NOTES

A NEW DERIVATION OF THE METHOD OF CHARACTERISTICS FOR AXIALLY SYMMETRICAL SUPERSONIC FLOW*

BY ANDREW VAZSONYI (*North American Aviation, Inc.*)

The flow of supersonic fluids can be studied theoretically with the aid of the so-called method of characteristics. This method is well known for two-dimensional flows[1] and was extended recently to axially symmetrical flows.[2] The basis of this method

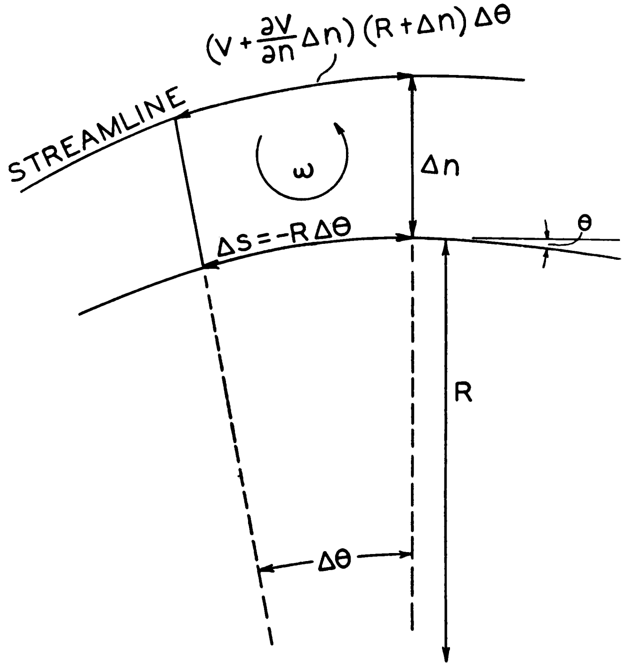

FIG. 1

is a differential equation [Eq. (17) of this paper] expressing the change in the magnitude and the direction of the velocity vector along a Mach wave. The purpose of this note is to give a simple derivation of this equation. Two-dimensional flows and irrotational flows can be obtained by specializing Eq. (17).

1. Bernoulli's equation. If V denotes the velocity, p the pressure, ρ the density, and s the arc length of the stream line, Bernoulli's equation can be written in the form

$$V \frac{\partial V}{\partial s} + \frac{1}{\rho} \frac{\partial p}{\partial s} = 0. \tag{1}$$

* Received June 18, 1947.

[1] *Cf.* H. W. Liepman and A. E. Puckett, *Introduction to aerodynamics of a compressible fluid*, John Wiley & Sons, Inc., 1947, pp. 210–232.

[2] *Cf.* R. Sauer, *Theoretische Einführung in die Gasdynamik*, Springer, Berlin, 1943; reprinted by J. W. Edwards, Ann Arbor, 1945, pp. 137–144.

Introducing the velocity of sound defined by

$$a^2 = \frac{dp}{d\rho} \tag{2}$$

we may write this as follows:

$$V\frac{\partial V}{\partial s} - a^2\frac{1}{\rho}\frac{\partial p}{\partial s}\cdot \tag{3}$$

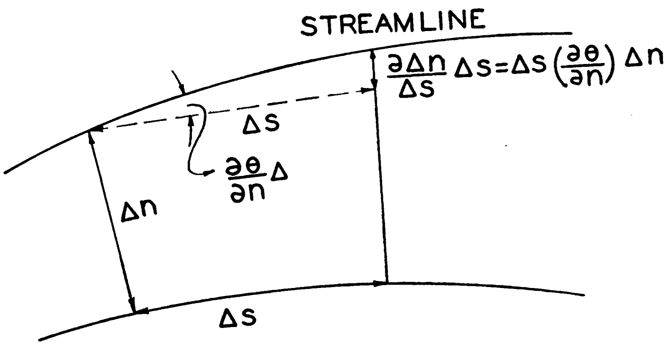

FIG. 2

2. The expression for the rotation. The rotation can be obtained with the aid of Fig. 1 as follows. The circulation around the small element is given by

$$\Gamma = 2\omega\Delta n\Delta s = \left(V + \frac{\partial V}{\partial n}\Delta n\right)(R + \Delta n)\Delta\theta - VR\Delta\theta. \tag{4}$$

Thus,

$$\frac{\partial V}{\partial n} - V\frac{\partial \theta}{\partial s} = -\,2\omega \tag{5}$$

where R is the radius of curvature of the streamline.

