ON GRAEFFE'S METHOD FOR SOLVING ALGEBRAIC EQUATIONS*

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In the usual descriptions of the methods of solving numerical algebraic equations, Graeffe's method takes a minor place as compared with the methods of Newton, Horner, and others. It is not useful, of course, for correcting a single approximate value, as the other methods are, but has the advantage that no first approximation need be known. A second advantage is that approximations to *all* roots are obtained simultaneously, in contradistinction to the other methods which furnish approximations to *one* root at a time. In spite of this, the computations required by Graeffe's method are not much more laborious than those necessary to obtain an approximation to a single root by one of the other methods if allowance is made for the time necessary to find the first approximation. Yet this slight increase in labor may be the reason that Graeffe's method is somewhat neglected. Its third and perhaps its main advantage is that it also affords a means of finding the complex roots. It is true that by certain other methods, such as that of Newton, an approximation to a complex root can be improved, but obtaining the first approximation is rather difficult in the case of complex roots. A last advantage of Graeffe's method is that it automatically separates roots which are close together, such as $\sqrt{5/2}$ and $3/2$. It is known that Lagrange claimed this same advantage for this method of developing a root into a continued fraction, in contradistinction to Newton's method; Lagrange's method fails, however, in the case of complex roots.

These advantages are well known, though not sufficiently appreciated in practice. But so far as can be seen, it is not known that Graeffe's method also gives the *multiple* (*real or complex*) *roots*, in a manner *essentially simpler* than is generally pointed out in more elaborate descriptions of the procedure. Further, of all the methods it is the only one for solving an equation having *several pairs of complex roots of the same modulus*.

To derive these and other properties, for example the convergence of the process, we discuss the whole method in a somewhat simpler form than is generally used.

Preliminary remarks. The method consists in deriving from an equation

$$x^n + a_1x^{n-1} + \cdots + a_n = 0 \tag{1}$$

with the roots $x_1, x_2, \cdots, x_n$ another equation

$$X^n + A_1X^{n-1} + \cdots + A_n = 0 \tag{2}$$

having the roots $X_i = x_i^p$, where p is a large number (and for practical reasons a power of 2), so that the distinct roots of (2) are widely separated and can thus be easily calculated in the following manner.

Splitting of equation (2).

First case. All the roots of (2) are *positive* and *simple*.

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Then, from

$$X_1 \gg X_2 \gg \cdots \gg X_n,$$

we have

$$\begin{aligned} -A_1 &= \sum X_i \approx X_1 \\ A_2 &= \sum X_i X_j \approx X_1 X_2 \\ -A_3 &= \sum X_i X_j X_k \approx X_1 X_2 X_3 \text{ etc.,} \end{aligned}$$

since the first members of the sums predominate. Thus

$$X_1 \approx -A_1/1, \ X_2 \approx -A_2/A_1, \ X_3 \approx -A_3/A_2, \cdots,$$

that is, equation (2) is split into the n approximate linear equations

$$X + A_1 = 0, \ A_1X + A_2 = 0, \ A_2X + A_3 = 0, \cdots, A_{n-1}X + A_n = 0.$$

Second case. Several of the roots of (2) have the *same absolute value.*

(ia) There is one pair of complex roots, say

$$X_3 = Re^{i\mathrm{A}}, \qquad X_4 = Re^{-i\mathrm{A}},$$

so that

$$\begin{aligned} A_2 &= \sum X_i X_j \approx X_1 X_2 \\ -A_3 &= X_1X_2X_3 + X_1X_2X_4 + \cdots \approx X_1X_2(X_3 + X_4) = X_1X_2 \cdot 2R \cos \mathrm{A} \\ A_4 &\approx X_1X_2X_3X_4 = X_1X_2 \cdot R^2. \end{aligned}$$

Thus X_3, X_4 are approximately the roots of the quadratic equation

$$A_2X^2 + A_3X + A_4 = 0, \tag{3}$$

so that equation (2) is split into the $n-2$ linear equations

$$X + A_1 = 0, \ A_1X + A_2 = 0, \ A_4X + A_5 = 0, \cdots, A_{n-1}X + A_n = 0$$

and the quadratic (3).

(ib) There are two pairs of complex roots, having the same absolute value R,

$$X_{3,4} = Re^{\pm i\mathrm{A}}, \qquad X_{5,6} = Re^{\pm i\mathrm{B}}.$$

Then

$$\begin{aligned} A_2 &\approx X_1X_2 \\ -A_3 &\approx X_1X_2(X_3 + X_4 + X_5 + X_6) = X_1X_2 \cdot 2R(\cos \mathrm{A} + \cos \mathrm{B}) \\ A_4 &\approx X_1X_2(X_3X_4 + \cdots + X_5X_6) = X_1X_2 \cdot 2R^2(1 + 2 \cos \mathrm{A} \cos \mathrm{B}) \\ -A_5 &\approx X_1X_2(X_3X_4X_5 + \cdots + X_4X_5X_6) = X_1X_2 \cdot 2R^3(\cos \mathrm{A} + \cos \mathrm{B}) \\ A_6 &\approx X_1X_2X_3X_4X_5X_6 = X_1X_2 \cdot R^4. \end{aligned}$$

Therefore $X_3, \cdots, X_6$ satisfy the equation

$$A_2X^4 + A_3X^3 + A_4X^2 + A_5X + A_6 = 0. \tag{4}$$

To solve it, let us first compute R from

$$R^4 = A_6/A_2. \tag{5}$$

With this R let

$$X = R \cdot Y, \quad \text{where} \quad Y = e^{i\phi}. \tag{6}$$

Then all the roots of Eq. (4) in Y have the modulus 1, and if Y is a root, then $1/Y$ is also a root, that is, Eq. (4) is "reciprocal":

$$Y^4 + BY^3 + CY^2 + BY + 1 = 0.$$

By

$$Y + Y^{-1} = 2 \cos \phi = Z \tag{7}$$

it becomes a quadratic equation in Z.

