QUARTERLY OF APPLIED MATHEMATICS

Vol. IV APRIL, 1946 No. 1

ON THE PROPAGATION OF SMALL DISTURBANCES IN A MOVING COMPRESSIBLE FLUID*

BY

G. F. CARRIER AND F. D. CARLSON

Harvard University

1. Introduction. Although the propagation of sound waves in moving media has received considerable attention [1, · · · , 9],[1] little information is available concerning the propagation of such disturbances in rotational streams or concerning the propagation of transient rotational phenomena. It is shown in the present paper that the wave fronts associated with those parts of a disturbance which are derivable from a potential propagate in a rotational stream according to those laws which they are already known to obey in an irrotational stream. It is further shown that the rotational disturbances drift with the stream rather than propagate relative to the moving fluid.

The analysis consists of an application of conventional perturbation procedures to the Navier-Stokes and continuity equations. The equations so derived are treated according to the theory of characteristics. The results obtained lead to a general expression for the Mach lines of an arbitrary supersonic flow and also suggest a new method of wind tunnel calibration which eliminates the necessity of placing an obstacle in that portion of the stream being calibrated. Finally, predictions are carried out as to the nature of pulses which are formed at a surface and then propagate through a boundary layer into a uniform stream.

2. The equations of motion. In this analysis, we shall consider the propagation of small disturbances in fluid streams which are characterized by three functions of the space coordinates and the time, namely: ρ_0 (the density), p_0 (the pressure), and $\mathbf{v}_0$ (the velocity). No restrictions will be applied to these functions except that they obey the differential equations implying the conservation of momentum, mass, and energy. These equations, known familiarly as the Navier-Stokes, continuity, and energy equations, may be written in the forms:

$$(\mathbf{v}\cdot\text{grad})\mathbf{v} + \partial\mathbf{v}/\partial t + \frac{1}{\rho}\,\text{grad}\,p = \frac{\mu}{\rho}L(\mathbf{v}) \tag{1}$$

$$\text{div}\,\mathbf{v} + \partial\ln\rho/\partial t + \mathbf{v}\cdot\text{grad}\ln\rho = 0. \tag{2}$$

$$dU/dt + pd(\rho^{-1})/dt = Q + \frac{\mu}{\rho}\chi. \tag{3}$$

* Received Jan. 10, 1946.

[1] Numbers in brackets refer to the bibliography.

In the foregoing equations, μ is the viscosity of the fluid, L symbolizes $(\Delta + \frac{1}{3}\,\text{grad div})$ where Δ is the Laplacian operator; Q is the rate of heat accumulation, U is the internal energy, and χ abbreviates the viscous dissipation terms. Discussions of these equations are conveniently found in [7] and [8]. The necessity of manipulating the energy equation in the investigation may be eliminated by using the following assumption. The changes in pressure and density accompanying the disturbance are taken to obey the law

$$p/p_0 = (\rho/\rho_0)^\gamma \tag{3'}$$

where ρ, p, $\mathbf{v}$, characterize the disturbed stream; that is, the disturbance is a phenomenon such that the changes in state from undisturbed to disturbed stream are isentropic. Note that this in no way restricts p_0 and ρ_0. The appendix indicates briefly the fact that while this assumption is by no means rigorously justified, it leads to valid results.

It is convenient at this point to introduce the small parameter ϵ. Although this may be done in a fairly arbitrary manner, we shall define it in the following way in order to avoid any possible ambiguities. Let the initial conditions of any particular problem be such that at time zero, $\rho = \rho_0 + \epsilon\rho_1$, where the maximum value of ρ_1/ρ_0 over the region under consideration is unity. Thus, since we are considering small disturbances, ϵ is a small number compared to unity. Consistent with this notion, we shall write $\rho = \rho_0 + \epsilon\rho_1 + + \epsilon^2\rho_2 + \cdots$, $p = p_0 + \epsilon p_1 + \cdots$, and $\mathbf{v} = \mathbf{v}_0 + \epsilon\mathbf{v}_1 + \cdots$, at time t; and shall require that the series be valid over a range of ϵ. Since disturbances can usually be expected to attenuate, it is certainly reasonable to expect that the series will converge for sufficiently small values of this parameter.

If we now substitute the foregoing forms of ρ, p, and $\mathbf{v}$, into Eqs. (1) and (2), eliminate the p_i (except for p_0) by using Eq. (3′), and collect the coefficients of each power of ϵ, we obtain:

$$\begin{aligned}(\mathbf{v}_0\cdot\text{grad})\mathbf{v}_0 &+ \partial\mathbf{v}_0/\partial t + \frac{1}{\rho_0}\,\text{grad}\, p_0 - \frac{\mu}{\rho_0}L(\mathbf{v}_0)\\ &+ \epsilon\left\{(\mathbf{v}_0\cdot\text{grad})\mathbf{v}_1 + (\mathbf{v}_1\cdot\text{grad})\mathbf{v}_0 + \partial\mathbf{v}_1/\partial t + \text{grad}\left(a_0^2\frac{\rho_1}{\rho_0}\right)\right.\\ &\left. - \frac{\rho_1}{\rho^2}\,\text{grad}\, p_0 + a_0^2\frac{\rho_1}{\rho_0}\,\text{grad}\ln\rho_0 - \frac{\mu}{\rho_0}\left[L(\mathbf{v}_1) + \frac{\rho_1}{\rho_0}L(\mathbf{v}_0)\right]\right\} + \cdots = 0,\end{aligned} \tag{4}$$

