AERODYNAMICS OF THIN QUADRILATERAL WINGS AT SUPERSONIC SPEEDS*

BY

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1. Introduction. The method of conical fields, due to Busemann,[1] furnishes a powerful tool for the study of linearized supersonic flow around obstacles of special type. An appropriate application is to plane (infinitely thin) wings at small angles of attack. The present paper is devoted to a study of the supersonic aerodynamics of some of the more interesting special cases of quadrilateral planform. The application of the formulae obtained to a more general polygonal planform is suggested, and a procedure is given for accurately taking account of a dihedral bend in the wing. Modifications due to wing thickness, viscous effects, and interference effects (with a fuselage or with other wings) are beyond the scope of this method.

Since the general theory has been discussed by Stewart[2] recently in this journal, a brief description of the method will suffice. A conical field corresponds to a linear homogeneous solution of the linearized (or Prandtl-Glauert) potential equation for supersonic flow:

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}-(M^2-1)\frac{\partial^2\Phi}{\partial z^2}=0. \tag{1}$$

Here M is the Mach number of the main stream, which is moving along the z-axis. The perturbation velocity components (u, v, w) are also solutions of Eq. (1) and are homogeneous of degree zero, i.e., u, v, and w are constant along any ray emanating from the origin. The particular simplicity of conical fields lies in the fact that after a transformation[3] the perturbation velocity components are obtained a ssolutions of Laplace's equation in two variables. Let x, y, z be rectangular coordinates with respect to a set of axes fixed in the wingtip, with the z-axis pointing downstream, the y-axis normal to the wing, and the x-axis directed spanwise away from the wing. If

$$\phi = \text{arc tan } y/x, \qquad R=\sqrt{x^2+y^2}/z, \qquad A=\tan\mu=(M^2-1)^{-1/2},$$

then the transformation

$$r=\left\{\frac{1-\sqrt{1-R^2/A^2}}{1+\sqrt{1-R^2/A^2}}\right\}^{1/2}=\frac{A}{R}-\sqrt{\frac{A^2}{R^2}-1}, \qquad R=2Ar/(1+r^2) \tag{2}$$

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[1] A. Busemann, *Infinitesimale kegelige Ueberschallstroemung*, Schriften Deutsch. Akad. Luftfahrtforschung, **7**, 105–121 (1943).

[2] H. J. Stewart, *The lift of a Delta wing at supersonic speeds*, Quart. Appl. Math. **4**, 246–254 (1946).

[3] Busemann credits this transformation to Chaplygin, who made use of it in a formally similar problem. Cf. S. Chaplygin. *Gas jets*, Part V, Scientific Memoirs, Moscow University, 1902. Translated from the Russian as NACA Tech. Memo. No. 1063. See also the translation published in mimeographed form by Brown University in 1944.

is such that homogeneous functions of degree zero which satisfy Eq. (1) also satisfy Laplace's equation in the polar coordinates (r, ϕ). The evaluation of the streamwise component (w) of perturbation velocity is of primary importance since the aerodynamic forces on the wing are determined by w alone. This follows from the linearized Bernoulli equation,

$$p = p_0 - \rho_0 W w, \tag{3}$$

which like Eq. (1), results from neglecting squares and products of perturbation velocities in the corresponding exact equation, In many problems of this type, including those considered here, the boundary conditions are such that w may be determined directly without further reference to the components u and v.

2. **Characteristic cones. Boundary conditions.** To illustrate the type of boundary condition needed to determine the conical field, consider the special case of a rectangular wing. This rectangular wing may be regarded as the result of cutting the ends off of a two-dimensional airfoil. This operation causes a modification in the (originally two-dimensional) flow; this modification may be referred to as the "tip effect." In this connection a fundamental distinction should be made between subsonic flow and supersonic flow. For subsonic flow (differential equation of elliptic type) the tip effect dies off asymptotically with increasing distance inboard. For

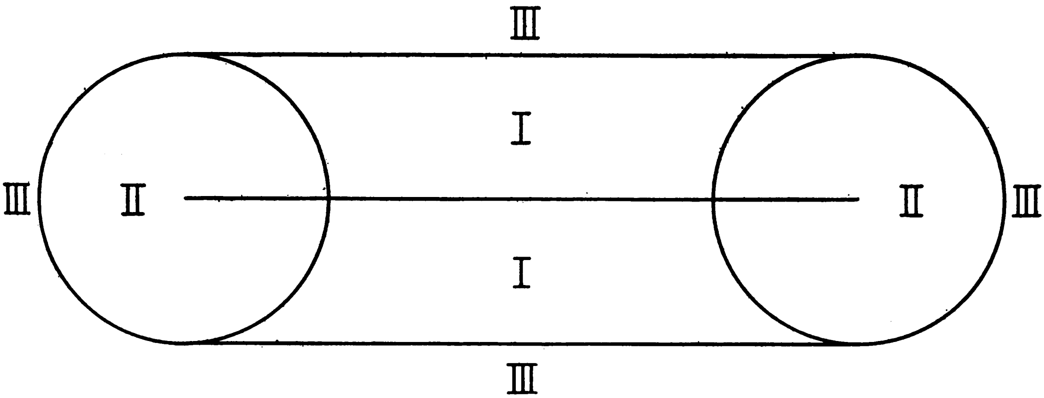

FIG. 1. Flow for rectangular wing, section by plane perpendicular to stream.
I—Flow same as for flow about infinite span airfoil.
II—Conical field.
III—Free stream.

