3 < 0$



HAPPY VALENTINE'S DAY!

Inspirations

Ann Marielson's mathematical art, http://www.aisonart.co.uk/.

elementary mathematical problem solving, http://math.ucsd.edu/~dmeyer/teaching/elementary.html.

Grant W. Allen, *Semi-Algebraic Entanglement Consistency Relations: Fundamental and Dynamical*, UCSD Physics Ph.D. thesis (2017).

Poetry

Ludwig Uhland, *Des Goldschmieds Töchterlein* (1815).

George Gordon, Lord Byron, *Childe Harold's Pilgrimage*, Canto the Fourth (1818).

James Joseph Sylvester, *The Laws of Verse, or Principles of Versification Exemplified in Metrical Translations: together with an Annotated Reprint of the Inaugural Presidential Address to the Mathematical and Physical Section of the British Association at Exeter* (1870).

Mathematics

James Joseph Sylvester, "XVIII. On a theory of the syzygetic relations of two rational functions, comprising an application to the theory of Sturm's functions, and that of the greatest algebraical common measure", *Philosophical Transactions of the Royal Society of London* **143** (1853) 407–548.

Luigi Cremona, "*Sur l'hypocycloïde à trois rebroussements*", *Journal für Mathematik* **LXIV** (1865) 101–123.

Raymond Clare Archibald, "The cardioid and tricuspid: quartics with three cusps", *Annals of Mathematics* **4** (1930) 95–104.