and

$$\begin{aligned}\partial\ln\rho_0/\partial t &+ \text{div}\,\mathbf{v}_0 + \mathbf{v}_0\cdot\text{grad}\ln\rho_0\\ &+ \epsilon\left\{\text{div}\,\mathbf{v}_1 + \mathbf{v}_1\cdot\text{grad}\ln\rho_0 + \mathbf{v}_0\cdot\text{grad}\frac{\rho_1}{\rho_0} + \frac{\partial}{\partial t}\left(\frac{\rho_1}{\rho_0}\right)\right\} + \cdots = 0,\end{aligned} \tag{5}$$

where, $a_0 = \gamma p_0/\rho_0$.

When, in equations (4) and (5), the coefficients of ϵ^0 are equated to zero, we find two of the necessary conditions that the functions ρ_0, p_0, $\mathbf{v}_0$, characterize a possible fluid stream. Since these quantities must vanish identically, we may omit them from Eqs. (4) and (5), and divide the remaining equalities through by ϵ. When we allow ϵ to approach zero, we see that the mathematically exact solution to the problem is found by setting the coefficients of ϵ in Eqs. (4) and (5) to zero. Hence, we may expect that the functions ρ_1, $\mathbf{v}_1$, so determined will provide a good first approximation to the behaviour of small amplitude disturbances. This, of course, is the conventional perturbation reasoning.

If we had been willing to assume at the outset a functional relationship $p=p(\rho)$ applicable both to the stream and the disturbance, the perturbation procedure would have been unnecessary. The forthcoming techniques could have been applied directly to Eqs. (1) and (2). However, the solution possesses the desired generality only when we refrain from such restrictions on the nature of the stream. This leads to a choice between working with the energy equation or using the foregoing procedure; the latter seems more convenient. As a matter of fact, some of the results of this analysis differ from those of previous investigators only in that they are obtained for any stream wherein the medium behaves as a continuum rather than one of a very restricted character.

Recalling now that any vector may be expressed as the gradient of a scalar plus the curl of a vector and that

$$(\mathbf{B}\cdot\text{grad})\mathbf{C} + (\mathbf{C}\cdot\text{grad})\mathbf{B} = \text{grad}\,(\mathbf{B}\cdot\mathbf{C}) + (\text{curl}\,\mathbf{B}\times\mathbf{C}) + (\text{curl}\,\mathbf{C}\times\mathbf{B}),$$

one may write

$$\mathbf{v}_1 = \text{grad}\,\phi + \text{curl}\,\mathbf{A},$$

and the differential equations defining ρ_1 and $\mathbf{v}_1$ become

$$\text{grad}\left[(\mathbf{v}_0\cdot\mathbf{v}_1) + a_0^2\frac{\rho_1}{\rho_0} + \frac{\partial\phi}{\partial t}\right] + \frac{\partial}{\partial t}\text{curl}\,\mathbf{A} + a_0^2\frac{\rho_1}{\rho_0}\text{grad}\ln\rho_0$$
$$-\frac{\rho_1}{\rho_0^2}\text{grad}\,p_0 + \omega_1\times\mathbf{v}_0 + \omega_0\times\mathbf{v}_1 = \frac{\mu}{\rho_0}\left[L(\mathbf{v}_1) + \frac{\rho_1}{\rho_0}L(\mathbf{v}_0)\right] \tag{6}$$

and

$$\Delta\phi + \frac{d}{dt}\left(\frac{\rho_1}{\rho_0}\right) + \mathbf{v}_1\cdot\text{grad}\ln\rho_0 = 0 \tag{7}$$

where $\omega_i = \text{curl}\,\mathbf{v}_i$, and $d/dt = [\mathbf{v}_0\cdot\text{grad} + \partial/\partial t]$. When the operation "curl" is performed on Eq. (6), the following equality arises:

$$\frac{\partial\omega_1}{\partial t} = \text{curl}\left\{\frac{\mu}{\rho_0}\left[L(\mathbf{v}_1) + \frac{\rho_1}{\rho_0}L(\mathbf{v}_0)\right] - a_0^2\frac{\rho_1}{\rho_0}\text{grad}\ln\rho_0\right.$$
$$\left.+\frac{\rho_1}{\rho_0^2}\text{grad}\,p_0 - \omega_1\times\mathbf{v}_0 - \omega_0\times\mathbf{v}_1\right\}. \tag{8}$$

It is evident by inspection of Eq. (8) that an identically vanishing initial choice of ω_1 does not imply that this function will vanish for all time, as for example, is the case in an irrotational stream. Thus, we cannot omit ω_1 in this investigation.

It will prove useful to define an (artificial) auxiliary potential ψ in the following manner[2] (Eq. (6) implies the existance of this quantity).