supersonic flow (differential equation of hyperbolic type) the tip effect falls to zero at a certain finite distance, and the entire effect is contained within a region bounded by real characteristic surfaces. For linearized supersonic flow, the domain of influence of any point is bounded by a "Mach cone," which is (one nappe of) a cone opening downstream with semi-vertex angle equal to the Mach angle μ. Fig. 1 represents a section by a plane perpendicular to the main stream. The Mach cones from the tips of the leading edge divide this plane into three types of regions. In the central region (I) the flow is in all respects the same as if the wing were of infinite span, since no point of this central region lies in the domain of influence of any point removed in the mental process of obtaining the rectangular wing from an airfoil of infinite span. On the other hand the perturbation velocity components are zero in the exterior region (III), since no point of this region lies in the domain of influence of any point on the rectangular wing. The requirement of continuity leads to boundary conditions which must be satisfied by the perturbation velocity components u, v, w on the boundary of each conical transition region (II). In the boundary between region (II) and (III), u, v, and w must vanish. On the boundary between regions (I) and (II), u, v, w take on the (constant, two-dimensional) values of region (I).

Similar remarks apply to the example of a swept-back leading edge (Fig. 2). The two lines forming the leading edge are of course finite in the actual case, but the effect of their finiteness, so to speak, will be confined by characteristic cones passing through the points at which the leading edge changes direction. The possibility of treating

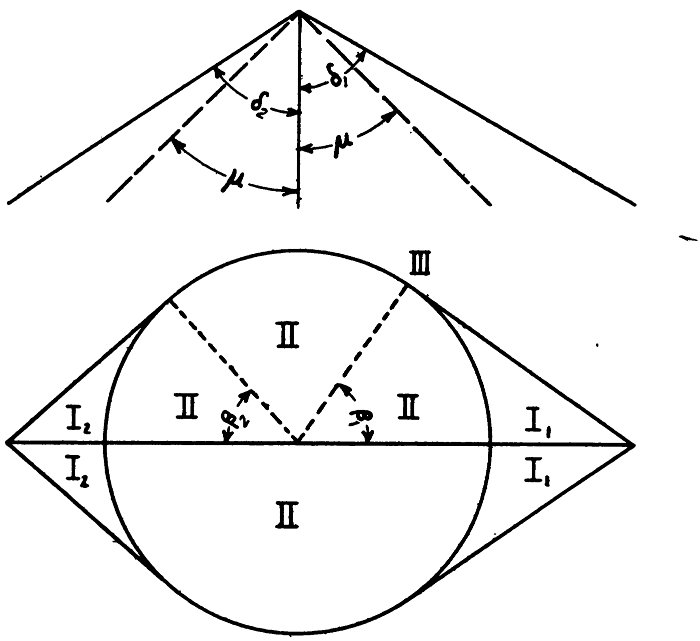

FIG. 2. Flow for swept-back leading edge.
Above, planform for definition of symbols. Airflow is toward bottom of page.
Below, section by plane perpendicular to stream.
I_1—Flow same as for infinite span airfoil at angle of yaw $\pi/2-\delta_1$.
I_2—Flow same as for infinite span airfoil at angle of yaw $\pi/2-\delta_2$.
II —Conical field.
III —Free stream.

more general polygonal wings by this method follows from these remarks. For the polygons which may be thus treated there are isolated regions of uniform flow (identical with the flow for an infinite wing at a certain angle of yaw) separated by regions of transition in the Mach cones which start from each vertex of the polygon and open downstream. It is convenient in the following to refer to these special Mach cones simply as "the Mach cones" or "the Mach cone."

Boundary conditions must also be given over that part of the wing which lies inside the Mach cone. The velocity component normal to the wing must have the same value as on the rest of the wing, because of the condition that there be no flow through the wing. The boundary condition for w is that the normal derivative of w be zero on the wing. This follows from the requirement of irrotationality (implicit in the use of the potential Φ), so that $\partial w/\partial y = \partial v/\partial z$, and the condition that v is constant on the wing. The fact that the wing does not lie exactly in the plane $y=0$ is neglected; this has no effect on the first order perturbation. This simplification is made throughout, so that the angle of attack enters only in the boundary values and not in the position of the boundaries.

