THE ANALOGY BETWEEN MULTIPLY-CONNECTED SLICES AND SLABS*

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1. Introduction. The analogy between the two-dimensional field of stress and the transverse flexure of a thin plate was first applied by K. Wieghardt[1] to the solution of a problem involving boundary loading of a simply-connected body. As is well known, the analogy establishes the proportionality of the curvatures of the surface of the plate to the components of stress in the two-dimensional field of stress. H. M. Westergaard[2] introduced the useful terminology of slab and slice, free slice and constrained slice, and gave the boundary conditions for the slab when the slice is multiply-connected and is stressed by boundary loads having no resultant force on an internal boundary. Westergaard also proposed the use of the analogy in the investigation of the stresses in the Boulder Canyon Dam,[3] a problem involving gravity and boundary loading of a simply connected body. An improvement in experimental technique was contributed by H. Cranz[4] in introducing an optical spherometer[5] for measuring the components of surface curvature. Cranz's application was to boundary load problems in simply connected bodies.

It is the purpose of this paper to give the general boundary conditions for the slab when the slice is multiply-connected and is stressed by any combination of boundary loading, body forces, dislocations and thermal dilatations. The analogy has, in fact, its most useful applications in the last three cases as they are either difficult to reproduce, or the resulting stresses are difficult to measure, in an experimental model of the slice itself, while the analogous conditions for the slab, developed below, are easy to handle.

In order to proceed, it is necessary, first, to set down the general boundary value problem for the slice. It is convenient to do this along the lines established by Michell,[6] with the additional consideration of dislocations and thermal dilatations.

2. Airy's stress function and its differential equations. In a state of plane strain defined by setting

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[1] K. Wieghardt, *Über ein neues Verfahren, verwickelte Spannungsverteilungen in elastischen Körpern auf experimentellem Wege zu finden*, Mitteilungen über Forschungsarbeiten a. d. Gebiete d. Ingenieurwesens, **49**, 15–30 (1908).

[2] H. M. Westergaard, *Graphostatics of stress functions*, Transactions, Amer. Soc. Mech. Eng., **56**, 141–150 (1934).

[3] United States Department of the Interior, Bureau of Reclamation, Boulder Canyon Project, Final Reports (1938), Part V, Technical Investigations: Bulletin 2, *Slab analogy experiments*; Bulletin 4, *Stress studies for Boulder Dam.*

[4] H. Cranz, *Die experimentelle Bestimmung der Airyschen Spannungsfunktion mit Hilfe des Plattengleichnisses*, Ingenieur-Archiv, **10**, 159–166 (1939).

[5] E. Einsporn, *Ebenheit*, Zeitschrift für Instrumentenkunde, **57**, 265–285 (1937).

[6] J. H. Michell, *On the direct determination of stress in an elastic solid, with application to the theory of plates*, Proc. London Math. Soc., **31**, 100–124 (1899).

$$\gamma_{yz} = \gamma_{zx} = \epsilon_z = 0$$

and restricting the displacements u and v to be functions of x and y only, the relations between strain, displacement, stress and temperature in an isotropic elastic body are

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{1}{E_1}\left[(1 - \nu_1^2)\sigma_x - \nu_1(1 + \nu_1)\sigma_y\right] + (1 + \nu_1)\alpha_1 T, \tag{2.1a}$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{1}{E_1}\left[(1 - \nu_1^2)\sigma_y - \nu_1(1 + \nu_1)\sigma_x\right] + (1 + \nu_1)\alpha_1 T, \tag{2.1b}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{2(1 + \nu_1)}{E_1}\tau_{xy}. \tag{2.1c}$$

These are the relations for a constrained slice. The notations for stress, strain and displacement are the usual ones and E_1, ν_1 are Young's Modulus and Poisson's ratio for the material of the slice, α_1 is the coefficient of linear thermal expansion, and T is the temperature in excess of a uniform initial temperature.