---

[2] This will allow us to eliminate ρ_1/ρ_0 and thus obtain equations in which each unknown has the dimensions of a velocity potential.

$$- \operatorname{grad} \frac{\partial \psi}{\partial t} + \operatorname{curl} \frac{\partial \mathbf{A}}{\partial t} = \frac{\mu}{\rho_0}\left[L(\mathbf{v}_1) + \frac{\rho_1}{\rho_0} L(\mathbf{v}_0)\right] - a_0^2 \frac{\rho_1}{\rho_0} \operatorname{grad} \ln \rho_0$$

$$+ \frac{\rho_1}{\rho_0^2} \operatorname{grad} p_0 - \boldsymbol{\omega}_1 \times \mathbf{v}_0 - \boldsymbol{\omega}_0 \times \mathbf{v}_1 = 0. \tag{9}$$

Upon substitution of Eq. (9) into Eq. (6), the latter becomes

$$\operatorname{grad}\left[\mathbf{v}_0 \cdot \mathbf{v}_1 + a_0^2 \frac{\rho_1}{\rho_0} + \frac{\partial \phi}{\partial t} + \frac{\partial \psi}{\partial t}\right] = 0.$$

This however, we may solve for ρ_1/ρ_0 arbitrarily choosing the "constant of integration" to be zero.[3] We obtain

$$\rho_1/\rho_0 = - a_0^{-2}\left[\mathbf{v}_0 \cdot \operatorname{curl} \mathbf{A} + \frac{d\phi}{dt} + \frac{\partial \psi}{\partial t}\right]. \tag{10}$$

This may be combined with Eq. (7) to give

$$\Delta \phi - \frac{d}{dt}\left(\frac{1}{a_0^2} \frac{d\phi}{dt}\right) + \mathbf{v}_1 \cdot \operatorname{grad} \ln \rho_0 = \frac{d}{dt}\left[\frac{1}{a_0^2}\left(\frac{\partial \psi}{\partial t} + \mathbf{v}_0 \cdot \operatorname{curl} \mathbf{A}\right)\right]. \tag{11}$$

It will be shown directly that the wave front propagation can be derived from Eqs. (8), (9), and (11), provided we can justify the omission of the term $(\mu/\rho_0)L(\mathbf{v}_1)$ from Eqs. (8) and (9). We note, considering Eq. (8), that if only the terms $\partial\boldsymbol{\omega}_1/\partial t$ and curl $(\mu/\rho_0)L(\mathbf{v}_1)$ were non-vanishing, we would have virtually the equation for the conduction of heat, that is $\partial\boldsymbol{\omega}_1/\partial t \cong (\mu/\rho_0)\Delta\boldsymbol{\omega}_1$. The "conduction coefficient" is very small (in air the spreading of vorticity is known to be very slow) so that the term in question may be thought of as one which causes a small dispersive effect. It is to be understood, then, that this effect is to be superimposed on any results which are obtained by treating the equations from which this term has been omitted. With this omission we are now ready to apply the method of characteristics.

Hadamard [1] has shown the following facts concerning second order differential equations which will be useful in the analysis of the foregoing equations. He considers the equation

$$\sum_{i,k=1}^{n} a_{ik} p_{ik} + h = 0$$

in the n independent variables $x_1, \cdots, x_n$, where the a_{ik} and h are functions of the unknown quantity z, the x_i, and the first partial derivatives of z with regard to the x_i; $p_{ik} = \partial^2 z/\partial x_i \partial x_k$. The differential equation which defines the characteristic surfaces (wave fronts) of this equation is given by

$$B = \sum_{i,k=1}^{n-1} a_{ik} P_i P_k - \sum_{i=1}^{n-1} a_{in} P_i + a_{nn} = 0 \tag{12}$$

where $P_i = \partial x_n/\partial x_i$ when the "surface" is written in the form

$$x_n = x_n(x_1, \cdots, x_{n-1}). \tag{12a}$$

[3] Any such (actually time dependent) constant could be absorbed in $\partial\psi/\partial t$ and would contribute nothing to $\mathbf{v}_1$.

Furthermore, let there be s unknown functions $z_1, \cdots, z_s$, and s equations of the form

$$\sum_{i,k} a_{ik}p_{ik} + b_{ik}q_{ik} + \cdots + c_{ik}g_{ik} + h = 0.$$

Here the $a_{ik}, b_{ik}, \cdots$ are respectively the coefficients of the second derivatives of $z_1, z_2, \cdots$. The characteristic surfaces of this system of equations are determined by the relation

$$\begin{vmatrix} B_{11}, & B_{12}, \cdots, & B_{1s} \\ B_{21} & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ B_{s1} & & B_{ss} \end{vmatrix} = 0. \tag{13}$$

The $B_{\alpha\beta}$ are analogous to the quantity B of Eq. (12). In fact, when α takes the values $1, 2, \cdots, s$, B is derived respectively from the first, second, $\cdots$ sth, equations. When β takes these values, the $B_{\alpha\beta}$ are obtained from the $a_{ik}, b_{ik}, \cdots, c_{ik}$, respectively. The present problem deals with the five unknown quantities ϕ, ψ, and the three components of **A**. Eq. (9) is equivalent to four scalar equations[4] if we specify (for example) that ψ and **A** are to be those solutions for which div $\mathbf{A}=0$. This is no restriction since only curl **A** appears in $\mathbf{v}_1$. We may, then, apply the foregoing type of analysis to Eqs. (9) and (11) (with ρ_0/ρ_1 replaced by the expression given in Eq. (10)). In fact, in order to determine the characteristic surfaces which define a motion involving the function ϕ, we need only a brief inspection of Eq. (9). It is evident that no second derivatives of ϕ appear in this equation. Thus, when it is split into its four subdivisions, we find that the four quantities $B_{21}, \cdots, B_{51}$, which appear in the left column of Eq. (13), vanish. This implies that Eq. (13) is satisfied when either B_{11} or the minor associated with this quantity vanishes. Since the vanishing of the former involves only the coefficients of derivatives of ϕ, we may assume that this surface will be associated with the potential type of disturbance. The vanishing of the minor will correspond to the propagation of disturbances of the rotational type.