3. The continuity equation. This can be written as

$$2\pi r V\rho\Delta n = \text{const}, \tag{6}$$

where the factor $2\pi r$ enter when the flow is axially symmetrical. By logarithmic differentiation of Eq. (6) we obtain

$$\frac{1}{\rho}\frac{\partial p}{\partial s} + \frac{1}{V}\frac{\partial V}{\partial s} + \frac{1}{\Delta n}\frac{\partial \Delta n}{\partial s} + \frac{\sin\theta}{r} = 0, \tag{7}$$

where

$$\frac{\partial r}{\partial s} = \sin\theta \tag{8}$$

was used. (For two-dimensional flows $r = \infty$.) From Eq. (3) it follows that

$$\frac{1}{\rho}\frac{\partial p}{\partial s}+\frac{1}{V}\frac{\partial V}{\partial s}=\left(1-\frac{V^2}{a^2}\right)\frac{1}{V}\frac{\partial V}{\partial s}=(1-M^2)\frac{1}{V}\frac{\partial V}{\partial s}, \tag{9}$$

where M is the Mach number. From Fig. 2 it can be seen that

$$\frac{1}{\Delta n}\frac{\partial \Delta n}{\partial s}=\frac{\partial \theta}{\partial n}. \tag{10}$$

From Eqs. (7), (9), and (10) follows finally that[3]

$$(M^2-1)\frac{1}{V}\frac{\partial V}{\partial s}-\frac{\partial \theta}{\partial n}-\frac{\sin\theta}{r}=0. \tag{11}$$

4. Let us compute the infinitesimal change in V and θ when moving from point A to point B on Fig. 3. Obviously,

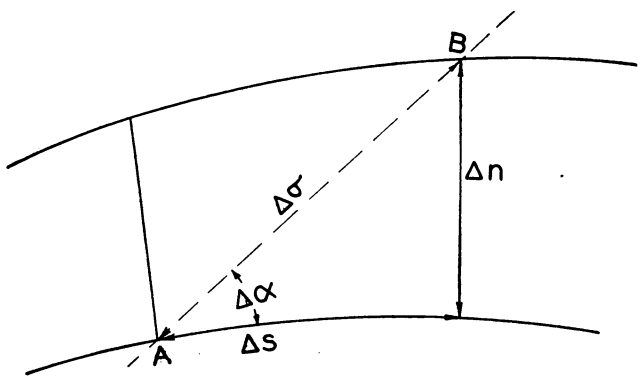

FIG. 3

$$\Delta V=\frac{\partial V}{\partial s}\Delta s+\frac{\partial V}{\partial n}\Delta n=\left(\frac{\partial V}{\partial s}\cos\alpha+\frac{\partial V}{\partial n}\sin\alpha\right)\Delta\sigma \tag{12}$$

and

$$\Delta\theta=\frac{\partial \theta}{\partial s}\Delta s+\frac{\partial \theta}{\partial n}\Delta n=\left(\frac{\partial \theta}{\partial s}\cos\alpha+\frac{\partial \theta}{\partial n}\sin\alpha\right)\Delta\sigma. \tag{13}$$

Using Eqs. (5) and (11), we may transform the last equation into

$$\Delta\theta=\frac{1}{V}\frac{\cos\alpha}{\sin\alpha}\left[(M^2-1)\frac{\partial V}{\partial s}\frac{\sin^2\alpha}{\cos\alpha}+\frac{\partial V}{\partial n}\sin\alpha\right]\Delta\sigma$$
$$+\left(\frac{2\omega\cos\alpha}{V}-\frac{\sin\theta\sin\alpha}{r}\right)\Delta\sigma. \tag{14}$$

Finally from Eqs. (12) and (14) it follows that

---

[3] A similar equation for two dimensional flows is found in T. von Kármán, *Compressibility effects in aerodynamics*, J. Aero. Sci. **8**, 337–356 (1941).

$$\Delta\theta = \frac{\Delta V}{V} \cot\alpha \frac{(M^2 - 1)\cdot \tan^2\alpha \cdot \dfrac{\partial V}{\partial s}\cos\alpha + \dfrac{\partial V}{\partial n}\sin\alpha}{\dfrac{\partial V}{\partial s}\cos\alpha + \dfrac{\partial V}{\partial n}\sin\alpha} + \left(\frac{2\omega\cos\alpha}{V} - \frac{\sin\theta\sin\alpha}{r}\right)\Delta\sigma. \tag{15}$$

