CONCERNING A NEW TRANSCENDENT, ITS TABULATION AND APPLICATION IN ANTENNA THEORY*

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1. Introduction. As is well known, the integral sine and cosine functions $Si(z)$ and $Ci(z)$, respectively, are frequently met with in problems of applied mathematics. As an example we may mention the theory of antenna radiation, though in this field one preferably uses a slightly different pair of functions $S(z)$, $C(z)$ defined by

$$E(z) = C(z) + iS(z) = \int_0^z (1 - e^{-it})dt/t, \tag{1}$$

wherein i denotes the imaginary unit. Obviously $S(z)$ is identical to $Si(z)$:

$$S(z) = Si(z) = \int_0^z \sin t \, dt/t. \tag{2}$$

Further,[1] if γ denotes Euler's constant,

$$C(z) = \gamma + \log z - Ci(z) = \int_0^z (1 - \cos t)dt/t. \tag{3}$$

Recently, the author was led to the study of another transcendental function closely related to that defined by (1). This new function $E_1(z)$ is related to $E(z)$ in the same way as the latter is to the ordinary trigonometric functions, viz.

$$E_1(z) = \int_0^z E(t)dt/t = \int_0^1 \int_0^1 (1 - e^{-izst})dsdt/st. \tag{4}$$

The function $E_1(z)$ was encountered in antenna theory but may possibly be of some value in other fields as well. Therefore it is thought worth while to treat some of its features here. In addition, a short table of numerical values may be of general interest. Finally, the function $\alpha_2(x)$, as it occurs in Hallén's antenna theory, is shown to be expressible in terms of the functions $E(x)$ and $E_1(x)$.

2. Power series and asymptotic expansion for $E_1(z)$. With respect to numerical evaluation, especially for small values of z, a power-series development may serve the purpose. After expanding the integrand in (1) into powers of t, one simple integration leads to a power series for $E(z)$. Using the latter in the left-hand integral of (4) we obtain, after another term-by-term integration, the required expansion immediately, viz.

$$E_1(z) = -\sum_{n=1}^{\infty} \frac{(-iz)^n}{n^2 \cdot n!} \cdot \tag{5}$$

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[1] In our opinion, the short notation $C(z)$ for the integral (3) is to be preferred to those like $\overline{Ci}(z)$ or $Cin(z)$, as suggested by some authors. Then $E(z)$ may be a suitable abbreviation for the combination (1) analogous to the familiar exp $(iz) = \cos z + i \sin z$.

For large values of z, however, an asymptotic expansion is more desirable. In this respect we have found the following development:

$$E_1(z) \sim A + B \log z + (\log z)^2/2 + \frac{e^{-iz}}{iz} \sum_{n=1}^{\infty} n! \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)(i/z)^n, \quad (6)$$

where the constants A and B are given by

$$A = \gamma^2/2 - \pi^2/24 + \pi\gamma i/2 = -0.24464\ 45548 + 0.90668\ 45943i,$$

$$B = \gamma + \pi i/2 = 0.57721\ 56649 + 1.57079\ 63268i.$$

Formula (6) may be proved as follows: An equivalent definition of $E_1(z)$ is

$$E_1(z) = -\int_0^1 (1 - e^{-izt}) \log t\, dt/t, \quad (7)$$

as can be verified by a partial integration of the left-hand integral in (4), and an obvious change in the variable of integration. Once more integrating by parts one is led to

$$E_1(z) = \frac{1}{2} iz \int_0^1 e^{-izt}(\log t)^2 dt. \quad (8)$$

Now, for large[2] values of z, the main contribution to the integral (8) comes from the values of the integrand in the neighbourhood of $t=0$. It is therefore reasonable to consider the integral (8) as the sum of two terms:

$$E_1(z) = H(z) + h(z), \quad (9)$$

where the "main term" and the "correction term" are defined by

$$H(z) = \frac{1}{2} iz \int_0^{-i\infty} e^{-izt}(\log t)^2 dt, \quad (10)$$

$$h(z) = \frac{1}{2} iz \int_{-i\infty}^1 e^{-izt}(\log t)^2 dt, \quad (11)$$

respectively.