If we now compute B_{11} using, of course, the a_{ik} of Eq. (11), we find the same wave front equation which was found by Hadamard for the isentropic stream. That is, the time-position correlation of a wave front does not depend on the character of the stream but only (as the following equation will show) on the local values of the quantities u_0, v_0, w_0, and a_0. The first three of these are the components of $\mathbf{v}_0$. This wave front equation, in a form somewhat more convenient for our purposes than Hadamard's, is shown below.

$$\partial y/\partial t + u_0\partial y/\partial x + w_0\partial y/\partial z - v_0 \pm a_0[1 + (\partial y/\partial x)^2 + (\partial y/\partial z)^2]^{1/2} = 0. \tag{14}$$

Eq. (8) indicates that whenever $\boldsymbol{\omega}_0$ and ϕ are each non-vanishing in a given region, a rotational motion $\boldsymbol{\omega}_1$, is generated continuously. This being so, there is always a possible "vorticity wave front" coincident with the wave front associated with ϕ. Hence, if we treat Eq. (8) according to the foregoing method, using the components of curl **A** as the unknown function, we find that the determinant vanishes identically. This is to be expected since the operation which led to Eq. (8) eliminated the higher derivatives of ϕ while retaining the higher derivatives of **A**. Hence, formally, the char-

[4] Any equation of the form curl $\mathbf{M}+$grad $Q=\mathbf{C}$ can be reduced to the forms grad $Q=\mathbf{P}$ and curl $\mathbf{M}=\mathbf{N}$ if one is ingenious enough to separate **C** into the required parts.

acteristics method fails to give the desired information. We note that in this method, however, the only terms which affect the positional nature of the propagation are those containing second derivatives of the unknown functions. If, in Eq. (8), we segragate these terms of the required order we find that they comprise exactly the single term $d\omega_1/dt$. Therefore, in so far as the *position* of the disturbance is concerned, we have $d\omega_1/dt=0$; that is, the time rate of change of vorticity, relative to an observer moving with the particle, vanishes. In other words, the rotational disturbance drifts with the stream instead of propagating relative to it. This statement must be modified, of course, by the results of the diffusion-implying viscous term which was omitted in this analysis.

3. The two-dimensional problem. Since, in general, the functions ρ_0, p_0, $\mathbf{v}_0$ associated with any given stream are not known (even approximately in many cases), it seems of interest to describe a method of wind tunnel calibration based on the foregoing analysis (in particular on Eq. 14). This proposed procedure will be seen to have the advantage that it does not require the insertion of an obstacle into that portion of the stream being calibrated. Let us consider only tunnels which are bounded by the side walls $z=\pm b$, where b is some constant. In this two-dimensional wind tunnel, the flow in the neighborhood of $z=0$ is essentially independent of z. Let us also restrict our consideration to disturbances having reflective symmetry about the plane $z=0$. Then at $z=0$, Eq. (14) reduces to

$$\partial y/\partial t \pm a_0[1+(\partial y/\partial x)^2]^{1/2} - v_0 + u_0\partial y/\partial x = 0. \tag{15}$$

We now have an equation, linear in the three quantities which we wish to determine; u_0, v_0, and a_0. Suppose we generate pulses at several points along some boundary of the stream, say by the use of an electric spark. The wave fronts of these pulses may be observed (photographed) at successive time intervals. The values of $\partial y/\partial x$ and $\partial y/\partial t$ can be determined from the photographs for each pulse throughout the region it traverses. For each point traversed by at least three pulses, we may form three simultaneous equations in the unknown quantities by using these experimentally determined values as coefficients in Eq. (15). Figures 1 to 4 illustrate such photographs of sound pulses in a fairly uniform stream of air. The development of the techniques used in obtaining these Schlieren photographs should be credited to the authors of [9]. In [9] the details of the experimental procedure are explained quite fully.

For an isentropic region of the stream (where the stagnation condition is known) only two pulses are needed since (see [11])

$$a_0^2 = (a_{st})^2 - (\gamma-1)(u_0^2+v_0^2)/2.$$

Finally, for an essentially one-dimensional stream (e.g., a jet or a slowly converging channel) Eq. (15) becomes

$$v_0/a_{st} = \frac{[2(\gamma+1)-2(\gamma-1)\mu^2]^{1/2}-2\mu}{\gamma+1} \tag{15a}$$

where $\mu=a_{st}^{-1}\partial y/\partial t$.