3. Bent leading edge. The basic flow problem which we shall solve is sketched in Fig. 2. The angles δ_1 and δ_2 are not necessarily acute, as in the case drawn, but each is assumed to lie in the range $\mu < \delta < \pi - \mu$. To simplify the immediate discussion it is assumed that the angle of the leading edge points upstream, so that $\delta_1 + \delta_2 < \pi$; it is shown in the next section that the formulae obtained are valid without this restriction. Evidently the wing cleaves the problem into two separate problems which may be treated independently. Attention may be confined to the upper half, since the solution for the lower half differs only in sign. The points on the circle for which $\phi = \beta_1$ and $\phi = \pi - \beta_2$ mark the tangency of the plane Mach waves from the leading edge with the Mach cone. By elementary geometry, $\cos \beta_1 = \tan \mu / \tan \delta_1$, $\cos \beta_2 = \tan \mu / \tan \delta_2$. From a consideration of the "domains of influence," it is seen that the flow in the regions I_1, I_2, and III is as indicated in the legend to Fig. 2. The normal derivative of w vanishes on the wing. All three components of perturbation velocity vanish on the arc $\beta_1 < \phi < \pi - \beta_2$; on the other two arcs the boundary conditions are to be obtained from the essentially two-dimensional problem of an infinite span airfoil at an angle of yaw, to which we now turn.

Let δ be the angle between the main stream direction and the leading edge of an airfoil of infinite span, so that $(\pi/2) - \delta$ is the angle of yaw. It is assumed that $\mu < \delta < \pi - \mu$. The uniform flow W may be considered as a superposition of a uniform flow at velocity $W \cos \delta$ parallel to the leading edge, giving rise to no perturbation, and a flow perpendicular to the leading edge at velocity $W_1 = W \sin \delta$ and effective Mach number $M_1 = M \sin \delta$. The effective angle of attack (α_1) for this second flow is measured in a plane perpendicular to the leading edge; it is related to the streamwise angle of attack (α), measured in a plane containing the stream direction and perpendicular to the plane of the wing for zero angle of attack, by the formula

$$\tan \alpha = \tan \alpha_1 \sin \delta.$$

Within the limits of validity of the linear theory we need not distinguish between the angle of attack and its tangent, so that $\alpha = \alpha_1 \sin \delta$. For a plane airfoil of infinite span and not yawed, the streamwise component (w) of perturbation velocity is

$$w = \alpha W \tan \mu = \alpha W / \sqrt{M^2 - 1} \equiv w_\infty.$$

To obtain the *chordwise* component of perturbation velocity for an airfoil at an angle of yaw $(\pi/2) = \delta$, replace α, W, M by α_1, W_1, M_1. The *streamwise* component of perturbation velocity follows from multiplication by $\sin \delta$:

$$w = \alpha_1 W_1 \sin \delta / \sqrt{M_1^2 - 1} = \alpha W \sin \delta / \sqrt{M^2 \sin^2 \delta - 1},$$

or

$$w = w_\infty / \sin \beta, \tag{4}$$

where, as before, $\cos \beta = \tan \mu / \tan \delta$.

To summarize, the boundary conditions for w on the upper semicircle are

$$\begin{aligned} w &= w_\infty / \sin \beta_1 \equiv K_1 \quad \text{for} \quad 0 < \phi < \beta_1, \qquad r = 1, \\ w &= 0 \qquad\qquad\qquad\quad \text{for} \quad \beta_1 < \phi < \pi - \beta_2, \ r = 1, \\ w &= w_\infty / \sin \beta_2 \equiv K_2 \quad \text{for} \quad \pi - \beta_2 < \phi < \pi, \ r = 1. \end{aligned}$$

The potential problem which is now uniquely determined in the upper semicircle may be written out immediately as a Fourier series. This is a cosine series only, because of the condition that $\partial w/\partial n = 0$ for $\phi = 0$ and for $\phi = \pi$.

$$\begin{aligned} w &= \frac{\beta_1 K_1 + \beta_2 K_2}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} r^n \cos n\phi \left\{ K_1 \int_0^{\beta_1} \cos nu du + K_2 \int_{\pi-\beta_2}^{\pi} \cos nu du \right\} \\ &= \frac{\beta_1 K_1 + \beta_2 K_2}{\pi} + \frac{K_1}{\pi} \left\{ \arctan \frac{r \sin (\phi + \beta_1)}{1 - r \cos (\phi + \beta_1)} - \arctan \frac{r \sin (\phi - \beta_1)}{1 - r \cos (\phi - \beta_1)} \right\} \\ &\quad - \frac{K_2}{\pi} \left\{ \arctan \frac{r \sin (\phi + \beta_2)}{1 + r \cos (\phi + \beta_2)} - \arctan \frac{r \sin (\phi - \beta_2)}{1 + r \cos (\phi - \beta_2)} \right\}. \end{aligned} \tag{5}$$

In Eq. (5) each inverse tangent is restricted to its principal values ($-\pi/2$ to $+\pi/2$).