(ic) There are μ distinct pairs of complex roots having the same modulus R.

Then in a manner similar to (ib) above an equation M of degree 2μ is split off. The modulus R is obtained from the equation

$$R^{2\mu} = \text{quotient of the last coefficient to the first coefficient } A_2 \text{ of the equation } M. \tag{8}$$

To compute the arguments $\mathrm{A}, \mathrm{B}, \cdots$ of the μ pairs of roots, we again set $X = RY$ as in (6), and divide by the leading coefficient $A_2R^{2\mu}$. The new equation in Y is again reciprocal:

$$(Y^{2\mu} + 1) + B(Y^{2\mu-1} + Y) + C(Y^{2\mu-2} + Y^2) + \cdots + NY^\mu = 0. \tag{9}$$

By substituting (7) into (9) it can be seen that (9) is an equation in Z of degree μ having μ real roots of $Z<2$ which can be found by Graeffe's method. Complications cannot arise since all the roots are distinct. The transformed equation therefore will break up into a system of μ linear equations.

(iia) There are three roots with the same modulus R, one positive and the others complex, say

$$X_3 = R, \qquad X_{4,5} = Re^{\pm i\mathrm{A}}.$$

Then

$$\begin{aligned} A_2 &\approx X_1X_2 \\ -A_3 &\approx X_1X_2(X_3 + X_4 + X_5 + X_6) = X_1X_2 \cdot R(1 + 2\cos \mathrm{A}) \\ A_4 &\approx X_1X_2(X_3X_4 + X_3X_5 + X_4X_5) = X_1X_2 \cdot R^2(1 + 2\cos \mathrm{A}) \\ -A_5 &\approx X_1X_2X_3X_4X_5 = X_1X_2 \cdot R^3, \end{aligned}$$

so that X_3, X_4, X_5 are approximately the roots of the cubic equation

$$A_2X^3 + A_3X^2 + A_4X + A_5 = 0. \tag{10}$$

The value of R is obtained from

$$R^3 = -A_5/A_2,$$

or more simply from

$$R = -A_4/A_3. \tag{11}$$

Thus equation (2) is split into $n-3$ linear equations and the cubic equation (10). The last equation can be broken up into the linear equation (11), that is $A_3X + A_4 = 0$, and a quadratic equation.

(iib) There are μ pairs of complex roots and one positive root, all of the modulus R.

An equation M of degree $2\mu+1$ results and from that a linear equation L is again split off. It consists of the two middle terms of M, that is

$$L: \quad A_{\mu+2}X + A_{\mu+3} = 0 \tag{12}$$

with the solution R.

To find the arguments of the complex roots, we again set $X=RY$ and obtain a reciprocal equation in Y of degree $2\mu+1$. It has the solution $Y=1$, so that by dividing it by $Y-1$ there results a reciprocal equation of degree μ in Z that may be solved by Graeffe's method, yielding μ pairs of complex roots.

(iii) There are *multiple roots.*

Let X_2 have the multiplicity ν. Then equation M, mentioned above in (ic) and (iib), will be divisible by $(X-X_2)^\nu=(X-R)^\nu$, if X_2 is real, and by $[(X-X_2)(X-\overline{X}_2)]^\nu$ $=(X^2-2R\cos \mathrm{A}X+R^2)^\nu$, if X_2 is complex. The reciprocal equation in Y is then divisible by $(Y-1)^\nu$ or $(Y^2-2\cos \mathrm{A}Y+1)^\nu$, respectively.

Note. It is not always possible to eliminate the multiple roots of (2) by eliminating at once the multiple roots of (1), for distinct roots of (1) may, by the successive squarings, give the same roots of (2).

In summary we can say, if the absolute values of the roots of (2) are partly equal, partly different, then Eq. (2) is split up into several approximate equations M_i of lower degrees. The degree of an M is equal to the number of roots having the same modulus R. Thus *there are as many equations M as there are distinct moduli.* To every simple root there corresponds a linear equation.

Determination of the equations M_i. It is well known that by squaring the roots of (1) a series of equations $G_1, G_2, G_3, \cdots$ having the roots $x_i^2, x_i^4, x_i^8, \cdots$ results.

The *problem* now is to decide which equation G first breaks up into equations M_i, and what these M_i are.