Let us first transform the main term $H(z)$. Evidently one can transform the expression (10) into

$$H(z) = \frac{1}{2} iz \left[\frac{d^2}{ds^2} \int_0^{-i\infty} t^{s-1} e^{-izt} dt\right]_{s=1}.$$

Now, from gamma-function theory, we have

$$\int_0^{-i\infty} t^{s-1} e^{-izt} dt = \Gamma(s)/(iz)^s.$$

Consequently, upon performing the differentiations,

$$H(z) = \Gamma''(1)/2 - (\log z + \pi i/2)\Gamma'(1) + (\log z + \pi i/2)^2\Gamma(1)/2.$$

[2] Henceforth we suppose $z>0$.

Finally, after substitution of the known numerical constants

$$\Gamma(1) = 1, \qquad \Gamma'(1) = -\gamma, \qquad \Gamma''(1) = \gamma^2 + \pi^2/6,$$

one easily finds

$$H(z) = A + B \log z + (\log z)^2/2,$$

wherein the coefficients A, B are as specified above.

Concerning the correction term $h(z)$ we proceed as follows: Let

$$g(t) = (\log t)^2/2;$$

then, by successive partial integrations of (11),

$$h(z) = e^{-iz}\left[g(1) + \frac{g'(1)}{iz} + \cdots + \frac{g^{(n)}(1)}{(iz)^n}\right] - \frac{1}{(iz)^n}\int_1^{-i\infty} g^{(n+1)}(t)e^{-izt}dt.$$

Further, by induction, or otherwise,

$$g^{(n+1)}(t) = (-1)^n n!\, t^{-n-1}\left(\log t - 1 - \frac{1}{2} - \cdots - \frac{1}{n}\right),$$

$$g(1) = g'(1) = 0, \qquad g^{(n+1)}(1) = (-1)^{n+1} n!\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right).$$

We thus obtain for $h(z)$, after $N = n-1$ terms,

$$h(z) = \frac{e^{-iz}}{iz}\sum_{n=1}^{N} n!\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)(i/z)^n + R_N(z), \tag{12}$$

where the remainder $R_N(z)$ is given by

$$R_N(z) = \frac{i^{N+1}(N+1)!}{z^{N+1}}\int_1^{-i\infty}\frac{e^{-izt}}{t^{N+2}}\left\{1 + \frac{1}{2} + \cdots + \frac{1}{N+1} - \log t\right\}dt. \tag{12a}$$

Moreover, it can be shown that

$$|R_N(z)| \leqq \left(1 + \frac{2/(N+1)}{1 + 1/2 + \cdots + 1/(N+1)}\right) \times |\text{last term taken into account}|,$$

$$|R_N(z)| \leqq \left(1 + \frac{1/z}{1 + 1/2 + \cdots + 1/(N+1)}\right) \times |\text{first term not taken into account}|.$$

Further details are left to the reader.

We have thus proved the validity of the asymptotic expansion (6) for positive values of z, though (6) holds for $\mathrm{Re}(z) > 0$ as well.

3. A third development for the function $E_1(z)$. For moderate values of z (for instance $z = 10$) neither the power series (5) nor the asymptotic development (6) is very useful for numerical purposes, as then too many terms are required. For such values of z it is better to apply the Taylor series.

To that end we use the obvious formula

$$E_1(z + \Delta) = E_1(z) + \sum_{n=0}^{\infty}\frac{d^n}{dz^n}\left\{\frac{E(z)}{z}\right\}\frac{\Delta^{n+1}}{(n+1)!} \equiv E_1(z) + \sum_{n=1}^{\infty}\frac{\Delta^n}{n!}c_n(z). \tag{13}$$

The series (13) converges for all values of Δ because $E_1(z+\Delta)$ is an integral function of Δ. The following recurrence relations for the coefficients $c_n(z)$, $(n>0)$, can be established:

$$z^2c_{n+2}(z) + (2n+1)zc_{n+1}(z) + n^2c_n(z) = (-1)^{n+1}i^ne^{-iz}, \tag{14}$$

with initial values

$$c_1(z) = E(z)/z, \qquad c_2(z) = [1 - e^{-iz} - E(z)]/z^2.$$

Given z, the functions $c_n(z)$ can be calculated successively. As at present there exist very accurate tables for the integral sine and cosine functions,[3] it is not difficult to prepare an auxiliary table for the function $E(z)$.