In the symmetrical case $\delta_1 = \delta_2$, the expression for w on the wing ($\phi = 0, \pi$) simplifies to

$$w = \frac{2K}{\pi} \left\{ \beta + \arctan \frac{r^2 \sin 2\beta}{1 - r^2 \cos 2\beta} \right\} = \frac{2}{\pi} \frac{w_\infty}{\sin \beta} \arctan \frac{\tan \beta}{\sqrt{1 - R^2/A^2}} \cdot \tag{6}$$

Returning to the general case (δ_1 and δ_2 not necessarily equal) we seek the average value of w along the segment of a spanwise line (i.e., perpendicular to the stream) cut off by the Mach cone. It is necessary to evaluate integrals such as

$$\frac{1}{A} \int_0^A \arctan \frac{r \sin \gamma}{1 - r \cos \gamma} dR.$$

Since $R/A = 2r/(1 + r^2)$, integration by parts leads to

$$\frac{1}{A} \int_0^A \arctan \frac{r \sin \gamma}{1 - r \cos \gamma} dR = (\pi \tan \gamma)/4 + (1 - \sec \gamma)(\pi - \gamma)/2, \qquad (0 < \gamma < 2\pi).$$

Using this result it is found that the average value sought is

$$\bar{w} = [K_1(1 + \tan \beta_1 - \sec \beta_1) + K_2(1 + \tan \beta_2 - \sec \beta_2)]/2. \tag{7}$$

4. Case of intersecting envelopes. For treating forward sweep or dihedral, it is necessary to discuss the plane waves from the leading edge in more detail. On one side there is a weak shock wave and on the other side a weak expansion wave; however, in the linear theory the distinction between shock waves and expansion waves disappears.* Both are regarded merely as surfaces of discontinuity, which can occur only across the *envelope* of Mach cones with vertex on the leading edge. As the simplest example Fig. 3 shows a section perpendicular to the main stream for the case of a wing with dihedral and with the leading edge perpendicular to the main stream. The situation for a plane wing with forward sweep would differ only in that the trace LMN of the wing in Fig. 3 would be straight, and overlapping of the plane waves would occur over the bottom arc as well as the top.

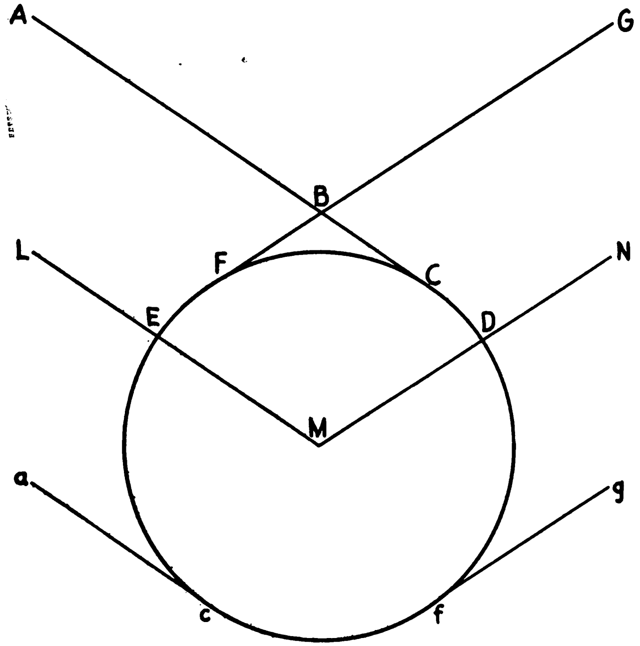

Fig. 3. Wing with dihedral, section by plane perpendicular to stream. In case drawn, leading edge is perpendicular to stream.

The envelope of all Mach cones with vertex on LM consists of two half planes with traces AC and ac. Similarly the Mach cones with vertex on MN give rise to the envelope represented by FG and fg. It is important to notice that no arc of the circle is part of either envelope. Within the bounds of the linear theory, shock waves or rarefaction waves intersect without mutual interference, and the perturbations caused by each are additive. In the region $GBCDN$ the flow is uniform; in the region $ABFEL$ there is another uniform flow; in the region between FBC and the circle the flow is also uniform, since the components of the perturbation velocity are obtained by addition of the components of perturbation velocity for the other two uniform flows. This completes the specification of the boundary conditions for the upper part of the circle. Whether we are dealing with u, v, or w, the sought function assumes a constant value of EF, another constant value of CD (both of these constants ob-

* The entropy increase is of the third order in the perturbation velocity components, whereas the linear theory retains only the first order. It is this fact which makes possible the usual two-dimensional linear and second order calculations, in which the pressure is determined by *local* conditions and does not depend on the history of the flow up to that point.

tainable from two dimensional theory), and a constant value of the arc FC, namely the sum of the other two constants. The boundary conditions for the lower part of Fig. 3 present nothing new; u, v, and w take on calculable (constant) values on the arcs Ec and fD, and the value zero on the arc cf.

