This formula holds even when k is large, provided that the function $F(x)$ can be fitted with reasonable accuracy over the range $2h$ by parabolic arcs.

To avoid an infinity at the origin, the integral actually evaluated was

$$I = \int_0^{12} \left[\frac{1}{1.4\alpha} - \frac{K_1^2(\alpha)}{\alpha D(\alpha)} \right] \sin \frac{c}{a} \alpha d\alpha,$$

and when this had been found, the required integral was given by

$$\frac{1}{1.4} \operatorname{Si}\left(\frac{12c}{a}\right) - I.$$

As a check that the substitution of the asymptotic series did not lead to unacceptable errors, the range of integration was also divided into 0 to 10, 10 to infinity and the infinite integral was similarly computed on this basis. Little extra work was involved and excellent agreement was obtained.

The results are shown below, together with those given by the approximate analysis by Westergaard. It is seen that even his second approximation is quite crude.

$\frac{c}{a}$	Values of $(u_{max})_{r=a}/(u')_{r=a}$		
	Westergaard		Present Method
	First Approximation	Second Approximation	
0.25	0.557	0.537	0.450
0.50	0.806	0.770	0.633

ON THE REPEATED INTEGRALS OF BESSEL FUNCTIONS*

By J. C. JAEGER (*University of Tasmania*)

It is well known that

$$L\left\{\frac{nJ_n(t)}{t}\right\} = [(p^2+1)^{1/2} - p]^n, \qquad n > 0, \tag{1}$$

and

$$L\{J_n(t)\} = \frac{[(p^2+1)^{1/2} - p]^n}{(p^2+1)^{1/2}}, \qquad n \geqq 0, \tag{2}$$

* Received Jan. 25, 1946.

where $L\{f(t)\}$ is written for the Laplace transform of $f(t)$, that is,

$$L\{f(t)\} = \int_0^\infty e^{-pt}f(t)dt. \tag{3}$$

These results have important applications in the theory of the semi-infinite dissipationless artificial transmission line with simple terminations, and thus in the expression of the solutions of corresponding problems on finite lines in terms of multiply reflected waves.

In an important class of similar problems in which the line is terminated by a matching resistance, the Laplace transforms of the solutions contain powers of $[1+(p^2+1)^{1/2}]$ or $[p+1+(p^2+1)^{1/2}]$ in the denominator, and the functions which have such Laplace transforms do not seem to have been given. The object of this note is to show that they can be expressed in terms of repeated integrals of Bessel functions and that numerical values of these can readily be obtained.

We use the notation

$$\begin{aligned} Ji_n^{(r)}(t) &= \int_0^t dt \cdots \int_0^t \frac{J_n(t)dt}{t}, \qquad n > 0, \\ Ji_{n,r}(t) &= \int_0^t dt \cdots \int_0^t J_n(t)dt, \qquad n \geqq 0, \end{aligned} \tag{4}$$

for the r-ple integrals of $J_n(t)/t$ and $J_n(t)$ respectively.

It is convenient to use both these types of integral though there are many relations between them, the simplest being

$$Ji_{n-1,r}(t) + Ji_{n+1,r}(t) = 2nJi_n^{(r)}(t) \tag{5}$$

and

$$Ji_{n-1,r}(t) - Ji_{n+1,r}(t) = 2Ji_{n,r-1}(t), \tag{6}$$

which follow immediately from the recurrence formulae for $J_n(t)$. $Ji_{0,1}(t)$ is tabulated[1] and $Ji_n^{(1)}(t) = (1/n) + Ji_n(t)$ where $Ji_n(t)$ is the ordinary Bessel integral function. For all values of n and r repeated application of the result

$$\int_0^t J_n(t)dt = 2\sum_{m=0}^\infty J_{n+2m+1}(t)$$

gives the formulae

$$Ji_{n,r}(t) = 2^r \sum_{m=0}^\infty \binom{m+r-1}{m} J_{n+2m+r}(t) \tag{7}$$

$$nJi_n^{(r)}(t) = 2^{r-1}\sum_{m=0}^\infty \frac{(2m+r-1)}{(m+r-1)}\binom{m+r-1}{m} J_{n+2m+r-1}(t). \tag{8}$$

For integral values of t, which are in fact close enough for many practical purposes, (7) and (8) may be evaluated rapidly from the Tables in Gray and Mathews.[2]

[1] Lowan and Abramowitz, J. Math. and Phys., **22**, 2 (1943).

[2] Gray and Mathews, *Treatise on Bessel functions*, 2nd ed., 1922, Table II.

The Laplace transforms referred to above may now be written down. Firstly we have immediately from (1) and (2)

$$L\{nJi_n^{(r)}(t)\} = p^{-r}[(p^2+1)^{1/2} - p]^n \tag{9}$$

$$L\{Ji_{n,r}(t)\} = \frac{[(p^2+1)^{1/2}-p]^n}{p^r(p^2+1)^{1/2}}. \tag{10}$$

Then, since

$$\frac{1}{1+(p^2+1)^{1/2}} = \left(1+\frac{1}{p^2}\right)\frac{1}{(p^2+1)^{1/2}} - \frac{1}{p^2},$$

it follows that

$$L\{J_0(t) + Ji_{0,2}(t) - t\} = \frac{1}{1+(p^2+1)^{1/2}}. \tag{11}$$

In the same way if $n>0$

$$L\{J_n(t) + Ji_{n,2}(t) - nJi_n^{(2)}(t)\} = \frac{[(p^2+1)^{1/2}-p]^n}{1+(p^2+1)^{1/2}}. \tag{12}$$