We have seen that if the equation M has m roots with equal moduli R, then the absolute member of the normalized M (that is an M whose leading coefficient is 1) is equal to $\pm R^m$. In the following transformed equation it will be equal to $\pm R^{2m}$, that is, from a certain equation G_k the absolute member of a (normalized) M is (to the required degree of accuracy) squared when passing to the following transformed equation. This or a similar relation does not hold for the other coefficients of M, for they involve cos A. Since cos A changes to cos 2A in the following equations, the coefficients not only irregularly change their quantity, but often their signs too.

To find the various M_i into which an equation G_k is eventually split up, we must therefore seek only those coefficients that are squared when passing to the next equation G. That is, we begin with the leading coefficient 1 of all G_k and choose the first member A_i that by the last root-squaring is itself squared. The coefficients from 1 to A_i form the first equation M_1. The next equation M_2 has the leading coefficient A_i. Since A_i is itself squared when going a step further, it is unnecessary to normalize the supposed equation M_2, but we may choose immediately the first coefficient A_j after A_i that is squared by the root-squaring. Equation M_2 now has the coefficients lying between A_i and A_j. If A_k is the next coefficient after A_j that is squared by root-squaring, M_3 extends from A_j to A_k, and so on.

It is not necessary that the same equation G_k yield all the various M_i. The higher the degree of an M is, the later the mentioned quality of the extreme coefficients will generally appear. However, from the stage where an M is split off, it is no longer necessary to keep it during the further calculation. It is sufficient to treat only the rest of the G_k that remain after cancelling the M_i.

Resolution of an equation M. Every M of degree m must be solved separately, in the following manner.

(i) Normalize M to M'.

(ii) Find the modulus R of all roots of M from (8), that is, in the normalized M', from

$$R^m = -\text{absolute member of } M' \text{ if } m \text{ is even,}$$

and from the linear equation L (12), if m is odd

(iii) With this R set $X=RY$, where $Y=e^{i\phi}$, into M' and normalize again to M''. This new equation in Y is reciprocal.

(iv) If this is possible, divide M'' by $Y-1$ (that is, is $Y=1$ a root?), and repeat this division as often as possible. If this is possible s times, then M has the root R of multiplicity s. If m is odd, s is at least 1.

(v) Form the quotient $Q=M''/(Y-1)^s$; this is also a reciprocal polynomial.

(vi) In $Q=0$ set $Z=Y+Y^{-1}$. Then $Q=0$ is transformed into an equation $Q'=0$ in Z of degree $(m-s)/2$. The equation $Q'=0$ has all its roots real.

(vii) $\cos\phi=Z/2$ yields $(m-s)/2$ values of ϕ and therefore $m-s$ values of X: $X=Re^{\pm i\phi}$, and this together with the root $X=R$ of multiplicity s yields the complete system of roots of M.

Solution of equation (1). To every root X_i of M there corresponds one root x_i of (1). Since $X_i=x_i^p$, every X_i yields p tentative values of x_i from which the right root must be chosen. We do not, of course, calculate with the complex values of x_i, but take only the real component of Eq. (1) that is:

$$R^n \cos n\phi + R^{n-1}a_1 \cos (n-1)\phi + R^{n-2}a_2 \cos (n-2)\phi + \cdots + a_n = 0, \quad (13)$$

where ϕ is given by

$$\phi = (2q\pi + \phi_1)/p, \qquad q = 0, 1, 2, \cdots, p-1. \quad (14)$$

After having found the first ϕ satisfying (13), we stop the calculation with this ϕ_i.

If equation (1) is not too complicated, that is, if the M's have a low degree, the process can be abbreviated in the usual manner.

In all other cases the process of finding the complex roots can always be abbreviated in the following manner. In the chain of the transformed equations G_k we go back to the first equation $G_m(m\ll k)$, where M starts to split off, that is, where the double products have no more influence on the first (or second) decimal place of the corner-coefficients of M. This equation M is solved to one or two decimal places only. The root of Eq. (1) is now—to one or two decimal places—to be selected from a group of 2^m members, instead of from the 2^k members of (14). This decreases the work involved.

In addition, the process is abbreviated by the fact that not all 2^k equations of (13) have to be computed, since the members of the majority of them differ from those of the others only in the sign. An example will make this clearer.

Nevertheless this part of the labor is the most tedious of the whole process if the first transformed equation G_m, from which M is first split, has a large m, that is, if two or more roots of (1) are close together. It is good, however, to have a method that yields these roots at all.

In cases where there are *only one pair or two pairs of complex roots* this whole process is superfluous as will be seen in the example.

Convergence of the process. If the coefficients of the transformed equation G_k are determined exactly and the roots of G_k are calculated by splitting G_k into several M, then the roots of the M's are only approximately the roots of G_k. If, thereby, a root of G_k is found with the relative error ϵ_k, then the error of the same root in the following equation is

$$\epsilon_{k+1} \approx \epsilon_k^2,$$

as is easily seen. Let us suppose that we are dealing with the largest root X_1 and there are m complex pairs of roots of the same modulus R, then we see that R is calculated from the absolute member a of the equation M_k. Let now $X_2 = R \cdot 10^{-i}$ be the following root and let the m pairs of complex roots be

$$X' = Re^{\pm i\mathrm{A}}, \quad X'' = Re^{\pm i\mathrm{B}}, \cdots, \quad X^{(m)} = Re^{\pm i\Gamma}.$$

Then a, being the sum of the combinations of all roots of G_k by $2m$, is equal to

$$\begin{aligned} a &= R^{2m} + 2R^{2m-1}(\cos \mathrm{A} + \cos \mathrm{B} + \cdots + \cos \Gamma) \cdot X_2 + \cdots \\ &\approx R^{2m}(1 + 2S \cdot 10^{-i}), \end{aligned}$$

where S is the sum of the cosines. Thus

$$R \approx \sqrt[2m]{a}\,(1 - (1/m)10^{-i}S),$$

that is, since $S < m$, the relative error ϵ_k of R is less than 10^{-i}.