Review of the problem illustrated by Fig. 2 now shows that the analysis given holds also for the case of a wing with forward sweep, i.e., with the angle pointing downstream; the only difference is a slight modification of Fig. 2.

5. Trapezoidal wing. We are now in a position to study the lift coefficient and center of pressure for the symmetrical trapezoidal wing shown in Fig. 4. The leading and trailing edges are perpendicular to the main stream, and the tip angle (δ) is

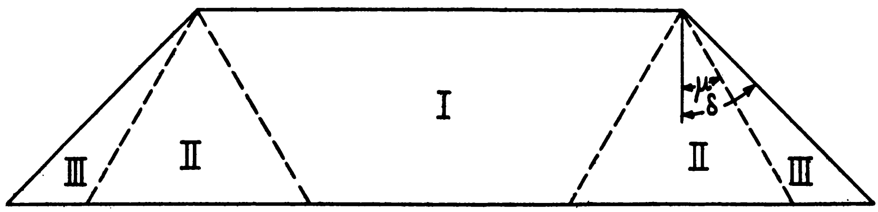

FIG. 4.

greater than the Mach angle. Since the leading edge of perpendicular to the stream, $\delta_2=\beta_2=\pi/2$, and the subscript may be dropped from δ_1 and β_1. In the region I, $\bar{w}/w_\infty=1$. In the region II, the average w is found from Eq. (7):

$$\bar{w}/w_\infty = 1/2 + (1 + \tan\beta - \sec\beta)/2\sin\beta.$$

In the region III, $\bar{w}/w_\infty=1/\sin\beta$. On taking the average of these quantities, weighted according to the area in which each applies, it is found that

$$C_L/C_{L\infty} = 1, \tag{8}$$

where $C_{L\infty}$ is the lift coefficient for an infinite span airfoil with leading edge perpendicular to the stream. Similarly the center of pressure is found to lie behind the leading edge by the distance

$$z = c(1 + c\tan\delta/3s)/2. \tag{9}$$

Thus in this case *the lift coefficient and center of pressure are the same as if the wing were subject to the uniform lift distribution of an infinite span airfoil.* The actual lift is not uniform; in the region I the lift is that of an infinite span airfoil; in the region II the

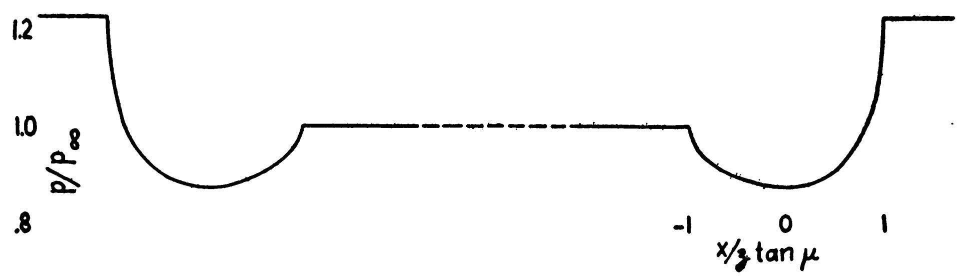

FIG. 5. Spanwise pressure distribution for wing shown in Fig. 4. $\delta=45°$, $\mu=30°(M=2)$.

life is less, and in the region III the lift is greater by just enough to compensate for the decreased lift in II. As an example, Fig. 5 shows the spanwise pressure distribution for the case $\delta=45°$, $\mu=30°$ ($M=2$).

The above discussion, and the manner of drawing Figs. 4 and 5, assumed that the two Mach cones from the tips do not intersect on the wing. However, this restriction is seen to be unnecessary. If Φ_1 is the (disturbance) potential inside one of the cones, Φ_2 the potential inside the other cone, and Φ_∞ the potential for a wing of infinite span with leading edge perpendicular to the stream, then in the region common to the two cones the potential is $\Phi=\Phi_1+\Phi_2-\Phi_\infty$. To verify this, it may be noted that Φ_1, Φ_2 and Φ_∞ are solutions of the Prandtl-Glauert Eq. (1), and since that equation is linear, Φ is a solution as well. Also Φ and its first derivatives (u, v, w) are continuous across the conical surfaces bounding the region in question. We may say that the "tip effects" from the two tips are additive, since the equation defining Φ may be written

$$(\Phi_\infty - \Phi) = (\Phi_\infty - \Phi_1) + (\Phi_\infty - \Phi_2).$$

(It will be noticed that the flow in the region in question is not a conical field. In general, the result of superposing two conical fields with different vertices is not a conical field, although it approaches a conical field asymptotically downstream.) Utilizing this result, it is easily seen that the results for lift coefficient and center of pressure are unaffected by the overlapping of the two conical fields.