4. A short table for the function $E_1(z)$. We have prepared a short six-decimal table for $E_1(z)$ for values of z between 0.0 and 20.0 at intervals of length 0.2. For $z \leqq 5.0$ the power series was applied, up to 8 decimals. For $5.0 \leqq z \leqq 20.0$ the function was computed by means of (13), the functions $c_n(z)$ being pre-calculated (8 decimals) for $z=5, 7, \cdots, 19$. Accordingly, (13) was successively applied with values of $|\Delta|$ not exceeding 1.0.

The value of $E_1(20)$, obtained in this way, was checked by application of the asymptotic series for $z=20$. The difference appeared only two units of the eighth decimal. The values of $E_1(10)$ and $E_1(15)$ were also checked, by comparison with the power-series values. Moreover the eight-decimal numbers on the worksheet were checked by calculating sixth-order differences, and then rounded off to six decimals. Therefore, it will be very unlikely that the error therein exceeds half a unit of the last decimal.

Tables I, II contain the real and imaginary parts of the function $E_1(z)$, respectively. Thus they give

$$\text{Re } E_1(z) = \int_0^z [\gamma + \log t - Ci(t)]dt/t = z\int_0^1 \sin zt\,(\log t)^2\,dt/2, \tag{15}$$

$$\text{Im } E_1(z) = \int_0^z Si(t)dt/t = z\int_0^1 \cos zt\,(\log t)^2\,dt/2. \tag{15a}$$

5. Hallén's second-order function $\alpha_2(x)$. Hallén[4] derived the following expression for the self-impedance of the center-fed perfectly conducting cylindrical antenna

$$Z(x) = -60i\Omega\frac{\cos x + \alpha_1(x)/\Omega + \alpha_2(x)/\Omega^2 + \cdots}{\sin x + \beta_1(x)/\Omega + \beta_2(x)/\Omega^2 + \cdots}.$$

In this formula Ω denotes a large constant, depending on the radius a and the half-length l of the antenna: $\Omega = 2\log(2l/a)$. Further $x = kl = 2\pi l/\lambda$, where λ is the wave-length.

Only the first-order coefficients $\alpha_1(x)$ and $\beta_1(x)$ can be given explicitly in terms of known functions, namely

$$\alpha_1(x) = \tfrac{1}{2}e^{ix}E(4x) - \cos xE(2x). \tag{16}$$

$$\beta_1(x) = \tfrac{1}{2}ie^{ix}\{E(4x) - 4E(2x)\} + \sin x\{\log 4 - E(2x)\}. \tag{16a}$$

[3] Tables of sine, cosine and exponential integrals, vols. I, II; New York (1940). Table of sine and cosine integrals; New York (1942). (Federal Works Agency, W.P.A., City of New York.)

[4] Erik Hallén, Nova acta reg. soc. sci. Upsaliensis (4) 11, 1044 (1938).

Rather intricate formulae were given for the second-order coefficients $\alpha_2(x)$ and $\beta_2(x)$, which were evaluated by graphical methods.[5] It may be noticed that in the refined theory[6] the same second-order coefficients occur.

Recently $\alpha_2(x)$ was found to be expressible in terms of $E(x)$ and $E_1(x)$ by means of a fairly simple formula, viz.

$$\begin{aligned}\alpha_2(x) = &- \alpha_1(x)\{\log 4 + E(2x)\} - \cos xE^2(2x)/2 \\ &+ 2i \sin xE_1(4x) + \cos x\{E_1(4x) - 2E_1(2x)\}.\end{aligned} \tag{17}$$

With the help of our tables for the function $E_1(x)$, and the American tables for $Si(x)$, $Ci(x)$, we have calculated α_1 and α_2 to six decimals for 0.0(0.1)5.0. After careful checking, these results were rounded off to four decimals. The final data are given in tables III, IV, whereby

$$\alpha_1(x) = \overset{\mathrm{I}}{\alpha_1}(x) + i\overset{\mathrm{II}}{\alpha_1}(x), \qquad \alpha_2(x) = \overset{\mathrm{I}}{\alpha_2}(x) + i\overset{\mathrm{II}}{\alpha_2}(x).$$

Comparison of our table for the first-order coefficient $\alpha_1(x)$ with those of King and Blake[7] shows only small differences in the last decimal. Also the values of the second-order coefficient $\alpha_2(x)$ are in good agreement with the corresponding two-decimal values obtained by graphical integration.[5]

As for the other second-order coefficient, we do not think it possible to express $\beta_2(x)$ in such a simple way; unfortunately, more intricate functions seem to play a part.