When the stresses are expressed in terms of Airy's stress function (ϕ) and a body force potential (V) by

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} + V, \qquad \sigma_y = \frac{\partial^2\phi}{\partial x^2} + V, \qquad \tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}, \tag{2.2}$$

the equations of equilibrium are satisfied and the strain relation

$$\frac{\partial^2\epsilon_x}{\partial y^2} + \frac{\partial^2\epsilon_y}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\partial y} \tag{2.3}$$

yields the differential equation governing ϕ:

$$\nabla^4\phi = -\frac{1 - 2\nu_1}{1 - \nu_1}\nabla^2 V - \frac{1 + \nu_1}{1 - \nu_1}\alpha_1\nabla^2 T. \tag{2.4}$$

In a state of plane stress, defined by

$$\sigma_z = \tau_{yz} = \tau_{zx} = 0,$$

the strain-displacement-stress-temperature relations become

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{1}{E_1}(\sigma_x - \nu_1\sigma_y) + \alpha_1 T, \tag{2.5a}$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{1}{E_1}(\sigma_y - \nu_1\sigma_x) + \alpha_1 T, \tag{2.5b}$$

$$\epsilon_z = \frac{\partial w}{\partial z} = -\frac{\nu_1}{E}(\sigma_x + \sigma_y) + \alpha_1 T, \tag{2.5c}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{2(1 + \nu_1)}{E_1}\tau_{xy}. \tag{2.5d}$$

These are the relations for a free slice. If the components of stress are again expressed in terms of an Airy function and a body force potential by (2.2), the equilibrium equations are identically satisfied and the strain relations reduce to

$$\nabla^4\phi = -(1-\nu_1)\nabla^2 V - (1+\nu_1)\alpha\nabla^2 T \tag{2.6}$$

if terms associated with the coordinate z are neglected.

In what follows, the case of plane strain (constrained slice) will be treated, but the results are directly applicable to plane stress (neglecting z-dependent terms) if Young's modulus E_1, Poisson's ratio ν_1 and the linear thermal expansion coefficient α_1 are replaced by E_1', ν_1' and α_1' where

$$E_1' = \frac{E_1(1+2\nu_1)}{(1+\nu_1)^2}, \qquad \nu_1' = \frac{\nu_1}{1+\nu_1}, \qquad \alpha_1' = \frac{\alpha_1(1+\nu_1)}{1+2\nu_1}. \tag{2.7}$$

3. Conditions on ϕ at a point on a boundary of the slice. Michell[6] gave the conditions to be satisfied, at each point of each boundary, by ϕ and its derivative normal to the boundary:

$$\phi = \int_0^s (Bl - Am)ds + \alpha x + \beta y + \gamma, \tag{3.1}$$

$$\frac{d\phi}{dn} = Al + Bm + \alpha l + \beta m, \tag{3.2}$$

where α, β, γ are constants, in general different for each boundary, ds is an element of arc of a boundary, dn an element of normal to that boundary, and

$$l = \frac{dy}{ds}, \qquad m = \frac{-dx}{ds}, \tag{3.3}$$

$$A = -\int_0^s \overline{Y}ds + \int_0^s Vmds, \qquad B = \int_0^s \overline{X}ds - \int_0^s Vlds, \tag{3.4}$$

$$\overline{X} = \sigma_x l + \tau_{xy}m, \qquad \overline{Y} = \tau_{xy}l + \sigma_y m. \tag{3.5}$$

In a simply connected body, α, β, γ may be assigned arbitrary (including zero) values as the addition of a linear function of x and y to ϕ does not affect the stresses. In a multiply-connected body, three additional conditions on ϕ are required for determining α, β, γ, on each additional boundary. Equations (3.1) to (3.5) are not altered by introducing thermal dilatations and dislocations of the type considered here.

4. Conditions on ϕ for each boundary of the slice. The additional conditions on ϕ are obtained by assuming the strains (and hence the stresses) to be continuous and requiring the rotations and displacements (a) to be single-valued or (b) to have prescribed discontinuities (dislocations). Michell[6] gave the conditions for case (a). The conditions for case (b), including, also, thermal dilatations, are derived by following Michell's procedure with modifications along the lines indicated by Volterra.[7]

(*i*) *Rotation condition.* Considering the rotation

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \tag{4.1}$$

[7] Love, *Theory of elasticity*, 4th ed., Cambridge Univ. Press, Cambridge, 1927, pp. 221–228.

we require that the line integral of its differential have a value, say c, after one complete circuit around (and along) a boundary. Thus,

$$c = \oint d\omega_z. \tag{4.2}$$

Now,

$$\oint d\omega_z = \oint \frac{\partial \omega_z}{\partial x} dx + \frac{\partial \omega_z}{\partial y} dy$$

$$= \oint \left(\frac{1}{2} \frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \epsilon_x}{\partial y}\right) dx + \left(\frac{\partial \epsilon_y}{\partial x} - \frac{1}{2} \frac{\partial \gamma_{xy}}{\partial y}\right) dy. \tag{4.3}$$