Similarly

$$L\{J_n(t) + 3Ji_{n,2}(t) + 2Ji_{n,4}(t) - 2nJi_n^{(2)}(t) - 2nJi_n^{(4)}(t)\} = \frac{(p^2+1)^{1/2}[(p^2+1)^{1/2}-p]^n}{[1+(p^2+1)^{1/2}]^2}, \tag{13}$$

if $n>0$, and if $n=0$ the term $nJi_n^{(r)}(t)$ is to be replaced by $t^{r-1}/(r-1)!$. Again with this convention we have

$$L\{\tfrac{1}{2}nJi_n^{(1)}(t) - \tfrac{1}{2}(n+1)Ji_{n+1}^{(1)}(t)\} = \frac{[(p^2+1)^{1/2}-p]^n}{p+1+(p^2+1)^{1/2}}, \tag{14}$$

$$L\{J_{n+2}(t) - 2J_{n+1}(t) + J_n(t) + Ji_{n+2,2}(t) - 2Ji_{n+1,2}(t) + Ji_{n,2}(t)\} = \frac{4(p^2+1)^{1/2}[(p^2+1)^{1/2}-p]^n}{[p+1+(p^2+1)^{1/2}]^2}. \tag{15}$$

These expressions may be transformed in many ways using (5) and (6) and general results for higher powers in the denominators[3] may be obtained in the same way.

As an example of the way in which the above functions arise, we consider a semi-infinite artificial transmission line with mid-series termination, in which the series elements are inductances L and the shunt elements are condensers of capacity C. Suppose that all condensers are charged to unit potential, and that at time $t=0$ the line is discharged through the matching resistance $\sqrt{(L/C)}$. Then if I_0 is the current in the resistance, I_n that in the nth inductance L, and Cv_n is the charge on the nth condenser, applying the Laplace transformation method in the usual way we find that

$$L\{I_r\} = \frac{aC[(1+p^2/a^2)^{1/2} - p/a]^{2r}}{2p[1+(1+p^2/a^2)^{1/2}]}, \qquad r = 0, 1, \cdots \tag{16}$$

[3] The extension of (15) is trivial; for that of (13) the results needed are given in Chrystal, *Textbook of algebra*, 2nd ed., 1906, vol. 2, pp. 204–205.

$$L\{v_r\} = \frac{1}{p} - \frac{[(1 + p^2/a^2)^{1/2} - p/a]^{2r-1}}{p[1 + (1 + p^2/a^2)^{1/2}]}, \qquad r = 1, 2, \cdots, \tag{17}$$

where $a = 2(LC)^{-1/2}$.

It follows from (12) that

$$v_r = 1 - Ji_{2r-1,1}(at) - Ji_{2r-1,3}(at) + (2r - 1)J\overset{(3)}{i}_{2r-1}(at) \tag{18}$$

$$I_r = (C/L)^{1/2}\{Ji_{2r,1}(at) + Ji_{2r,3}(at) - 2rJ\overset{(3)}{i}_{2r}(at)\} \tag{19}$$

$$I_0 = (C/L)^{1/2}\{Ji_{0,1}(at) + Ji_{0,3}(at) - \tfrac{1}{2}a^2t^2\} \tag{20}$$

$$= \tfrac{1}{2}(C/L)^{1/2}\{(1 + a^2t^2)Ji_{0,1}(at) - a^2t^2(1 + J_1(at)) + atJ_0(at)\}, \tag{21}$$

where (21) follows from (20) by integration by parts.

If the line is discharged into inductance $\frac{1}{2}L$ and resistance $\sqrt{(L/C)}$ in series, the solution follows from (14) in place of (12).

ON CERTAIN INTEGRALS IN THE THEORY OF HEAT CONDUCTION*

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In a recent note[1] W. Horenstein evaluates the integrals

$$\phi \equiv \int_0^t x^{-3/2} \exp\left(-\frac{a^2}{x} - b^2x\right)dx, \tag{1}$$

$$\psi \equiv \int_0^t x^{-1/2} \exp\left(-\frac{a^2}{x} - b^2x\right)dx, \tag{2}$$

in terms of the tabulated exponential and error functions. The evaluation of the more general integral, viz.

$$\int_k^l \exp\,(-s^2 - n^2/s^2)ds$$

from which ϕ and ψ are easily derived, was given by Riemann.[2]

Integrals of the above type arise in the solution by classical methods of various heat conduction problems. It is the purpose of this note to point out that treatment of many such problems by the Heaviside "operational" or equivalent Laplace transform method leads directly and naturally to the required solution in tabulated functions.

Thus, to take a simple case, the classical solution of

$$\frac{\partial\theta}{\partial t} = \frac{1}{4}\frac{\partial^2\theta}{\partial a^2} - b^2\theta; \qquad \theta \to 0,\ t \to 0,\ \theta \to 1,\ a \to 0+, \tag{3}$$

* Received Nov. 24, 1945.

[1] W. Horenstein, Quart. Appl. Math. **3**, 183–184 (1945).

[2] B. Riemann, *Partielle Differentialgleichungen*, 2nd ed., 1876, p. 173.