In the following sections a proof of formula (17) will be given. As a rather large amount of analysis seems necessary to establish such proof, we may once more emphasize the usefulness of the short abbreviation $E(x)$ as was adopted here for the combined integral sine and cosine functions.

6. Some auxiliary functions. We introduce the following four functions:

$$\phi_1(x) = \int_0^x \frac{E(x) - E(t)}{x - t} dt = - \int_0^x \log(1 - t/x)(1 - e^{-it})dt/t, \tag{18}$$

$$\phi_2(x) = \int_0^x E(x - t)(1 - e^{-it})dt/t, \tag{19}$$

$$\phi_3(x) = \int_0^x \{E(x - t) - E(x)\}e^{-it}dt/t, \tag{20}$$

$$\phi_4(x) = \int_0^{2x} \{\cos(x - t) - \cos x\} \log(1 - t/2x)e^{-it}dt/t. \tag{21}$$

Between these functions the following relations exist:

$$2\phi_1(x) + \phi_2(x) = 2E_1(x), \tag{22}$$

$$\phi_1(x) + \phi_2(x) + \phi_3(x) = E^2(x), \tag{23}$$

$$\phi_4(x) = e^{ix}\phi_1(4x)/2 - \cos x\, \phi_1(2x). \tag{24}$$

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[5] C. J. Bouwkamp, Physica **9**, 609–631 (1942).

[6] R. King and D. Middleton, Quart. Appl. Math. **3**, 302–335 (1945).

[7] R. King and F. G. Blake, Proc. Inst. Radio Engrs. **30**, 335–349 (1942).

Formula (22) is especially noteworthy as it does not seem to be some trivial equality. A straightforward proof of it may be established by expanding both sides into powers of x. Evidently, φ_1 and φ_2 are integral functions; their respective power series converge for all finite values of x. One will get

$$\phi_1(x) = -\sum_{n=1}^{\infty} \frac{(-ix)^n}{n\cdot n!}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right), \tag{18a}$$

$$\phi_2(x) = 2\sum_{n=1}^{\infty} \frac{(-ix)^{n+1}}{(n+1)(n+1)!}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right), \tag{19a}$$

and then (22) follows at once on account of (5).

A detailed proof of (24) only will be given as an example. Firstly,

$$\begin{aligned}\{\cos(x-t)-\cos x\}e^{-it}/t &= -\tfrac{1}{2}e^{ix}(1-e^{-2it})/t+\cos x\,(1-e^{-it})/t\\ &= \frac{d}{dt}\{-\tfrac{1}{2}e^{ix}E(2t)+\cos x\,E(t)\}.\end{aligned}$$

Therefore,

$$\phi_4(x) = -\int_0^{2x}\log(1-t/2x)\frac{d}{dt}\left[\tfrac{1}{2}e^{ix}\{E(2t)-E(4x)\}-\cos x\{E(t)-E(2x)\}\right]dt.$$

Secondly, by partial integration,

$$\phi_4(x) = \frac{1}{2}e^{ix}\int_0^{2x}\frac{E(2t)-E(4x)}{t-2x}dt-\cos x\int_0^{x}\frac{E(t)-E(2x)}{t-2x}dt,$$

and, after some trivial transformation and by the use of (18), this reduces to (24).

7. Proof of formula (17). Instead of Hallén's functions $F_n(z)$ we take $f_n(z) = F_n(z/k)$. These functions are recurrently defined by

$$\begin{aligned}f_0(z) &= \cos z,\\ f_{n+1}(z) &= \{f_n(x)-f_n(z)\}\log(1-z^2/x^2)\\ &\quad+\int_{-x}^{x}\frac{\{f_n(x)-f_n(\zeta)\}\exp(-i|z+\zeta|)-\{f_n(x)-f_n(z)\}}{|z-\zeta|}d\zeta.\end{aligned}$$

Apart from f_0, only f_1 can be given explicitly in terms of known functions, viz.