Replacing the strain components by their expressions in terms of ϕ, V and T, we find

$$\frac{E_1 c}{1+\nu_1} = (1-\nu_1) \oint \left(\frac{\partial}{\partial x}(\nabla^2\phi)dy - \frac{\partial}{\partial y}(\nabla^2\phi)dx\right) + (1-2\nu_1) \oint \left(\frac{\partial V}{\partial x} dy - \frac{\partial V}{\partial y} dx\right)$$
$$+ E_1\alpha_1 \oint \left(\frac{\partial T}{\partial x} dy - \frac{\partial T}{\partial y} dx\right).$$

Then

$$\oint \frac{d(\nabla^2\phi)}{dn} ds = \frac{E_1 c}{1-\nu_1^2} - \frac{1-2\nu_1}{1-\nu_1} \oint \frac{dV}{dn} ds - \frac{E_1\alpha_1}{1-\nu_1} \oint \frac{dT}{dn} ds. \tag{4.4}$$

This is the first of Michell's three conditions on ϕ for each boundary of the slice. It may be observed that, if the circuit of the line integral in (4.3) were reducible, the integral would vanish because, by Green's theorem,

$$\oint \left(\frac{1}{2} \frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \epsilon_x}{\partial y}\right) dx + \left(\frac{\partial \epsilon_y}{\partial x} - \frac{1}{2} \frac{\partial \gamma_{xy}}{\partial y}\right) dy$$
$$= \iint \left(\frac{\partial^2 \epsilon_y}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial y^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}\right) dx dy; \tag{4.5}$$

and the surface integral vanishes by virtue of (2.3).

(*ii*) *Displacement conditions.* We admit a translational dislocation a parallel to x and set

$$a = \oint du = \oint \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \oint \left(\epsilon_x dx + \frac{1}{2} \gamma_{xy} dy\right) - \oint \omega_z dy.$$

Now

$$\oint \omega_z dy = y_0 \oint d\omega_z - \oint y d\omega_z = y_0 c - \oint y\, d\omega_z,$$

where y_0 is the y-coordinate of the starting point of integration. Also

$$\oint y d\omega_z = \oint y \left(\frac{\partial \omega_z}{\partial x} dx + \frac{\partial \omega_z}{\partial y} dy\right)$$
$$= \oint y \left(\frac{1}{2} \frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \epsilon_x}{\partial y}\right) dx + \oint y \left(\frac{\partial \epsilon_y}{\partial x} - \frac{1}{2} \frac{\partial \gamma_{xy}}{\partial y}\right) dy.$$

Hence

$$a + y_0 c = \oint \left[\epsilon_x + y\left(\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \epsilon_x}{\partial y}\right)\right]dx + \oint \left[\frac{1}{2}\gamma_{xy} + y\left(\frac{\partial \epsilon_y}{\partial x} - \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial y}\right)\right]dy. \tag{4.6}$$

We now note that

$$\oint \epsilon_x dx + \frac{1}{2}\gamma_{xy}dy = [x\epsilon_x]_0^0 + \frac{1}{2}[y\gamma_{xy}]_0^0 - \oint \left(x\frac{\partial \epsilon_x}{\partial x}dx + \frac{1}{2}y\frac{\partial \gamma_{xy}}{\partial y}dy\right)$$
$$= -\oint \left(x\frac{\partial \epsilon_x}{\partial x}dx + \frac{1}{2}y\frac{\partial \gamma_{xy}}{\partial y}dy\right),$$

the terms outside the integrals vanishing because of the assumption of continuous strains. Equation (4.6) then becomes

$$a + y_0 c = \oint \left[y\left(\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \epsilon_x}{\partial y}\right) - x\frac{\partial \epsilon_x}{\partial x}\right]dx + \oint y\left(\frac{\partial \epsilon_y}{\partial x} - \frac{\partial \gamma_{xy}}{\partial y}\right)dy. \tag{4.7}$$