From the following equation G_{k+1}, equation M_{k+1} is set up, and if b is the absolute member of M_{k+1}, then

$$b \approx R^{4m} + R^{4m-2} \cdot 2S'X_2^2, \quad \text{where} \quad S' < m.$$

Thus

$$R' \equiv R^2 \approx \sqrt[2m]{b}\,(1 - 10^{-2i}).$$

That is, the relative error of R' is

$$\epsilon_{k+1} < 10^{-2i} \approx \epsilon_k^2.$$

Now the roots $x_1, x_2, x_3, \cdots$ of (1) are the 2^kth roots of the roots X_i of G_k. Thus if ϵ is the relative error of X_1 of G_k, then

$$x_1 = \sqrt[2^k]{X_1(1 + \epsilon)} \approx \sqrt[2^k]{X_1}\,(1 + \epsilon/2^k).$$

That is, the relative error of x_1 is $\epsilon/2^k$. But from G_{k+1} we obtain, as we have seen

$$x_1 = \sqrt[2^{k+1}]{X_1^2(1 + \epsilon^2)} \approx \sqrt[2^k]{X_1} \cdot (1 + \epsilon^2/2^{k+1}),$$

that is, the relative error of x_1 is $\epsilon^2/2^{k+1}$ or the square of that of the equation G_k. Thus, we have:

The relative error of the roots x_i of (1) *as they are computed from the transformed equations G_k decreases quadratically with every following equation G_k, that is, if the roots x_i of* (1) *following from G_k are exact to r decimals, then equation G_{k+1} will yield them to 2r decimals exactly. Roughly, every following equation yields tiwce as many exact decimals of the roots x_i of* (1).

It must be taken into account that this property holds only if the roots are already sufficiently separated, for instance, if the difference of any two neighboring roots of G_k is at least equal to 100, or else the approximations above are invalid.

From this property it follows that Graeffe's method has its *greatest efficiency* if it is carried out to *many decimal* places. If a calculating machine is used this does not require more computational work than required for fewer decimal places. Now, it is true that in this case one must calculate more transformed equations at least if the number of decimals are to be fully used. But for this purpose it will be sufficient to have *one or two equations* more. If, for instance, the equation G_5 yields 5 exact decimals of the x_i of (1) then the next smaller root of G_5 has a modulus $r \approx 10^{-5}R$, where R is the modulus of the greatest root. In G_6 the relation of the two greatest roots will then be 10^{-10}, that is, only the 10th decimal place of the coefficients of G_6 will be influenced.

From this it follows that—apart from exceptional cases—*the same number of transformed equations will in general be necessary if a certain exactness is required.* For, suppose we have an equation with two roots having the ratio 1.1. Then by 3 or 4 transformations the ratio will become 3. Thus to have a certain exactness, it will be necessary to calculate 3 or 4 equations more than for an original equation with two roots having the ratio 3. The example is very unfavorable, for there will be few equations having roots of the ratio 1.1. At a ratio 1.5, there are only one or two more transformations required.

Since by raising to powers all roots with distinct moduli will be separated automatically by a quadratically convergent process, Graeffe's method is more powerful than other ones.

Influence of rounding off. These considerations of convergence are strictly valid only if the coefficients of the transformed equations are exact, but in reality the calculation is carried out to a fixed number ν of decimals. The errors of rounding off are, of course, increased by every squaring and multiplication. Now, these errors can be estimated by adding the proper inequality to every coefficient of the scheme. But in general this tedious supplement will be superfluous, for on the whole the error will be annulled by the process of extracting the 2^kth roots of the roots of G_k. That is, *if the calculation is carried out to ν decimals, the roots of* (1) *will on the whole be exact to ν decimals too.*

Moduli or roots lying close together. When the equations of the chain do not soon show signs of an approaching splitting up, this will signify that some moduli of even roots are close to each other.

Various procedures have been proposed for accelerating the convergence. But they are all unpractical. For they require much more work than does the Graeffe's method when carried on two or three steps further. Add to this that by these devices the calculation of the chain is interrupted which is very undesirable when at the end of the calculation the roots of (1) are to be computed from those of the M_i.

These procedures make no allowance for the quadratic convergence (or divergence) of Graeffe's method. For before it becomes obvious that several moduli or roots are close to each other several transformations are already effected. By then the roots will be separated so far that the greatest difficulties have been overcome, and the further calculation will proceed rapidly. It is not advisable, therefore, to disregard the entire calculation performed up to this point and instead, to apply Graeffe's method to a transform of the original equation.

For instance *Encke* (Gesammelte mathematische und astronomische Abhandlungen, vol. 1, Berlin, 1888, p. 185), when dealing with a certain equation states that "after 6 or 7 operations we have got the conviction that two trinomial factors lying close together are existing here." Then he abandons Graeffe's method and starts on a new calculation. Yet, if Graeffe's method is carried 2 or 3 steps further, all roots separate automatically.