$$\begin{aligned}f_1(z) &= (\cos x-\cos z)\log(1-z^2/x^2)\\ &\quad+\tfrac{1}{2}\{e^{iz}E(2x+2z)+e^{-iz}E(2x-2z)\}-\cos x\{E(x+z)+E(x-z)\}.\end{aligned} \tag{25}$$

The functions $\alpha_1(x)$ and $\alpha_2(x)$ are obtained when $z=x$ is substituted in $f_1(z)$ and $f_2(z)$, respectively. Therefore the required expression for $\alpha_2(x)$ has to be derived from

$$\alpha_2(x) = \int_0^{2x}\{f_1(x)-f_1(x-t)\}\frac{e^{-it}}{t}dt. \tag{26}$$

We first write

$$f_1(x)-f_1(x-t) = T_1+T_2+\cdots+T_7,$$

wherein

$$T_1 = \{\cos(x - t) - \cos x\} \log(2t/x), \tag{27}$$

$$T_2 = - e^{ix} \cdot e^{it} E(2t)/2, \tag{28}$$

$$T_3 = \cos x E(t) \tag{29}$$

$$T_4 = e^{ix} E(4x)(1 - e^{-it})/2, \tag{30}$$

$$T_5 = \{\cos(x - t) - \cos x\} \log(1 - t/2x), \tag{31}$$

$$T_6 = \cos x \{E(2x - t) - E(2x)\}, \tag{32}$$

$$T_7 = - e^{ix} \cdot e^{-it} \{E(4x - 2t) - E(4x)\}/2. \tag{33}$$

Let the corresponding contributions to the integral in (26) be denoted by I_n, thus

$$I_n = \int_0^{2x} T_n e^{-it} dt/t$$

Then one will find consecutively

$$I_1 = - \alpha_1(x) \log 4 + e^{ix} E_1(4x)/2 - \cos x E_1(2x), \tag{27a}$$

$$I_2 = - e^{-ix} E_1(4x)/2, \tag{28a}$$

$$I_3 = \cos x \{E_1(2x) - E^2(2x)/2\} \tag{29a}$$

$$I_4 = e^{ix} E(4x) \{E(4x) - E(2x)\}/2, \tag{30a}$$

$$I_5 = e^{ix} \phi_1(4x)/2 - \cos x \phi_1(2x), \tag{31a}$$

$$I_6 = \cos x \{E^2(2x) - \phi_1(2x) - \phi_2(2x)\}, \tag{32a}$$

$$I_7 = e^{ix} \{\phi_1(4x) + \phi_2(4x) - E^2(4x)\}/2. \tag{33a}$$

It is thought unnecessary to give detailed proofs of the above formulae, as the general lines are the same as in the example of the preceeding section.

Upon substituting (27a) $\cdots$ (33a) in $\alpha_2(x) = I_1 + I_2 + \cdots + I_7$, we obtain

$$\alpha_2(x) = \cos x E^2(2x)/2 - e^{ix} E(2x) E(4x)/2 - \alpha_1(x) \log 4 + i \sin x E_1(4x)$$
$$- \cos x \{2\phi_1(2x) + \phi_2(2x)\} + e^{ix} \{2\phi_1(4x) + \phi_2(4x)\}/2.$$

On account of (22), the functions φ_1 and φ_2 can be eliminated. One then easily obtains the required formula (17).