When the strain components in (4.7) are replaced by their expressions in terms of ϕ, V and T, we find

$$\begin{aligned}\frac{E_1(a + y_0 c)}{1 + \nu_1} &= (1 - \nu_1)\oint y\left[\frac{\partial}{\partial x}(\nabla^2\phi)dy - \frac{\partial}{\partial y}(\nabla^2\phi)dx\right] \\ &+ (1 - 2\nu_1)\oint y\left(\frac{\partial V}{\partial x}dy - \frac{\partial V}{\partial y}dx\right) \\ &+ E_1\alpha_1\oint y\left(\frac{\partial T}{\partial x}dy - \frac{\partial T}{\partial y}dx\right) \\ &- \oint x\left[(1 - \nu_1)\frac{\partial}{\partial x}(\nabla^2\phi) + (1 - 2\nu_1)\frac{\partial V}{\partial x} + E_1\alpha_1\frac{\partial T}{\partial x}\right]dx \\ &+ \oint \left(x\frac{\partial^3\phi}{\partial x^3}dx + y\frac{\partial^3\phi}{\partial x\partial y^2}dy\right).\end{aligned} \tag{4.8}$$

Now,

$$\oint \left(x\frac{\partial^3\phi}{\partial x^3}dx + y\frac{\partial^3\phi}{\partial x\partial y^2}dy\right) = \left[x\frac{\partial^2\phi}{\partial x^2} + y\frac{\partial^2\phi}{\partial x\partial y}\right]_0^0 - \oint \left(\frac{\partial^2\phi}{\partial x^2}dx + \frac{\partial^2\phi}{\partial x\partial y}dy\right)$$
$$= -\oint \frac{d}{ds}\left(\frac{\partial\phi}{\partial x}\right)ds,$$

the term outside the integral vanishing because the stresses are continuous. But, from (2.2), (3.3) and (3.5),

$$\frac{d}{ds}\left(\frac{\partial\phi}{\partial x}\right) = Vm - \overline{Y}.$$

Hence (4.8) may be written:

$$\oint \left(y \frac{d(\nabla^2\phi)}{dn} - x \frac{d(\nabla^2\phi)}{ds} \right) ds = \frac{E_1(a + y_0c)}{1 - \nu_1^2} - \frac{1 - 2\nu_1}{1 - \nu_1} \oint \left(y \frac{dV}{dn} - x \frac{dV}{ds} \right) ds$$
$$- \frac{E_1\alpha_1}{1 - \nu_1} \oint \left(y \frac{dT}{dn} - x \frac{dT}{ds} \right) ds$$
$$- \frac{1}{1 - \nu_1} \oint (\overline{Y} - Vm) ds. \tag{4.9}$$

This is Michell's second condition on ϕ for each boundary of the slice.

Similarly, admitting a translational dislocation b in the y-component of displacement, we set

$$b = \oint dv$$

and we find

$$\oint \left(y \frac{d(\nabla^2\phi)}{ds} + x \frac{d(\nabla^2\phi)}{dn} \right) ds = - \frac{E_1(b - x_0c)}{1 - \nu_1^2} - \frac{1 - 2\nu_1}{1 - \nu_1} \oint \left(y \frac{dV}{ds} + x \frac{dV}{dn} \right) ds$$
$$- \frac{E_1\alpha_1}{1 - \nu_1} \oint \left(y \frac{dT}{ds} + x \frac{dT}{dn} \right) ds$$
$$- \frac{1}{1 - \nu_1} \oint (\overline{X} - Vl) ds, \tag{4.10}$$

which is the last of Michell's three conditions.

Corresponding to (4.5), a similar application of Green's theorem to (4.6) reveals that the right hand side of the latter vanishes for reducible circuits and the same result is found for the corresponding step in the development of Michell's third condition.

The differential equation (2.6), the boundary conditions (3.1) and (3.2), and the three conditions (4.4), (4.9) and (4.10) constitute a statement of the boundary value problem of plane elasticity for stresses induced by boundary loading, body forces, dislocations, and thermal dilatations. The general formulation of the problem reveals the analogies, discovered by M. A. Biot,[8] between gravity loading and boundary pressures, and between thermal loading and boundary pressures and dislocations.

5. The slab equations. In the approximate theory of the bending of thin plates[9] (slabs), the deflection (w) is governed by the differential equation

$$D\nabla^4 w = Z, \tag{5.1}$$

where D is the flexural rigidity of the plate and Z is the surface load, normal to the middle plane.

The components of curvature in the y, z and x, z planes are given by

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[8] M. A. Biot, *Distributed gravity and temperature loading in two-dimensional elasticity replaced by boundary pressures and dislocations*, J. Appl. Mech., **2**, A 41–A 95 (1935).