Thus the usual procedure of Graeffe will be always the most suitable one. Indeed, one of the chief advantages of the method is, that no special devices are required.

On the other hand, if some knowledge of the position of the roots of $f(x)=0$ is not furnished by the first steps of Graeffe's method, but *by other sources*, then this knowledge may be used from the beginning to accelerate the separation of the roots by transforming the equation first.

If, for instance, several moduli are close to each other and their absolute value ρ is known approximately and if also several roots are close to each other and their values, too, are known approximately, we may proceed as follows.

We have a group G of roots lying near a circle C with radius ρ around the origin of the Gaussian plane. And in this group G there exist several places $u, v, w, \cdots$ where the roots "accumulate" so that G is divided into the subgroups $U, V, W, \cdots$.

Now the slow convergence of G when applying Graeffe's method arises from the fact that the quotient of two moduli of G is lying close to 1. This difficulty will be partly overcome by chosing the origin of the coordinates in the neighbourhood of one of the points $u, v, w, \cdots$, say u. The circle C will still be a circle, but it does no longer have the new origin as its center. Thus the quotients of the moduli of the group G have essentially changed. For the relations of the distances of the subgroup U from the origin to each other as well as to those of the subgroups $V, W, \cdots$ differ now widely from 1. The same holds for the relations of the distances of the group V to the distances of $W, \cdots$. However the relations of the distances of V to each other remain nearly the same as they were before, and the same will hold for W to each other.

By this transformation the separation of the group G will, therefore, be accelerated, that is, the equation of the group G will break up into the equations for the subgroups $U, V, W, \cdots$ faster than would have been the case without this transformation, and the subgroup U will even be split into its individual elements. The method is to be repeated eventually, as far as the subgroups $V, W, \cdots$ are concerned.

The value of this method is largely theoretical because equations with several points of accumulation, $u, v, w, \cdots$, near the same circle C do not occur frequently.

The details of the transformation mentioned above will depend on the nature of the roots:

i. Several roots near the circle C are real. We may suppose that all these real roots are positive. Otherwise we form the first transformed equation and get a new equation with the assumed property. The convergence will be most rapid if the origin of the coordinates is chosen in the neighborhood of the least of these positive roots, say a, i.e., if the following transformation is made:

$$x = y + a', \quad \text{where} \quad a' \approx a \approx \rho.$$

ii. The roots near C are all complex, but "simple," i.e., no two of them are close together. In this case, too, the transformation

$$x = y + \rho', \quad \text{where} \quad \rho' \approx \rho,$$

will be sufficient as is evident geometrically.

iii. All roots are complex, but several of them are lying near $u=a+ib$ (and $u'=a-ib$, of course). Then it will not be convenient in general to bring u into the origin immediately, for this would require a complex transformation. It will be more suitable to do this by two real transformations. First we make the transformation

$$x = y + a', \quad \text{where} \quad a' \approx a,$$

so that u will come near the y-axis. The equation in y will have several roots near $\pm ib$. We transform it by Graeffe's procedure and obtain an equation in Y having several roots near the point $(0;\ -b^2)$. Then by the second transformation

$$Y = U - b^2$$

we obtain an equation in U having several roots near the origin. The ratios of the moduli will now differ widely from 1.

This method will be particularly useful when *all* the roots of the equation are lying near the circle C, for instance, for the equations M_i. In this special case the method may be brought into a more convenient form by applying a procedure of Ostrowski. [Recherches sur la méthode de Graeffe et les zéros des polynomes et des séries de Laurent, Acta mathematica, **72**, 245 (1940)].

In case *iii* above, i.e., if all roots of $f(x)=0$ are lying around $u=a+ib$ and $u'=a-ib$, the sum ma of all roots will be equal to the coefficient $-a_1$, so that the real part of all roots will be approximately $\bar{a} \approx -a_1/m$. After the transformation $x=y+a$, the coefficient of x^{m-1} will then vanish. The same holds for the equation in Y. That is:

If we know that $f(x)=0$ has all its roots near two conjugate complex numbers, we may apply the transformations of *iii* without knowing these points, by bringing $f(x)$ into the reduced form $\bar{f}(x)$, transforming $\bar{f}$ once by Graeffe's procedure into $F(x)$ and bringing the latter into its reduced form $\bar{F}(x)$. The roots of $\bar{F}(x)=0$ differ widely, and Graeffe's method will converge then rapidly.

Example. By the following example we show the efficiency of the method in the case of *roots lying close together* or having the *same moduli*.

$$8x^5 + 4x^4 + 18x^3 - 15x^2 - 18x - 81 = 0 \text{ or normalized:} \tag{A}$$
$$x^5 + 0.5^4 + 2.25x^3 - 1.875x^2 - 2.25x - 10.125 = 0.$$

In the coefficients of the transformed equations there is always a power of 10 omitted; its exponent is given in italics on the left of the coefficient, so that, for instance, the last coefficient of G_2 is in reality $-10^4 \cdot 1.0509 \cdots$.