When the stream is supersonic, we may generate stationary disturbances (Mach lines) by placing very small irregularities in the boundaries of the passage. For this case Eq. (15) reduces to

$$\partial y/\partial x = \frac{- u_0 v_0 \pm a_0^2[M^2 - 1]^{1/2}}{a_0^2 - u_0^2} \tag{16}$$

where M, the Mach number, is given by $M^2 = (u_0^2 + v_0^2)/a_0^2$. A mesh of Mach lines proceeding from both edges of the passage give sufficient information to calibrate any stream known to be isentropic with known stagnation condition. For a one-dimensional stream, we have the familiar formula for the Mach angle θ

$$\partial y/\partial x = \tan \theta = (M^2 - 1)^{1/2}. \tag{17}$$

Note that when one wishes to find the characteristics of a *given* supersonic stream, the classical Charpit procedure will always provide solutions for Eq. (16). The equation analogous to Eq. (16) in three dimensions follows directly from Eq. (14) by merely dropping the time dependent term.

4. The effect of boundary layers. A problem of considerable interest arises in connection with the ideas of the foregoing section when we inquire into the effect of the boundary layer on the form of the wave front when the pulse is generated at the surface of a boundary (or obstacle). We shall use the Charpit procedure to solve Eq. (15) for this case, using, of course, an idealized group of values for u_0, v_0, and a_0. The justification of the steps of this procedure are given in [10] and need not be given here, so we shall proceed formally with this method. The solution obtained will be in closed form and is readily verified (as a solution of Eq. 15) by mere substitution.

We characterize the stream by the functions

$$u_0 \equiv 0, \quad v_0 = vx/\delta, \quad \text{for} \quad x \leqq \delta, \quad v_0 = v \quad \text{for} \quad x \geqq \delta, \quad a_0 = a = \text{const.}$$

This simplification of an actual situation is somewhat drastic but useful information results. We first set $\xi = x/\delta$, $\eta = y/\delta$, $\tau = at/\delta$, $M = v/a$, $p = \partial\eta/\partial\xi$, $q = \partial\eta/\partial\tau$, and Eq. (15) in the notation of [10] becomes

$$F(\xi, \eta, \tau, p, q) = q \pm [1 + p^2]^{1/2} - M\xi = 0 \quad \text{for} \quad 0 \leqq \xi \leqq 1$$

or

$$F = q \pm [1 + p^2]^{1/2} - M = 0 \quad \text{for} \quad 1 \leqq \xi. \tag{18}$$

We proceed by considering the associated ordinary differential equations

$$\frac{dp}{F_\xi + pF_\eta} = \frac{dq}{F_\tau + qF_\eta} = \frac{d\eta}{- pF_p - qF_q} = \frac{d\xi}{- F_p} = \frac{d\tau}{- F_q} \tag{19}$$

and choose any solution which expresses p or q in terms of a parameter α. In our case, (formally)

$$\frac{- dp}{M} = dq/0 = - \left(\frac{p^2}{[1 + p^2]^{1/2}} + q\right)^{-1} d\eta = \cdots \tag{19a}$$

and $q = \alpha$ is the required solution. When this is substituted into Eq. (18), we obtain for p in the respective regions

$$p = [(\alpha - M\xi)^2 - 1]^{1/2} \quad \text{and} \quad p = [(\alpha - M)^2 - 1]^{1/2}. \tag{20}$$

We now determine η by the following integration

$$\eta - \beta = \int p d\xi + q d\tau \tag{21}$$

β is chosen to suit the initial and boundary conditions; α is determined by the relation $\partial\eta/\partial\alpha = 0$, and the sign of p must be taken consistent with that portion of the wave front under consideration.

For the initial condition $\eta = 0$ at $\xi = \tau = 0$ (a point source), and for $\xi \leqq 1$, we have for that downstream portion of the wave for which $p \leqq 0$,

$$\eta = \alpha\tau + \beta(\alpha - M\xi) - \beta(\alpha) \tag{22}$$

where

$$\beta(\alpha) = \alpha[\alpha^2 - 1]^{1/2} - \text{arc cosh } \alpha \tag{23}$$

$$\alpha = \frac{M\xi}{2} + \frac{M\tau}{2}\left[1 + \frac{4}{\tau^2 - \xi^2}\right]^{1/2}. \tag{24}$$

This solution is valid when $\xi_0 \leqq \xi \leqq \min(\tau, 1)$;

$$\xi_0 = [\tau^2 + M^{-2}]^{1/2} - 1/M.$$

When $\xi \leqq \xi_0$, $p \geqq 0$, and we must find η by writing

$$\eta(\tau, \xi) = \eta(\tau, \xi_0) + \int_{\xi_0}^{\xi} p d\xi$$

which results in the formula

$$\eta(\tau, \xi) = \alpha\tau - \beta(\alpha) - \beta(\alpha - M\xi) \tag{25}$$

where α and β are still the quantities defined by Eqs. (23) and (24).

When we consider the upstream portion of the wave front, α must be negative. By the foregoing procedure we obtain

$$\eta_1 = \alpha_1\tau - \beta(-\alpha_1) + \beta(M\xi - \alpha_1) \tag{26}$$

$$\alpha_1 = \frac{M\xi}{2} - \frac{M\tau}{2}\left[1 + \frac{4}{\tau^2 - \xi^2}\right]^{1/2} \tag{27}$$

or

$$\eta_1(\xi, \tau) = -\eta(\xi, \tau) + M\xi\tau. \tag{28}$$

This equation is again valid only when $\xi \geqq \xi_0$.