[9] Love, loc. cit., p. 487.

$$\kappa_x = \frac{\partial^2 w}{\partial y^2}, \qquad \kappa_y = \frac{\partial^2 w}{\partial x^2}. \tag{5.2}$$

On a boundary of the slab, the shearing force (N) normal to the middle plane, the flexural couple (G), and the torsional couple (H) (all per unit of arc length s) are

$$N = -D\frac{\partial}{\partial n}(\nabla^2 w), \tag{5.3a}$$

$$G = -D\left[\frac{\partial^2 w}{\partial n^2} + \nu_2\left(\frac{\partial^2 w}{\partial s^2} + \frac{1}{\rho'}\frac{\partial w}{\partial n}\right)\right], \tag{5.3b}$$

$$H = (1-\nu_2)D\frac{\partial}{\partial n}\left(\frac{\partial w}{\partial s}\right), \tag{5.3c}$$

where ρ' is the radius of curvature of the boundary of the unflexed slab and ν_2 is Poisson's ratio for the slab material.

The resultant force and the components, parallel to the x and y axes, of the resultant couple on a complete boundary are[10]

$$F_z = \oint\left(N - \frac{\partial H}{\partial s}\right)ds, \tag{5.4a}$$

$$M_x = \oint\left[y\left(N - \frac{\partial H}{\partial s}\right) + G\frac{dx}{ds}\right]ds, \tag{5.4b}$$

$$M_y = \oint\left[G\frac{dy}{ds} - x\left(N - \frac{\partial H}{\partial s}\right)\right]ds. \tag{5.4c}$$

Substituting (5.3) in (5.4) we find

$$F_z = -D\oint\left[\frac{\partial}{\partial n}(\nabla^2 w) + (1-\nu_2)\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial w}{\partial s}\right)\right]ds, \tag{5.5a}$$

$$\begin{aligned} M_x = -D\oint\Bigg\{&y\left[\frac{\partial}{\partial n}(\nabla^2 w) + (1-\nu_2)\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial w}{\partial s}\right)\right] \\ &+ \frac{dx}{ds}\left[\frac{\partial^2 w}{\partial n^2} + \nu_2\left(\frac{\partial^2 w}{\partial s^2} + \frac{1}{\rho'}\frac{\partial w}{\partial n}\right)\right]\Bigg\}ds, \end{aligned} \tag{5.5b}$$

$$\begin{aligned} M_y = -D\oint\Bigg\{&\frac{dy}{ds}\left[\frac{\partial^2 w}{\partial n^2} + \nu_2\left(\frac{\partial^2 w}{\partial s^2} + \frac{1}{\rho'}\frac{\partial w}{\partial n}\right)\right] \\ &- x\left[\frac{\partial}{\partial n}(\nabla^2 w) + (1-\nu_2)\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial w}{\partial s}\right)\right]\Bigg\}ds. \end{aligned} \tag{5.5c}$$

6. The analogy for singly-connected bodies. Noting the similarity between the differential equations (2.6) and (5.1) for ϕ and w, we set

$$w = K\phi, \tag{6.1}$$

[10] Love, loc. cit., p. 460.

where K is a conversion constant having the dimensions of length/force.

Then, from (6.1) and (2.6),

$$\nabla^4 w = -\frac{1-2\nu_1}{1-\nu_1} K\nabla^2 V - \frac{1+\nu_1}{1-\nu_1} K\alpha_1 \nabla^2 T \tag{6.2}$$

becomes the differential equation for the deflection of the analogous slab. Hence

$$Z = -\frac{1-2\nu_1}{1-\nu_1} KD\nabla^2 V - \frac{1+\nu_1}{1-\nu_1} KD\alpha_1 \nabla^2 T \tag{6.3}$$

is the normal surface loading to be applied to the face of the slab. In the case of a steady state temperature distribution,

$$\nabla^2 T = 0. \tag{6.4}$$

If, in addition, the body force potential is harmonic, the slab is subjected to edge loading only. If either V or T is not harmonic, transverse loading is required on the surface of the slab, and the load may vary slowly with time.