To avoid slips it is advisable to put under each transformed equation G_k the signs of the equations having the same roots, but of opposite signs. These signs are alternately equal or opposite to the signs of G_k.

As may be seen, it is not until G_5 that the pace of the coefficients begins to become more regular. This late start indicates that the moduli of the roots of (A) lie close together. Also from this point of view it is advisable to carry out the calculation to many decimals. Equation G_5 is the first equation where the approaching split is perceptible, for in the double products forming the coefficient of x^3 two zeros appear. From G_5 onwards the process goes rapidly, so that in G_8 the coefficients of x^5, x^3, x^0

1	*0*	+0,5000 00000 00000 0 −	*0*	+2,2500 00000 00000 0 +	*0*	−1,8750 00000 00000 0 −	*0*	−2,2500 00000 00000 0 +	*1*	−1,0125 00000 00000 0 +
1	*0*	−0,25 4,5	*0*	5,0625 1,875 −4,5	*0*	−3,5156 25 −10,1250 10,1250	*0*	+5,0625 −37,9687 5		$k=1$
1	*0*	4,25 −	*0*	2,4375 +	*0*	−3,5156 25 +	*1*	−3,2906 25000 −	*2*	−1,0251 56250 +
1	*1*	−1,8062 5 0,4875	*1*	+0,5941 40625 2,9882 81250 −6,5812 5	*2*	−0,1235 96191 40625 0 −1,6041 79687 5 8,7138 28125	*3*	1,0828 21289 06250 0 −0,7208 12988 28125 0		$k=2$
1	*1*	−1,3187 5 +	*1*	−2,9988 28125 −	*2*	+6,9860 52246 09375 0 −	*2*	3,6200 83007 81250 0 +	*4*	−1,0509 45336 91406 25 +
1	*2*	−1,7391 01562 5 −0,5997 65625	*2*	+8,9929 70123 29101 6 184,2571 27990 72265 6 7,2401 66015 625	*5*	−4,8804 92598 51515 3 −0,2171 20134 77325 4 −2,7718 68326 11084 0	*5*	1,3015 00098 34528 0 146,8391 80629 40478 3		$k=3$
1	*2*	−2,3388 67187 5 +	*4*	2,0049 02641 29638 7 +	*5*	−7,8694 81059 39924 7 +	*7*	1,4814 96807 27750 1 +	*8*	−1,1044 86101 18141 2 +
1	*4*	−5,4702 99720 76416 0 4,0098 05282 59277 4	*8*	4,0196 34601 07722 90 −3,6811 34206 49632 75 0,2962 99361 45550 02	*11*	−6,1928 73214 42434 95 5,9405 13723 96565 23 −0,5166 49260 22060 19	*14*	2,1948 32789 97342 90 −1,7383 46490 72336 84		$k=4$
1	*4*	−1,4604 94438 17138 6 +	*7*	6,3479 97560 36401 7 +	*10*	−7,6900 87506 79299 1 +	*13*	4,5648 62992 50060 6 +	*16*	−1,2198 89547 70291 6 +
1	*8*	−2,1330 44003 92955 24 1,2695 99512 07280 34	*15*	4,0297 07302 63875 1 −2,2462 66006 54448 5 0,0912 97259 85001 2	*21*	−5,9137 44586 21336 40 5,7955 47827 95796 66 −0,3563 28379 92070 33	*27*	2,0837 97414 03015 87 −1,8762 11474 09150 94		$k=5$
1	*7*	−8,6344 44918 56749 0 +	*15*	1,8747 38555 94427 8 +	*20*	−4,7452 51381 76100 7 +	*26*	2,0758 59399 38649 3 +	*32*	−1,4881 30508 59482 5 +
1	*15*	−7,4553 63905 17759 5 3,7494 77111 88855 6	*30*	3,5146 44653 14403 7 −0,0819 45223 36114 3 0,0004 15171 87987 7	*41*	−2,2517 41067 61047 5 7,7833 87305 49838 0 −0,2569 83618 16203 7	*52*	4,3091 92246 02125 1 −14,1231 06704 30060 7		$k=6$
1	*15*	−3,7058 86793 28903 9 +	*30*	3,4331 14601 66277 1 +	*41*	5,2746 62619 72586 8 −	*52*	−9,8139 14458 27935 6 −	*64*	−2,2145 32410 61069 3 +
1	*30*	−13,7335 96924 67411 6 6,8662 29203 32554 2	*61*	1,1786 27586 81501 3 0,0003 90946 05083 0 −0,0000 00019 62782 9	*83*	−2,7822 06575 19333 6 −6,7384 58605 23764 8 −0,0016 41361 28275 9	*105*	9,6312 91699 44245 8 23,3618 22652 83927 4		$k=7$
1	*30*	−6,8673 67721 34857 4 +	*61*	1,1790 18513 23801 4 +	*83*	−9,5223 06541 71374 3 +	*106*	3,2993 11435 22817 3 +	*128*	−4,9041 53787 64520 5 +
1	*61*	−4,7160 73942 02203 1 2,3580 37026 47602 8	*122*	1,3900 84654 55797 7 −0,0000 00130 78636 1	*167*	−9,0674 32187 43643 4 7,7798 98526 14379 7 −0,0000 00067 35725 5	*212*	10,8854 55946 62738 7 −9,3397 71157 77744 6		$k=8$
1	*61*	−2,3580 36915 54600 3	*122*	1,3900 84523 77161 6	*167*	−1,2875 33728 64989 2	*212*	1,5458 84788 84994 1	*257*	−2,4050 72447 09578 9

are fixed to 16 places since they are no longer influenced by the double products.