6. General symmetrical quadrilateral. We turn now to the problem of a quadrilateral which is symmetrical about a diagonal, that diagonal being parallel to the stream and of length c. The semi-vertex angles at the nose and tail, say δ and δ_1 respectively, are not necessarily acute angles (see Fig. 7 for the various possibilities); it is assumed only that each lies in the range μ to $\pi-\mu$. It is, of course, necessary that $\delta+\delta_1<\pi$.

The forward pointing triangle is a special case, $\delta_1=\pi/2$. It is also a special case of the trapezoid; setting $s_L=0$ or $s=c\tan\delta$ in Eqs. (8) and (9) leads to

$$C_L/C_{L\infty} = 1, \qquad z = 2c/3. \tag{10}$$

Here c is the distance from the vertex to the trailing edge. Fig. 6 shows the pressure distribution spanwise for the case $\delta=45°$, $\mu=30°$ ($M=2$). Fig. 6 is, of course, applicable to any other case for which $\tan\delta=\sqrt{3}\tan\mu$, by a uniform change of scale.

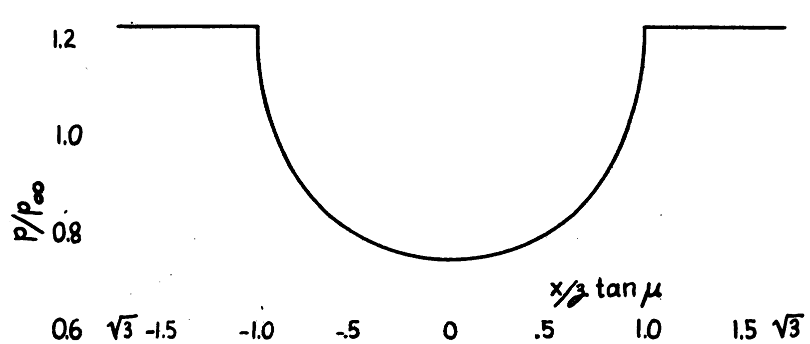

FIG. 6. Spanwise pressure distribution for forward pointing triangle. Semi-vertex angle $\delta=45°$. Mach angle $\mu=30°$ ($M=2$).

For the general quadrilateral of Fig. 7, the pressure distribution is given by Eq. (6), in connection with Eq. (3). In deriving Eq. (6) it was perhaps tacitly implied that the planform of the wing consists of two *semi-infinite* lines, but a consideration of "domains of influence" shows that the requirement $\delta_1 > \mu$ is sufficient to ensure

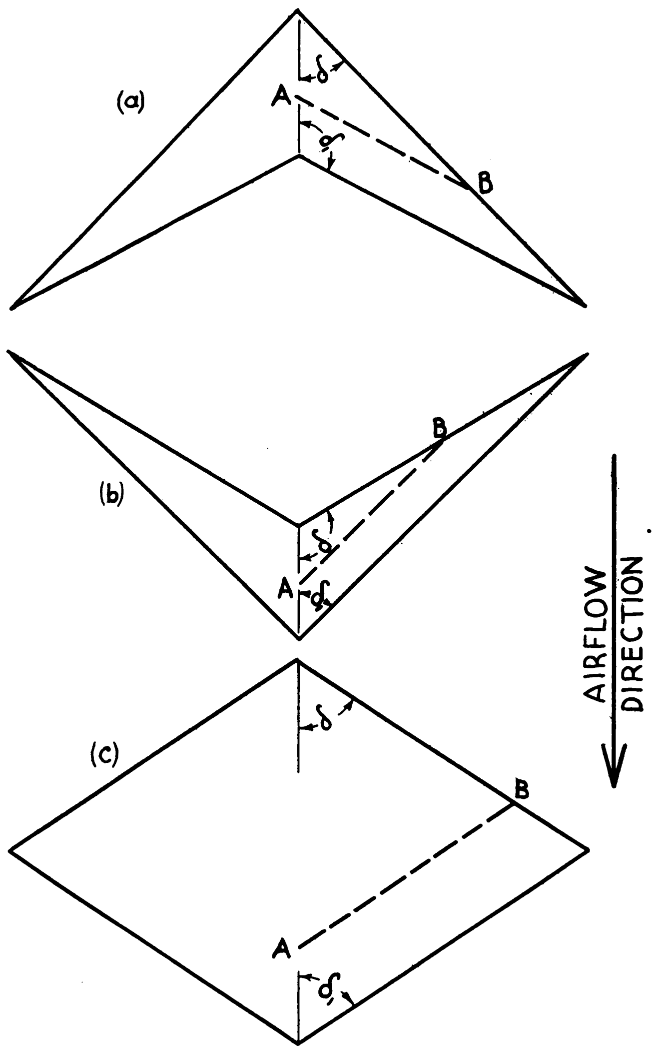

FIG. 7.