The edge conditions (i.e., the elevation and slope at each point of a boundary) of the slab are specified by substituting $w=K\phi$ in (3.1) and (3.2). Thus

$$\frac{w}{K} = \int_0^s (Bl - Am)ds + \alpha x + \beta y + \gamma, \tag{6.5}$$

$$\frac{1}{K}\frac{dw}{dn} = Al + Bm + \alpha l + \beta m. \tag{6.6}$$

The normal components of stress in the slice are obtained by combining (2.2), (5.2) and (6.1), with the result

$$\sigma_x = \frac{\kappa_x}{K} + V, \qquad \sigma_y = \frac{\kappa_y}{K} + V. \tag{6.7}$$

The principal stresses and their directions may be calculated from two sets of curvature measurements at each point.[4] If the boundary of the slab is a scale model of the boundary of the slice, e.g., if the ratio of a linear dimension of the slab to the corresponding linear dimension of the slice is k, the stress components in the slice are given by

$$\sigma_x = \frac{k^2\kappa_x}{K} + V, \qquad \sigma_y = \frac{k^2\kappa_y}{K} + V. \tag{6.8}$$

For a singly-connected body, (6.1) to (6.8) completely specify the analogy, since the unknown constants α, β, γ may be given arbitrary values.

7. Additional conditions on the slab for multiply-connected bodies. For a multiply-connected body, α, β, γ must be prescribed for each boundary. Now, it will be observed, from (6.5) and (6.6), that α, β, γ specify a rigid body translation and rotation of each complete boundary of the slab. Such rigid body movements may be effected by applying, on each boundary, a resultant force, normal to the middle plane of the slab, and a couple about an axis properly oriented in the plane of the slab. The magnitudes of the force and the x and y components of the couple on each boundary are determined by expressing F_z, M_x, M_y (see (5.5)) in terms of the specified boundary loadings, body forces, dislocations, and temperature distribution of the slice.

i. Resultant force on a boundary of the slab. Replacing w by $K\phi$ in (5.5a), we have

$$F_z = -KD\oint\left[\frac{\partial}{\partial n}(\nabla^2\phi) + (1-\nu_2)\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)\right]ds. \tag{7.1}$$

Now

$$\oint\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)ds = 0$$

because of the assumption of continuity of the components of stress in the slice. Hence

$$F_z = -KD\oint\frac{\partial}{\partial n}(\nabla^2\phi)ds. \tag{7.2}$$

Therefore, from (4.4),

$$\frac{(1-\nu_1)F_z}{KD} = -\frac{E_1c}{1+\nu_1} + (1-2\nu_1)\oint\frac{dV}{dn}ds + E_1\alpha_1\oint\frac{dT}{dn}ds, \tag{7.3}$$

whereby F_z is expressed in terms of known quantities.

ii. x-component of couple on a boundary of the slab. Substituting $K\phi$ for w in (5.5b):

$$M_x = -KD\oint\left\{y\left[\frac{\partial}{\partial n}(\nabla^2\phi) + (1-\nu_2)\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)\right]\right.$$
$$\left. + \frac{dx}{ds}\left[\frac{\partial^2\phi}{\partial n^2} + \nu_2\left(\frac{\partial^2\phi}{\partial s^2} + \frac{1}{\rho'}\frac{\partial\phi}{\partial n}\right)\right]\right\}ds. \tag{7.4}$$

Eliminating

$$\oint y\frac{\partial}{\partial n}(\nabla^2\phi)ds$$

between (7.4) and (4.9), we find

$$\frac{M_x}{KD} = -\oint\left\{x\frac{d(\nabla^2\phi)}{ds} + (1-\nu_2)y\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)\right.$$
$$\left. + \frac{dx}{ds}\left[\frac{\partial^2\phi}{\partial n^2} + \nu_2\left(\frac{\partial^2\phi}{\partial s^2} + \frac{1}{\rho'}\frac{\partial\phi}{\partial n}\right)\right]\right\}ds$$
$$-\frac{E_1(a+y_0c)}{1-\nu_1^2} + \frac{1-2\nu_1}{1-\nu_1}\oint\left(y\frac{dV}{dn} - x\frac{dV}{ds}\right)ds$$
$$+\frac{E_1\alpha_1}{1-\nu_1}\oint\left(y\frac{dT}{dn} - x\frac{dT}{ds}\right)ds + \frac{1}{1-\nu_1}\oint(\bar{Y} - Vm)ds. \tag{7.5}$$

Now,

$$\oint \left[x \frac{\partial(\nabla^2\phi)}{\partial s} + (1 - \nu_2) y \frac{\partial}{\partial s} \frac{\partial}{\partial n} \left(\frac{\partial \phi}{\partial s} \right) \right] ds = [x\nabla^2\phi]_0^0 + (1 - \nu_2) \left[y \frac{\partial}{\partial n} \left(\frac{\partial \phi}{\partial s} \right) \right]_0^0$$
$$- \oint \left[\frac{dx}{ds} \nabla^2\phi + (1 - \nu_2) \frac{dy}{ds} \frac{\partial}{\partial n} \left(\frac{\partial \phi}{\partial s} \right) \right] ds. \quad (7.6)$$