Thus G_8 is split into the two equations M:

$$M_1 \equiv X^2 - 2.358036915546003 \cdot 10^{61}X + 1.390084523771616 \cdot 10^{122} = 0$$

$$M_2 \equiv 10^{122} \cdot 1.390084523771616X^3 - 10^{167} \cdot 1.287533728649992X^2 + 10^{212} \cdot 1.545884788849941X - 10^{256} \cdot 2.405072447095789 = 0.$$

These two equations must now be solved. Since in G_8 the coefficient of X^4 is approximately twice as large as the coefficient of X^3 in G_7, this signifies that M_1 has a real double-root, so that M_1 may be put into the form

$$M_1 \equiv (X - m)^2 = X^2 - 2mX + m^2$$

as is approximately confirmed. Thus from the coefficient of X we have

$$X_1 = X_2 = 1.17901845777300 \cdot 10^{61},$$

whereas from the coefficient of X^0 we have

$$X_1 = X_2 = 1.17901845777393 \cdot 10^{61}.$$

The difference of the two values arises from the rounding off errors we mentioned above. To annul them slightly we could take the mid-value of the two values:

$$X_1 = X_2 = 1.17901845777346 \cdot 10^{61}.$$

Equation M_2 no longer splits into factors as can be seen from the course of its coefficients. Otherwise some of the double products should begin to converge to zero. Now, this is only the case with the product $2A_4A_0$, and there only because of the coefficient A_4. This follows from the fact that equation M_1 is already split off and has no influence on M_2. Thus M_2 has all its roots with the same modulus R. Since its degree is odd, we have according to p. 179, ii and according to (12):

$$R = 10^{212} \cdot 1.545884788849941/10^{167} \cdot 1.287533728649892 = 10^{45} \cdot 1.200500425311787.$$

We now set $X = RY$ in M_2 and get the reciprocal equation in Y

$$M'' \equiv BY^3 - 1.855595246409359Y^2 + 1.855595246407256Y - B = 0$$

or, to make the equation wholly reciprocal, instead of the two middle coefficients we put their arithmetic mean, and divide the resulting equation by $Y-1$:

$$Q = M''/(Y - 1) = Y^2 + 0.2284659663166439Y + 1 = 0.$$

Since this equation is no longer divisible by $Y-1$, we put $Z = Y + Y^{-1}$ and obtain instead of Q:

$$Z = 2 \cos \phi = - 0.2284659663166439,$$

$$\phi = 292^g.71149255575355.$$

(The notation "g" refers to the division of the quadrant into 100 parts, instead of into 90.) Thus the roots of M_2 are

$$X_3 = R, \qquad X_{4,5} = Re^{\pm i\phi}.$$

The roots of the original equation arise from the X_i by extracting the 256th root. Now

$$\sqrt[256]{X_1} = \sqrt{3}, \qquad r = \sqrt[256]{R} = 1.5,$$
$$\phi/256 = 1^g.143405439670912305,$$

so that the roots x_i are to be found among the values

$$\sqrt{3}\cdot(\cos 2k\pi/256 + i \sin 2k\pi/256)$$

and

$$1.5(\cos \phi_k + i \sin \phi_k), \quad \text{where} \quad \phi_k = (2k\pi + \phi)/256, \qquad k = 1, 2, \cdots, 256.$$

There are two roots of the first kind, thus either two conjugate-complex ones or two real opposite ones or a real double root, and three roots of the second kind, thus there is always one real root.

Because of the simple character of the moduli of our roots, the procedure of finding their arguments could be very much abbreviated. However, to show the general method, we do not make use of this special property. We begin with the more difficult part, namely equation M_2. Thus we look back in the chain of transformed equations G_1 until we come to the first equation where our M_2 starts to split off. That is G_5, where the coefficient of X^3 is determined to two places. For the transition to G_6 makes two zeros in each of the double products. That is, from G_5 an equation M is split off

$$1.87X^3 - 10^5\cdot 4.75X^2 + 10^{11}\cdot 2.08X - 10^{17}\cdot 1.49 = 0$$

or, putting $X = 1.5Y$:

$$Y^3 - 0.596Y^2 + 0.596Y - 1 = (Y - 1)(Y^2 + 0.404Y + 1).$$

This gives

$$\cos \mathrm{B} = -0.202 \quad \text{or} \quad \mathrm{B} = 287^g.\mathrm{B}/32 = 8^g.97.$$

Thus the argument of the roots equals

$$\phi_m = 8^g.97 + 2m\pi/32, \quad \text{where} \quad m = 1, 2, \cdots, 32;$$

by this the number of values to be tried has sunk from 256 to 32. We can *correct these values ϕ_m by comparing them with the former values ϕ_k*, that are nearly exact. For that we must determine the values of k from the equation $\phi_k \equiv (2k\pi + \phi)/256 \approx 8^g.97$, that is $k \approx 5$, thus $k = 5$ and $\phi_5 = 8^g.9559 \cdots \approx 8^g.956$.