TABLE I

z	$\mathrm{Re}E_1$	z	$\mathrm{Re}E_1$	z	$\mathrm{Re}E_1$	z	$\mathrm{Re}E_1$
0.0	0.000 000	5.0	1.972 538	10.0	3.726 338	15.0	4.982 566
0.2	0.004 996	5.2	2.066 256	10.2	3.784 619	15.2	5.025 625
0.4	0.019 933	5.4	2.157 214	10.4	3.842 420	15.4	5.068 443
0.6	0.044 664	5.6	2.245 360	10.6	3.899 692	15.6	5.111 035
0.8	0.078 943	5.8	2.330 699	10.8	3.956 381	15.8	5.153 410
1.0	0.122 434	6.0	2.413 282	11.0	4.012 436	16.0	5.195 569
1.2	0.174 714	6.2	2.493 205	11.2	4.067 808	16.2	5.237 508
1.4	0.235 281	6.4	2.570 598	11.4	4.122 452	16.4	5.279 217
1.6	0.303 564	6.6	2.645 618	11.6	4.176 332	16.6	5.320 681
1.8	0.378 933	6.8	2.718 446	11.8	4.229 418	16.8	5.361 884
2.0	0.460 706	7.0	2.789 276	12.0	4.281 693	17.0	5.402 804
2.2	0.548 165	7.2	2.858 310	12.2	4.333 147	17.2	5.443 422
2.4	0.640 563	7.4	2.925 751	12.4	4.383 783	17.4	5.483 715
2.6	0.737 142	7.6	2.991 799	12.6	4.433 614	17.6	5.523 663
2.8	0.837 139	7.8	3.056 642	12.8	4.482 661	17.8	5.563 247
3.0	0.939 800	8.0	3.120 458	13.0	4.530 954	18.0	5.602 453
3.2	1.044 392	8.2	3.183 403	13.2	4.578 533	18.2	5.641 267
3.4	1.150 210	8.4	3.245 618	13.4	4.625 441	18.4	5.679 683
3.6	1.256 591	8.6	3.307 218	13.6	4.671 727	18.6	5.717 694
3.8	1.362 916	8.8	3.368 298	13.8	4.717 442	18.8	5.755 304
4.0	1.468 623	9.0	3.428 929	14.0	4.762 639	19.0	5.792 515
4.2	1.573 207	9.2	3.489 159	14.2	4.807 370	19.2	5.829 339
4.4	1.676 231	9.4	3.549 016	14.4	4.851 684	19.4	5.865 788
4.6	1.777 320	9.6	3.608 507	14.6	4.895 628	19.6	5.901 879
4.8	1.876 168	9.8	3.667 622	14.8	4.939 244	19.8	5.937 632
5.0	1.972 538	10.0	3.726 338	15.0	4.982 566	20.0	5.973 068

TABLE II

z	$\mathrm{Im}E_1$	z	$\mathrm{Im}E_1$	z	$\mathrm{Im}E_1$	z	$\mathrm{Im}E_1$
0.0	0.000 000	5.0	3.467 907	10.0	4.526 334	15.0	5.157 090
0.2	0.199 852	5.2	3.527 976	10.2	4.559 056	15.2	5.178 576
0.4	0.398 818	5.4	3.584 497	10.4	4.590 878	15.4	5.199 864
0.6	0.596 026	5.6	3.637 923	10.6	4.621 791	15.6	5.220 929
0.8	0.790 627	5.8	3.688 683	10.8	4.651 802	15.8	5.241 742
1.0	0.981 811	6.0	3.737 180	11.0	4.680 931	16.0	5.262 280
1.2	1.168 815	6.2	3.783 779	11.2	4.709 208	16.2	5.282 518
1.4	1.350 936	6.4	3.828 813	11.4	4.736 676	16.4	5.302 436
1.6	1.527 537	6.6	3.872 571	11.6	4.763 386	16.6	5.322 019
1.8	1.698 057	6.8	3.915 302	11.8	4.789 398	16.8	5.341 255
2.0	1.862 017	7.0	3.957 213	12.0	4.814 776	17.0	5.360 140
2.2	2.019 023	7.2	3.998 470	12.2	4.839 587	17.2	5.378 671
2.4	2.168 772	7.4	4.039 198	12.4	4.863 898	17.4	5.396 855
2.6	2.311 048	7.6	4.079 485	12.6	4.887 779	17.6	5.414 701
2.8	2.445 729	7.8	4.119 385	12.8	4.911 291	17.8	5.432 223
3.0	2.572 779	8.0	4.158 921	13.0	4.934 494	18.0	5.449 442
3.2	2.692 246	8.2	4.198 089	13.2	4.957 441	18.2	5.466 378
3.4	2.804 259	8.4	4.236 865	13.4	4.980 178	18.4	5.483 057
3.6	2.909 021	8.6	4.275 206	13.6	5.002 741	18.6	5.499 504
3.8	3.006 798	8.8	4.313 058	13.8	5.025 158	18.8	5.515 747
4.0	3.097 916	9.0	4.350 357	14.0	5.047 448	19.0	5.531 813
4.2	3.182 750	9.2	4.387 037	14.2	5.069 623	19.2	5.547 727
4.4	3.261 713	9.4	4.423 033	14.4	5.091 683	19.4	5.563 513
4.6	3.335 250	9.6	4.458 283	14.6	5.113 624	19.6	5.579 192
4.8	3.403 823	9.8	4.492 731	14.8	5.135 432	19.8	5.594 782
5.0	3.467 907	10.0	4.526 334	15.0	5.157 090	20.0	5.610 298