The terms outside the integral vanish on account of the assumption of continuity of the stress components. Therefore the first integral on the right hand side of (7.5) becomes

$$\oint \left\{ (1 - \nu_2) \frac{dy}{ds} \frac{\partial}{\partial n} \left(\frac{\partial \phi}{\partial s} \right) + \frac{dx}{ds} \left[\nabla^2\phi - \frac{\partial^2 \phi}{\partial n^2} - \nu_2 \left(\frac{\partial^2 \phi}{\partial s^2} + \frac{1}{\rho'} \frac{\partial \phi}{\partial n} \right) \right] \right\} ds. \quad (7.7)$$

On a boundary

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial n^2} + \frac{1}{\rho'} \frac{\partial \phi}{\partial n} + \frac{\partial^2 \phi}{\partial s^2}, \quad (7.8)$$

so that (7.7) becomes

$$(1 - \nu_2) \oint \left[\frac{dy}{ds} \frac{\partial}{\partial n} \left(\frac{\partial \phi}{\partial s} \right) + \frac{dx}{ds} \left(\frac{\partial^2 \phi}{\partial s^2} + \frac{1}{\rho'} \frac{\partial \phi}{\partial n} \right) \right] ds. \quad (7.9)$$

However, along a boundary,

$$\frac{\partial}{\partial n} \left(\frac{\partial \phi}{\partial s} \right) = - \tau_{ns}, \quad (7.10) \qquad \frac{\partial^2 \phi}{\partial s^2} + \frac{1}{\rho'} \frac{\partial \phi}{\partial n} = \sigma_n - V, \quad (7.11)$$

$$\frac{dy}{ds} \tau_{ns} - \frac{dx}{ds} \sigma_n = \overline{Y}. \quad (7.12)$$

Hence, (7.9) becomes

$$- (1 - \nu_2) \oint \left(\overline{Y} + V \frac{dx}{ds} \right) ds.$$

Substituting back in (7.5), we have, finally,

$$\frac{(1 - \nu_1) M_x}{KD} = - \frac{E_1(a + y_0 c)}{1 + \nu_1} + (1 - 2\nu_1) \oint \left(y \frac{dV}{dn} - x \frac{dV}{ds} \right) ds$$
$$+ E_1 \alpha_1 \oint \left(y \frac{dT}{dn} - x \frac{dT}{ds} \right) ds$$
$$- [(1 - \nu_2)(1 - \nu_1) + 1] \oint (\overline{Y} - Vm) ds. \quad (7.13)$$

This gives M_x in terms of known quantities.

iii. y-component of couple on a boundary of the slab. Substituting $K\phi$ for w in (5.5c),

$$M_y = -KD\oint\left\{\frac{dy}{ds}\left[\frac{\partial^2\phi}{\partial n^2} + \nu_2\left(\frac{\partial^2\phi}{\partial s^2} + \frac{1}{\rho'}\frac{\partial\phi}{\partial n}\right)\right] - x\left[\frac{\partial(\nabla^2\phi)}{\partial n} + (1-\nu_2)\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)\right]\right\}ds. \quad (7.14)$$

Eliminating

$$\oint x\frac{\partial}{\partial n}(\nabla^2\phi)ds$$

between (7.14) and (4.10), we have

$$\frac{M_y}{KD} = -\oint\left\{y\frac{d}{ds}(\nabla^2\phi) - (1-\nu_2)x\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right) + \frac{dy}{ds}\left[\frac{\partial^2\phi}{\partial n^2} + \nu_2\left(\frac{\partial^2\phi}{\partial s^2} + \frac{1}{\rho'}\frac{\partial\phi}{\partial n}\right)\right]\right\}ds$$
$$- \frac{E_1(b - x_0c)}{1-\nu_1^2} - \frac{1-2\nu_1}{1-\nu_1}\oint\left(y\frac{dV}{ds} + x\frac{dV}{dn}\right)ds$$
$$- \frac{E_1\alpha_1}{1-\nu_1}\oint\left(y\frac{dT}{ds} + x\frac{dT}{dn}\right)ds - \frac{1}{1-\nu_1}\oint(X - Vl)ds. \quad (7.15)$$