We now consider that portion of the wave exterior to the boundary layer (i.e. $\xi \geqq 1$). We must again extend the integration of Eq. (21), this time into the uniform stream. Using now $p = -[(\alpha - M)^2]^{1/2}$, we obtain for the downstream portion

$$\begin{aligned}\eta &= \alpha\tau + \beta(\alpha - M) - \beta(\alpha) - \int_1^{\xi} [(\alpha - M)^2 - 1]^{1/2} d\xi \\ &= \alpha\tau + \beta(\alpha - M) - \beta(\alpha) - [(\alpha - M)^2 - 1]^{1/2}(\xi - 1).\end{aligned} \tag{29}$$

When $\partial\eta/\partial\alpha$ is equated to zero we find

$$\xi - 1 = \frac{[(\alpha - M)^2 - 1]^{1/2}}{\alpha - M}\left\{\tau + \frac{1}{M}[(\alpha - M)^2 - 1]^{1/2} - \frac{1}{M}[\alpha^2 - 1]^{1/2}\right\}. \tag{30}$$

Eqs. (29) and (30) may be considered as parametric equations for ξ and η in terms of the parameter α for each value of τ. In a similar manner we obtain for the upstream portion

$$\eta_1 = \alpha_1\tau - \beta(-\alpha_1) + \beta(M - \alpha_1) + [(\alpha_1 - M)^2 - 1]^{1/2}(\xi_1 - 1) \tag{31}$$

and

$$\xi_1 - 1 = \frac{[(M - \alpha_1)^2 - 1]^{1/2}}{M - \alpha_1}\left\{\tau + \frac{1}{M}\left[\alpha_1^2 - 1\right]^{1/2} - \frac{1}{M}\left[(M - \alpha_1)^2 - 1\right]^{1/2}\right\}. \tag{32}$$

In Eqs. (29) to (32), $1+M \leqq \alpha < \infty$ and $-1 \geqq \alpha_1 > -\infty$.

The solution is now complete except for surface reflections. Note that the velocity at which the point of contact between boundary and wave front moves is

$$\left.\frac{\partial\eta}{\partial\tau}\right|_{\xi=0} = \alpha(\tau, 0) = \left[1 + \frac{M^2\tau^2}{4}\right]^{1/2}$$

or

$$\frac{\partial y}{\partial t} = a_0\left[1 + \frac{v^2t^2}{4\delta^2}\right]^{1/2}. \tag{33}$$

Figure 5 illustrates the results of the foregoing section for a stream with Mach number .50. The peculiar behavior of the solution for $\tau > \sqrt{5}$ leads us to investigate the rays of the propagation. It has been shown [1, 2, 3] that Eq. (14) implies that the rays be defined by $dx = (la_0 + u_0)dt$, $dy = (ma_0 + v_0)dt$, and $dz = (na_0 + w_0)dt$, where l, m, n, are the direction cosines of the outwardly directed wave front normal with the coordinates axes. In particular, it has been shown that for the conditions prevailing in the boundary layer specified above, the rays are given by (see [9])

$$\eta = -\frac{1}{2M}\left[\text{arc cosh}\,(m_0 - M\xi) + (m_0 + M\xi)\{(m_0 - M\xi)^2 - 1\}^{1/2}\right]_0^{\xi} \tag{34}$$

where m_0 is the value of m^{-1} at the origin. The value of m_0 for which the ray becomes tangent to the line $\xi = 1$ is given by $m_0 = 1 + M$. However, any ray associated with a uniform stream which is directed parallel to that stream will maintain this orientation. Hence, this ray which just becomes tangent to the uniform stream bifurcates into the curves shown in Fig. 5. Note that no ray which is once reflected back into the boundary layer will ever leave this region. This implies that a sizeable portion of the energy of such pulses never leaves the boundary layer. Furthermore, as one can readily see from the few rays plotted in the figure, the reflections occur in such a manner that interference as well as the extremely turbulent conditions in such a region make the observation of waves in such regions improbable. This is borne out in Figs. 1 to 4. One large discrepancy between these pictures and the theory is easily noticed. The limiting upstream ray given by the theory does not agree too well with the evidence of Fig. 4. This is due to the fact that the pulse used in the experiment started as one of finite amplitude (probably traveling initially at a speed of about $2a_0$). This means that at first the wave travels as though it were a small amplitude wave in a slower stream; hence, less distortion from the shape which would be expected with no boundary layer (a family of semi-circles) is the logical result to expect. This, of course, is consistent with the observations. One other remark is essential in view of the initial idealization of the boundary layer. In the actually occurring physical situa-tion, the boundary layer thickens in the direction of flow and thus the sharp break in wave front predicted in this theory is not valid. However, the transition from large velocity gradient to uniform stream occurs in a sufficiently small region so that little energy transfer from the stream part of the wave to the boundary layer is to be ex-pected.