5. The basic equation of the method of characteristics. This follows immediately from Eq. (15). In this case the angle α equals the Mach angle or

$$\tan\alpha = \frac{1}{\sqrt{M^2 - 1}} \tag{16}$$

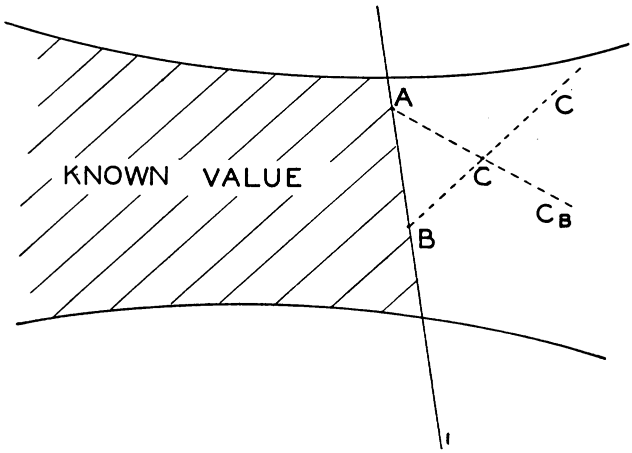

FIG. 4

and so Eq. (15) simplifies to

$$\Delta\theta = \frac{\Delta V}{V}\cot\alpha + \left(\frac{2\omega\cos\alpha}{V} - \frac{\sin\theta\sin\alpha}{r}\right)\Delta\sigma. \tag{17}$$

Thus, it is seen, that moving along characteristics involves an enormous simplification of the differential equations. In the case of two-dimensional, irrotational motion, $\omega = 0$, $r = \infty$ and Eq. (17) reduces to the well-known relation

$$\Delta\theta = \frac{\Delta V}{V}\cot\alpha. \tag{18}$$

6. The expression for the rotation. This is needed when the motion is rotational. It is shown[4] that ω is related to the rate of change of the entropy S normal to the streamline:

[4] *Cf.* A Vazsonyi, *On rotational gas flows*, Quart. Appl. Math. **3**, 29–37 (1945).

$$2\omega = \frac{p}{R\rho V} \frac{\partial S}{\partial n}, \tag{19}$$

where S is the specific entropy and R is the gas constant. It can also be expressed simply in terms of the gradient of the total pressure pt as follows:

$$2\omega = -\frac{p}{p_t \rho V} \frac{\partial pt}{\partial n}. \tag{20}$$

7. The essence of the method of characteristics is the following. Suppose the flow is to be determined in the channel of Fig. 4 and the flow is known up to the line l. Select points A and B on line l. At these points the velocity, the direction of flow, etc., is known. Thus characteristics C_a and C_b can be drawn. Equation (17) used along the two different characteristics gives two equations for the velocity and its direction at point C. Thus properties of the flow at C can be determined by solving these simultaneous equations. By repeating this process the complete flow field can be computed or constructed.

CONTINUOUS HEATING OF A HOLLOW CYLINDER*

By G. COMENETZ** (*Geophysical Laboratory, Carnegie Institution of Washington*)

1. Introduction. Formulas are given here for the temperatures within the wall of an infinitely long hollow cylinder which is supplied with heat through the inner surface. The thermal coefficients are assumed constant, the initial temperature of the wall is zero, and the outer surface is either maintained at zero or thermally insulated. Heat is transmitted at any one time uniformly all over the inner surface, but the rate of heat input is permitted to vary with the time, linearly, or at most quadratically. This, of course, is far from a general mode of variation. However, in an application in which a linear or quadratic rate is at least a permissible approximation, the formulas will be of value. They have been useful in connection with the heating of a gun firing steadily, in which, as the barrel grows hot, the heat input through the bore surface decreases nearly enough linearly with the time. Other possible applications are to the heating of a tubular furnace, a chimney, or the insulation on a wire carrying an electric current. It may not be out of place if the writer remarks that in his experience with physics, the occasions on which a first approximation is as much as is required are at least as numerous as those where only an exact theory will do.

The results follow quite directly from formulas given by Carslaw and Jaeger.[1] Certain slowly convergent infinite series arise in the derivations, however, and the main effort below is directed toward summing these series in finite terms. The re-

* Received April 11, 1947. The information described in this article was obtained under contract OEMsr-51 with the Office of Scientific Research and Development, under the supervision of Division One of the National Defense Research Committee.

** Now at Westinghouse Research Laboratories, East Pittsburgh, Pa.

[1] H. S. Carslaw and J. G. Jaeger, *Some two-dimensional problems in conduction of heat with circular symmetry*, Proceedings of the London Mathematical Society (2) **46,** 361–388 (1939) (referred to below as CJ).