Now,

$$\oint\left[y\frac{d(\nabla^2\phi)}{ds} - (1-\nu_2)x\frac{\partial}{\partial s}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)\right]ds = [y\nabla^2\phi]_0^0 - (1-\nu_2)\left[x\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)\right]_0^0$$
$$-\oint\left[\frac{dy}{ds}\nabla^2\phi - (1-\nu_2)\frac{dx}{ds}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)\right]ds. \quad (7.16)$$

The terms outside the integrals in (7.16) vanish on account of the assumption of continuity of stresses. Therefore the first integral on the right hand side of (7.15) becomes

$$\oint\left\{\frac{dy}{ds}\left[\nabla^2\phi - \frac{\partial^2\phi}{\partial n^2} - \nu_2\left(\frac{\partial^2\phi}{\partial s^2} + \frac{1}{\rho'}\frac{\partial\phi}{\partial n}\right)\right] - (1-\nu_2)\frac{dx}{ds}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial s}\right)\right\}ds. \quad (7.17)$$

Then, using (7.8), (7.10) and (7.11) and noting that

$$\tau_{ns}\frac{dx}{ds} + \sigma_n\frac{dy}{ds} = \overline{X}, \quad (7.18)$$

(7.17) may be written in the form

$$(1-\nu_2)\oint(\overline{X} - Vl)ds. \quad (7.19)$$

Substituting back in (7.15), we have

$$\frac{(1-\nu_1)M_y}{KD} = -\frac{E_1(b-x_0c)}{1+\nu_1} - (1-2\nu_1)\oint\left(y\frac{dV}{ds}+x\frac{dV}{dn}\right)ds$$

$$-\ E_1\alpha_1\oint\left(y\frac{dT}{ds}+x\frac{dT}{dn}\right)ds$$

$$+\ [(1-\nu_2)(1-\nu_1)-1]\oint(\overline{X}-Vl)ds. \qquad (7.20)$$

8. Recapitulation. The stresses, in a multiply-connected slice, resulting from boundary loadings $\overline{X}$, $\overline{Y}$, a body force potential V, dislocations a, b, c and temperatures T, are related to the curvatures of a slab according to

$$\sigma_x = \frac{\kappa_x}{K} + V, \qquad \sigma_y = \frac{\kappa_y}{K} + V$$

if the following conditions are satisfied on the slab:
(i) The surface loading on the slab is

$$Z = -\frac{1-2\nu_1}{1-\nu_1}KD\nabla^2V - \frac{1+\nu_1}{1-\nu_1}KD\alpha_1\nabla^2T; \qquad (6.3)$$

(ii) The boundaries of the slab are geometrically identical with those of the slice, with elevations and normal slopes given by

$$\frac{w}{K} = \int_0^s (Bl-Am)ds + \alpha x + \beta y + \gamma, \quad \frac{1}{K}\frac{dw}{dn} = Al + Bm + \alpha l + \beta m, \qquad (6.5)$$

at each point of each boundary;
(iii) There are a resultant force (F_z) and resultant couples (M_x) and (M_y), on each boundary, with magnitudes given by

$$\frac{(1-\nu_1)F_z}{KD} = -\frac{E_1c}{1+\nu_1} + (1-2\nu_1)\oint\frac{dV}{dn}ds + E_1\alpha_1\oint\frac{dT}{dn}ds, \qquad (7.2)$$

$$\frac{(1-\nu_1)M_x}{KD} = -\frac{E_1(a+y_0c)}{1+\nu_1} + (1-2\nu_1)\oint\left(y\frac{dV}{dn}-x\frac{dV}{ds}\right)ds$$

$$+\ E_1\alpha_1\oint\left(y\frac{dT}{dn}-x\frac{dT}{ds}\right)ds$$

$$-\ [(1-\nu_1)(1-\nu_2)-1]\oint(\overline{Y}-Vm)ds, \qquad (7.13)$$

$$\frac{(1-\nu_1)M_y}{KD} = -\frac{E_1(b-x_0c)}{1+\nu_1} - (1-2\nu_1)\oint\left(y\frac{dV}{ds}+x\frac{dV}{dn}\right)ds$$

$$-\ E_1\alpha_1\oint\left(y\frac{dT}{ds}+x\frac{dT}{dn}\right)ds$$

$$+\ [(1-\nu_1)(1-\nu_2)-1]\oint(\overline{X}-Vl)ds. \qquad (7.20)$$