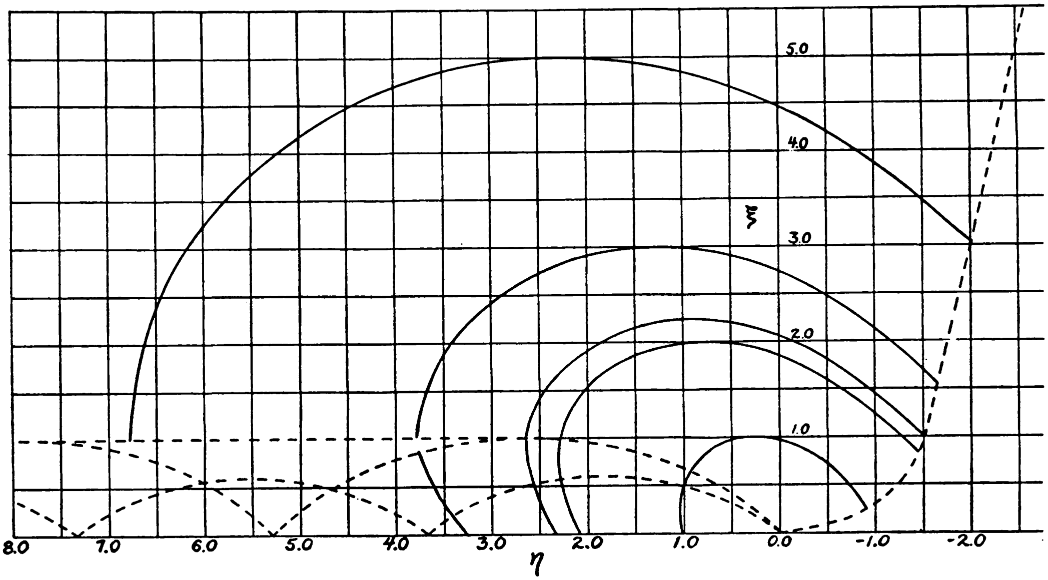

FIG. 5. Predicted wave fronts for the stream defined in section (4). Wave fronts are illustrated for $\tau =$ 2, $\sqrt{5}$, 3, 5. The dotted curves are rays of the propagation. The sound source is at the origin.

If one wishes to account for the large pulse velocity in a mathematical manner he can replace the constant value of a_0 by a function of the time, large at time zero but rapidly approaching the steady value. This, of course, makes the calculations tedious.

5. Intensity distribution. In general, it is difficult to obtain a solution for ϕ from Eq. (11). However, one very interesting solution for the isentropic uniform stream has recently appeared. Rott [6] has shown that when $v_0 = w_0 = 0$, $u_0 = \text{const} = -Ma_0$, Eq. (11) has a solution of the form

$$\phi_1 = \frac{C}{R} \cos \omega \left(t - \frac{Mx + R}{a_0(1 - M^2)} \right), \tag{35}$$

where $R = [x^2 + (1 - M^2)(y^2 + z^2)]^{1/2}$. This solution implies a continuous point source at the origin and, of course, assumes no boundary layer if the plane (say) $y = 0$ is to be a boundary.

The surfaces of constant phase (characteristic surfaces) are given by

$$t - \frac{Mx + R}{a_0(1 - M^2)} = \text{const.} = \lambda \tag{36}$$

a relationship which can be shown to satisfy Eq. (14) for the given stream.

In connection with our pulse problem, we see immediately that we may form a new solution as the integral

$$\phi = \int_0^\infty \frac{C(\omega)}{R} \cos \omega \left(t - \frac{Mx + R}{a_0(1 - M^2)} \right) d\omega. \tag{37}$$

If C is properly chosen, this solution corresponds to a pulse of any desired wave form originating at the origin. The surfaces of constant phase are again given by Eq. (36) and, for a given value of t, are circles with centers at the points $x = Ma_0t$, $y = z = 0$, and radii a_0t.

Suppose we now choose two values of λ (say λ_1 and λ_2) to represent two characteristic surfaces whose separation (along a radial line from the origin) we can call the wave length of the pulse. Then that upstream portion of the pulse at $y = z = 0$ will have a wave length $(\lambda_2 - \lambda_1)a_0(1 - M)$ and the downstream section at $y = z = 0$ will have a wave length $(\lambda_2 - \lambda_1)a_0(1 + M)$. That is, the ratio of the thicknesses of these two extreme portions of the pulse is $(1 + M)/(1 - M)$.

Inspection of Eq. (35) also leads to the conclusion that at these portions of the pulse the amplitudes vary in the ratio $(1 - M)/(1 + M)$. Thus the amplitude *gradients* at these two sections are in the ratio $(1 - M)^2/(1 + M)^2$. Since the density gradient is essentially the quantity observed in the Schlieren optical system, the foregoing constitutes an explanation of the far superior clarity of the wave front definition in the upstream portions of the photographs. This argument has assumed that the pulse started at time zero and has a small thickness to radius ratio at the time of observation.