With this value we try to verify the original equation, which on putting $x = 1.5\cdot y$, becomes

$$(y - 1)\left(3y^4 + 4y^3 + 7y^2 + \frac{16}{3}y + 4\right) = 0$$

or according to (13):

$$3 \cos 4\phi_m + 4 \cos 3\phi_m + 7 \cos 2\phi_m + \frac{16}{3} \cos \phi_m + 4 = 0.$$

Now the values ϕ_m are

$$\begin{aligned}\phi_1 &= 8^g.956,\ \phi_2 = \phi_1 + 12.5^g = 21.456^g, \cdots, \phi_8 = 96^g.456\\ \phi_9 &= 100^g + \phi_1, \cdots, \phi_{16} = 100^g + \phi_8,\\ \phi_{17} &= 200^g + \phi_1, \cdots, \phi_{24} = 200^g + \phi_8,\\ \phi_{25} &= 300^g + \phi_1, \cdots, \phi_{32} = 300^g + \phi_8.\end{aligned}$$

These 32 values must be tried. The calculation is carried out to 4–5 decimals. We find the solution ϕ_{13}, that is

$$\phi \equiv \phi_{13} = 146^g.4559054397.$$

(It is not necessary to compute all $32 \cdot 4 = 128$ values of cosines, since $\phi_{17}, \cdots, \phi_{32}$ yield values equal or opposite to those yielded by $\phi_1, \cdots, \phi_{16}$. Also, in the group belonging to ϕ_1 through ϕ_{16} not all values are different.) Thus all the roots of the modulus 1.5 are found.

For finding the roots of modulus $\sqrt{3}$, we do not need the somewhat tedious procedure above, but can abbreviate it in several ways. A method that is always applicable consists in dividing the original equation by the product of the three linear factors already found, thus by

$$(x - 1.5)(x^2 - 3x \cos \phi + 2.25).$$

In the quotient we put $x = y\sqrt{3}$ and get $y^2 + 1 = 0$, thus $y = i$; so we have as roots of the equation

$$x_1 = i\sqrt{3}, \qquad x_2 = -i\sqrt{3}, \qquad x_3 = 1.5, \qquad x_{4,5} = 1.5 \cdot e^{\pm i\phi}.$$

After this general and somewhat tedious way of finding the arguments of the roots x_4, x_5 having the modulus 1.5, we give in the following the simple *method appropriate to all cases where only two or four complex roots exist.*

From M_1, M_2 we determine the moduli of their respective roots, as we did above, that is $X_1 = X_2$ and R. Thus the moduli of the roots of the original equation are, as above,

$$\sqrt[256]{X_1} = \sqrt{3}, \qquad \sqrt[256]{R} = 1.5,$$

and the roots themselves are

$$x_{1,2} = \sqrt{3} \cdot e^{\pm i\psi}, \qquad x_3 = 1.5, \qquad x_{4,5} = 1.5 \cdot e^{\pm i\phi}.$$

Now we use the property of the coefficients of the original equation, namely that the sum of all the roots or their combinations at four are:

$$x_1 + x_2 + x_3 + x_4 + x_5 = -0.5 = 2\sqrt{3} \cos \psi + 1.5 + 3 \cos \phi.$$

$$x_1x_2x_3x_4 + x_1x_2x_3x_5 + \cdots = -18/8 = p/x_1 + p/x_2 + p/x_3 + p/x_4 + p/x_5,$$

where $p = x_1x_2x_3x_4x_5 = 81/8$. By inserting the above values we get the two equations

$$2\sqrt{3} \cos \psi + 3 \cos \phi = -2,$$

$$2\sqrt{3} \cos \psi + 4 \cos \phi = -8/3,$$

thus

$$\cos \phi = -2/3, \quad \text{or} \quad \phi = 146^g.4559054397.$$

Then

$$\cos \psi = 0 \qquad \text{or} \quad \psi = 100^g.$$

Conclusions. Graeffe's method has the following properties.

i. It yields not only one root at a time but *all roots* simultaneously, even the complex ones; this is accomplished by no other method.

ii. It is the only method that automatically discovers *roots lying close together* which easily escape attention. There is no method other than Graeffe's which solves, without special attention, an equation having, for instance, the roots $\sqrt{5/2} = 1.581 \cdots$ and $3/2 = 1.5$, not to mention theoretical cases such as $1.67324 \cdots$ and $1.67331 \cdots$ Lagrange's method is the only other one with the same advantage of separating those roots, but the process requires great attention and much computational work and fails entirely in the case of complex roots.

iii. An especially valuable property is that even *complex roots of the same modulus* are automatically obtained.

iv. These advantages are not due to special artifices. Any other method requires a first approximation which must then be corrected. But the finding of this first approximation is difficult, particularly in the case of complex roots. Only Bernoulli's method does not require a first approximation, but for that it yields only two real roots at a time, and the process of approximation may be very slow. Graeffe's method on the other hand does not require a first approximate value. Besides, it is *not necessary to use criteria of convergence* in order to determine if the approximate value is sufficiently close the actual root.

Therefore it seems to us that *Graeffe's method is by far the best for solving algebraic equations.* Only if one does not need all roots of the equation, but only a single one, will it be inferior to other methods.