that Eq. (6) holds at every point on the wing. The most convenient procedure for finding the average pressure over the wing is to integrate along any line (AB in Fig. 7) parallel to the trailing edge on one side. This gives for the average value of w, by Eq. (6),

$$\bar{w} = \left\{\frac{w_\infty}{\sin\beta}\left[\int_0^{t_1} \frac{2}{\pi} \arctan\frac{\tan\beta}{\sqrt{1-\lambda^2}}\, dt + \int_{t_1}^{t_2} dt\right]\right\} \Big/ \int_0^{t_2} dt,$$

where

$$\lambda = R/A, \qquad t = \lambda \sec \delta_1/(\lambda + \sec \beta_1),$$

$$t_1 = t]_{\lambda=1} = \sin \mu/\sin (\mu + \delta_1), \qquad t_2 = t]_{\lambda=\sec \beta} = \sin \delta/\sin (\delta + \delta_1).$$

The integration leads eventually to

$$C_L/C_{L\infty} = \bar{w}/w_\infty = \frac{2}{\pi} \frac{\beta_1 \sin 2\beta - \beta \sin 2\beta_1}{\sin \beta_1 \sin 2\beta - \sin \beta \sin 2\beta_1}. \tag{11}$$

The angles β and β_1 are, of course, to be measured in radians. This expression for $C_L/C_{L\infty}$ is symmetric in the two angles β and β_1, and therefore *is unchanged by interchanging δ and δ_1*.

The center of pressure is at $z=2c(1-\bar{t}\cos\delta_1)/3$, where $\bar{t}$ is the weighted average of t along AB, with weights proportional to the pressure. It is found that

$$\bar{t}\cos\delta_1 = \frac{\int_0^{t_1} t\cdot\frac{2}{\pi}\arctan\frac{\tan\beta}{\sqrt{1-\lambda^2}}\,dt + \int_{t_1}^{t_2} t\,dt}{\int_0^{t_1} \frac{2}{\pi}\arctan\frac{\tan\beta}{\sqrt{1-\lambda^2}}\,dt + \int_{t_1}^{t_2} dt}\cos\delta_1 \tag{12}$$

$$= \frac{1}{2}\frac{\cos^2\beta_1 + \cos^2\beta}{\cos^2\beta_1 - \cos^2\beta} + \frac{\sin 2\beta}{2\sin^2\beta_1}\cdot\frac{\sin 2\beta_1 - 2\beta_1\cos 2\beta_1}{2\beta_1\sin 2\beta - 2\beta\sin 2\beta_1}. \tag{13}$$

These formulae contain as special cases the forward pointing triangle ($\beta_1=\pi/2$) and the backward pointing triangle ($\beta=\pi/2$). For these triangles, the invariance of lift coefficient (and in this particular case, the center of pressure also) with respect to reversal of the flow direction may be easily verified. It has already been shown that lift coefficient and center of pressure for the forward pointing triangle are the same as if the pressure were uniform (which it is not; cf. Fig. 6). For the backward pointing triangle the pressure *is* uniform. Reversal of direction of flow thus causes a radical change in pressure distribution in general, even though it does not alter the lift coefficient, for these quadrilateral wings.

Another special case of interest is the diamond ($\delta_1=\delta$, $\beta_1=\beta$). In this case Eqs. (11) and (12) reduce to

$$C_L/C_{L\infty} = (\sin 2\beta - 2\beta\cos 2\beta)/\pi\sin^3\beta \tag{14}$$

$$\frac{z}{c} = \frac{1 - 2\beta\sin^2 2\beta/3(\sin 2\beta - 2\beta\cos 2\beta)}{1-\cos 2\beta}. \tag{15}$$

Using Eqs. (14) and (15), the lift coefficient and center of pressure for a diamond have been evaluated and are presented in the following table. It is seen at once that the property of invariance under reversal of flow direction, which was found to hold for C_L, does not in general apply to the center of pressure. If such were the case the center of pressure of the diamond would necessarily be at $z=c/2$. From Table 1 it is seen that the center of pressure of a diamond actually lies forward of the midpoint, though never forward of $7c/15$ as long as $\delta>\mu$.