TABLE III

x	α_1^{I}	α_1^{II}	x	α_1^{I}	α_1^{II}
0.0	0.0000	0.0000	2.5	0.2361	1.4528
0.1	−0.0100	0.0007	2.6	0.3550	1.3695
0.2	−0.0393	0.0053	2.7	0.4686	1.2690
0.3	−0.0865	0.0175	2.8	0.5756	1.1534
0.4	−0.1490	0.0407	2.9	0.6749	1.0247
0.5	−0.2235	0.0773	3.0	0.7661	0.8851
0.6	−0.3061	0.1292	3.1	0.8487	0.7362
0.7	−0.3924	0.1973	3.2	0.9225	0.5800
0.8	−0.4780	0.2815	3.3	0.9875	0.4180
0.9	−0.5583	0.3807	3.4	1.0436	0.2514
1.0	−0.6291	0.4930	3.5	1.0905	0.0816
1.1	−0.6866	0.6156	3.6	1.1280	−0.0904
1.2	−0.7278	0.7451	3.7	1.1556	−0.2634
1.3	−0.7503	0.8777	3.8	1.1730	−0.4363
1.4	−0.7528	1.0091	3.9	1.1795	−0.6080
1.5	−0.7345	1.1351	4.0	1.1743	−0.7771
1.6	−0.6957	1.2517	4.1	1.1569	−0.9422
1.7	−0.6376	1.3550	4.2	1.1265	−1.1016
1.8	−0.5618	1.4419	4.3	1.0828	−1.2535
1.9	−0.4708	1.5097	4.4	1.0253	−1.3960
2.0	−0.3672	1.5563	4.5	0.9542	−1.5271
2.1	−0.2540	1.5805	4.6	0.8695	−1.6446
2.2	−0.1343	1.5819	4.7	0.7720	−1.7468
2.3	−0.0109	1.5604	4.8	0.6625	−1.8315
2.4	0.1134	1.5170	4.9	0.5423	−1.8973
2.5	0.2361	1.4528	5.0	0.4130	−1.9426

TABLE IV

x	α_2^{I}	α_2^{II}	x	α_2^{I}	α_2^{II}
0.0	0.0000	0.0000	2.5	−2.5787	3.5091
0.1	−0.0359	0.0022	2.6	−2.2678	3.4640
0.2	−0.1415	0.0171	2.7	−1.9045	3.3755
0.3	−0.3099	0.0563	2.8	−1.4931	3.2385
0.4	−0.5311	0.1284	2.9	−1.0397	3.0487
0.5	−0.7919	0.2392	3.0	−0.5524	2.8034
0.6	−1.0784	0.3901	3.1	−0.0402	2.5018
0.7	−1.3762	0.5790	3.2	0.4869	2.1447
0.8	−1.6723	0.8004	3.3	1.0187	1.7349
0.9	−1.9560	1.0462	3.4	1.5450	1.2770
1.0	−2.2191	1.3070	3.5	2.0560	0.7771
1.1	−2.4564	1.5727	3.6	2.5430	0.2427
1.2	−2.6654	1.8344	3.7	2.9979	−0.3177
1.3	−2.8455	2.0844	3.8	3.4141	−0.8948
1.4	−2.9977	2.3171	3.9	3.7862	−1.4792
1.5	−3.1232	2.5291	4.0	4.1100	−2.0611
1.6	−3.2229	2.7192	4.1	4.3828	−2.6312
1.7	−3.2969	2.8875	4.2	4.6030	−3.1808
1.8	−3.3437	3.0355	4.3	4.7700	−3.7021
1.9	−3.3604	3.1644	4.4	4.8840	−4.1880
2.0	−3.3431	3.2752	4.5	4.9459	−4.6330
2.1	−3.2867	3.3680	4.6	4.9568	−5.0323
2.2	−3.1862	3.4413	4.7	4.9180	−5.3825
2.3	−3.0369	3.4923	4.8	4.8308	−5.6811
2.4	−2.8350	3.5166	4.9	4.6959	−5.9264
2.5	−2.5787	3.5091	5.0	4.5142	−6.1172