Specifically, the amplitude ratio (i.e., the ratio of amplitude at any point on the wave front divided into the amplitude at the upstream extremum) is given by

$$\text{Amp. ratio} = 1 + M - \frac{Mx}{a_0t(1 - M)} \cdot \tag{38}$$

APPENDIX

We wish to justify here the use of Eq. (3′). We write the energy equation in the form

$$\frac{R}{\gamma - 1} \frac{dT}{dt} + p \frac{d\rho^{-1}}{dt} = \frac{\alpha}{\rho} \Delta T + \frac{\mu}{\rho} \chi \tag{3}$$

where T is the temperature ($p = \rho RT$) and χ involves a sum of products of the form $(\partial u/\partial y)(\partial v/\partial x)$, $(\partial u/\partial y)^2, \cdots$. When, as in the foregoing work, the perturbation procedure is applied, the terms with coefficient ϵ can be written

$$\frac{1}{\gamma - 1} \frac{d(T/T_0)}{dt} - \frac{d(\rho/\rho_0)}{dt} = \frac{\alpha}{\rho_0 RT_0} \Delta T_1 + \frac{\mu}{\rho_0} \chi_1 + K \tag{3a}$$

where we have grouped the terms not containing derivatives of the unknown functions ρ_1, p_1, $\mathbf{v}_1$, T_1 in K. Such terms can obviously contribute nothing when the characteristics method is applied. In Eq. (3a) $(\gamma - 1)\alpha/R$ is a ratio of specific heat to thermal conductivity, and χ_1 involves products of the form $(\partial u_0/\partial y)(\partial v_1/\partial x)$, $(\partial u_0/\partial y)(\partial u_1/\partial y), \cdots$.

If we can show the third and fourth terms of this equation to be negligible, integration of Eq. (3a) leads to the terms with coefficient ϵ in the expansion of Eq. (3′). Using a dimensional treatment analogous to Prandtl's boundary layer analysis, we define a typical length l for the disturbance (say the wave length, if a continuous wave is considered, or the breadth of a pulse, etc.) and compare first the terms $(\gamma-1)^{-1}\mathbf{v}_0\cdot\text{grad}(T_1/T_0)$ and $(\mu/\rho_0)(\partial u_0/\partial y)(\partial u_1/\partial y)$. We note that $(\gamma-1)^{-1}\mathbf{v}_0\cdot\text{grad}$ $(T_1/T_0)\sim(\gamma-1)^{-1}(T_1/T_0)(|v_0|/l)$ and

$$\frac{\mu}{\rho_0}\frac{\partial u_0}{\partial y}\frac{\partial u_1}{\partial y}\approx\frac{\mu}{a_0\rho_0}\frac{|\mathbf{v}_0|}{\delta}\frac{|\mathbf{v}_1|}{l},$$

so we have the requirement, if the latter term is to be omitted, that $\rho_0\delta a_0/\mu|\mathbf{v}_1|\gg T_0/T_1$. Since T_1/T_0 and $|v_1|/a_0$ are of appreciable magnitude (of order unity) in the same region, this inequality is essentially $\rho_0\delta a_0/\mu\gg 1$. Here, δ is the boundary layer thickness or other typical dimension of the stream. Thus we see that except for very restricted regions the above inequality will hold and the term in question may be omitted. The other terms in χ_1 admit a similar treatment.

We now compare the terms $\partial(T_1/T_0)/\partial t$ and $\alpha\Delta T_1/RT_0\rho_0$. Note that these are exactly the terms which would need to be compared in the still air case; that is, no functions characterizing the stream appear except the slowly varying a_0^2. Using the typical length l and the fact that the propagation occurs at essentially velocity a_0, we obtain $\alpha\Delta T_1/RT_0\rho_0\sim\alpha T_1/RT_0l^2\rho_0\ll\partial(T_1/T_0)/\partial t\sim a_0T/lT_0$ as a necessary condition. That is, the number $\alpha/a_0lR\rho_0\ll 1$. This, of course, is the actual situation* and hence both terms in question may be omitted. This number is essentially the reciprocal of a Prandtl number-Reynolds number product.

Acknowledgment. The authors wish to express their appreciation to Prof. H. W. Emmons for valuable suggestions and for the use of the Harvard high speed wind tunnel.

Bibliography

1. J. Hadamard, *Leçons sur la propagation des ondes*, A. Hermann, Paris 1903, p. 362.
2. H. Bateman, *Sound waves as extremals*, J. Acoust. Soc. Amer. **2**, 468–475 (1930).
3. H. Bateman, Monthly Weather Rev. **46**, 4–11 (1918).
4. E. A. Milne, *Sound waves in the atmosphere*, Phil. Mag. Ser. 6, **42**, 96–114 (1921).
5. Lord Rayleigh, *Collected Papers*, Cambridge Univ. Press, Cambridge, 1920.
6. N. Rott, *Das Feld einer rasch bewegten Schallquelle*, Leemann, Zürich, 1945. 88 pp. (E. T. H., Inst. Aerodyn., No. 9).
7. H. Lamb, *Hydrodynamics*, Dover Publications, New York, 1945, p. 576.
8. A. Vazsonyi, *On rotational gas flows*, Quart. Appl. Math. **3**, 29–37 (1945).
9. J. C. Eisenstein, K. C. Clark, F. D. Carlson, O.S.R.D. Report No. 5369, Aug., 1945.
10. A. Cohen, *Differential equations*, Heath, 1933, p. 263.
11. Th. v. Kármán, *Compressibility effects in aerodynamics*, J. Aero Sci. **8**, 337–356 (1941).

* Numerical substitution indicates for air that $l\gg 10^{-6}$ inches is a sufficient condition.