TABLE 1

β	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
$C_L/C_{L\infty}$	.8488	.8511	.8592	.8720	.8897	.9120	.9376	.9646	.9885	1.0000
z/c	.4667	.4671	.4658	.4709	.4743	.4788	.4842	.4905	.4966	.5000
$\tan\delta/\tan\mu$	1.000	1.015	1.064	1.155	1.305	1.556	2.000	2.924	5.759	∞

7. **Dihedral.** The case of a wing with dihedral, which may be combined with forward or backward sweep at the dihedral point, is reducible by a conformal transformation to the case of a simple bend in the leading edge of a flat wing. In Fig. 8, the angle $\phi=0$ has been taken in the wing (with no loss of generality). Let γ be the radian measure of the arc of the circle subtended by the wing. The part of the circle not drawn refers to an independent problem of the same type, with a different γ. The angles β_1 and β_2 have the same meaning as in the plane case. In the case shown, $\beta_1+\beta_2<\gamma$ (which incidentally corresponds to considerable sweepback in this case), but this condition is not essential, and is assumed here merely to simplify the drawing.

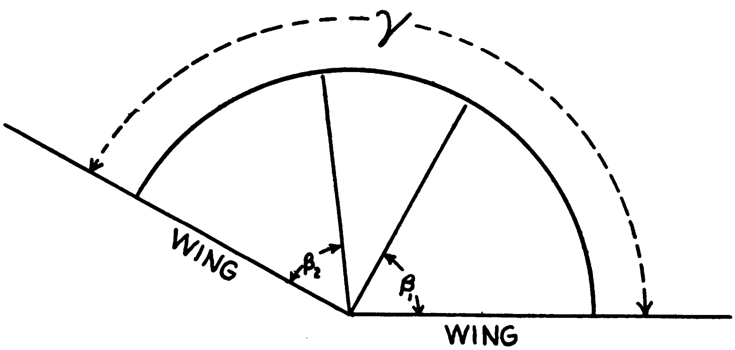

FIG. 8.

The potential problem to be solved in the r, ϕ plane involves the following boundary conditions:

$$\begin{aligned} \partial w/\partial\phi &= 0 \quad \text{for} \quad \phi = 0 \text{ or } \gamma, \ (0 < r < 1), \\ w &= K_1 \quad \text{for} \quad 0 < \phi < \beta_1, \qquad r = 1, \\ w &= 0 \quad \text{for} \quad \beta_1 < \phi < \gamma - \beta_2, \ r = 1, \\ w &= K_2 \quad \text{for} \quad \gamma - \beta_2 < \phi < \gamma, \ r = 1. \end{aligned}$$

The transformation $\epsilon'=\epsilon^{\pi/\gamma}$ maps the relevant part of the unit circle in the $\epsilon=re^{i\phi}$ plane into the upper half of the unit circle in the $\epsilon'=r'e^{i\phi'}$ plane. It is clear that w is given by Eq. (5) with r replaced by $r^{\pi/\gamma}$, and every angle (β_1, β_2, ϕ) multiplied by π/γ.

It remains to specify $\beta_1, \beta_2, K_1, K_2$ in terms of the geometry of the wing. As in the plane case,

$$\cos\beta_1 = \tan\mu/\tan\delta_1, \qquad \cos\beta_2 = \tan\mu/\tan\delta_2,$$

where δ_1 and δ_2 are the angles from the main stream direction to the leading edge. The boundary values K_1 and K_2 are given by the same expression as for a plane wing, except that the angle of attack is now, in general, different for each plane of the wing. Writing α_1 and α_2 for these "local" angles of attack, one finds that

$$K_1 = \alpha_1 W \tan \mu / \sin \beta_1, \qquad K_2 = \alpha_2 W \tan \mu / \sin \beta_2.$$

It is interesting to note that in the case of symmetry ($\delta_1 = \delta_2$, $\alpha_1 = \alpha_2$) there is a certain dihedral, namely $\gamma = 2\beta$, for which the boundary condition is $w = K$ over the entire arc γ so that the flow is uniform in the whole sector.

The restriction $\beta_1 + \beta_2 < \gamma$ is seen to be non-essential as in the previous case of a bent leading edge. Also it should be pointed out that for a wing with upswept dihedral ($\gamma < \pi$ for the upper surface) the lift is decreased in magnitude.

These formulae may be applied to a perpendicular vane at the tip of a rectangular wing, the vane being large enough to project through the Mach cone. This may be regarded as an example of a wing with dihedral, one of the angles of attack being zero.

If the vane extends both above and below the wing, the lift remains constant to the end of the wing. If the vane is confined to either the top or the bottom of the wing, the lift decreases moving along a span line toward the wingtip; the lift at the tip is 1/3 of the lift in the central region of the wing, and the spanwise average from tip to Mach cone is found by integration to be $(1 - 2/3\sqrt{3})$ of the lift in the central region.