ON AXIALLY SYMMETRIC FLOWS*

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1. Introduction. The determination of the irrotational flow of a perfect incompressible fluid around a given body constitutes a boundary value problem which can be solved by methods of Potential Theory. However, this theoretical solution has found little practical application.

In the case of a solid of revolution, an indirect but more efficient approach is given by the *method of sources and sinks*, which may be called an inverse method in the same sense in which this term is used in Elasticity and elsewhere in Hydrodynamics. The body of revolution cannot be prescribed but only approximated to a certain extent by the flow due to a suitable distribution of sources and sinks in joint action with a parallel uniform flow coming from infinity. As a compensation for this drawback, the approximating flows are given by explicit formulae which are often suitable for numerical computations.

The method of sources and sinks was originally confined to the case of plane motion until Stokes' generalization of the concept of stream function enabled Rankine in 1871 to adapt the method to axially symmetric flows. In the succeeding decades various examples were given and applied to the pressure distribution around airships. However, the possibilities implied by the method were far from being exhausted. In fact, up to the present day no other flows have been considered but such as are due to a distribution of sources, sinks, and doublets exclusively on the axis of symmetry. Let us consider, for instance, the case in which the direction of the parallel flow coincides with the axis of symmetry. In this case—the only one which will be discussed in the present paper—it has been already noticed by Munk[1] that blunt nosed bodies cannot be obtained by taking any distribution of sources and sinks on the axis.

The present paper deals with an *extension of the method of sources and sinks*. The sources and sinks are no longer confined to the axis but are distributed on circumferences, rings, discs and cylinders. The distribution must of course be symmetric with respect to the axis of revolution, but for practical purposes the choice is restricted to such cases in which the stream function can be explicitly computed in terms of known functions. We shall use here Beltrami's fundamental results obtained in a series of papers published in 1878–80. The importance of Beltrami's results for the theory of hydrodynamical flows has been completely overlooked. A serious error in his paper requiring a modification of nearly all his formulae does not seem to have been noticed. In fact, Beltrami, who applies his formulae chiefly to problems of Potential Theory and Electrostatics, fails to recognize that Stokes' stream function is, in many of his formulae, a many valued function. It is interesting to note that this

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[1] M. M. Munk in *Aerodynamic theory* (edited by W. F. Durand) vol. 1, J. Springer, Berlin, 1934, p. 266.

property of Stokes' function, which should be expected to hold rather as a rule than as an exception, is never mentioned in the literature on Hydrodynamics while the corresponding property of the stream function for plane motion is one of the basic concepts of the theory.

A superposition of the flow due to our sources and of a uniform flow in the direction of the axis of symmetry will give us some essentially new types of flows, in particular, flows around blunt nosed profiles. Up to the present date, only isolated limiting cases of these flows have been discussed. Incidentally, the use of sources and sinks outside the axis will enable us to obtain profiles consisting of piecewise analytic curves. As is well known, the profiles corresponding to a source distribution along the axis consist necessarily of a single analytic curve, the nose and the tail being the only possible singular points.

2. Stokes' stream function. In this paper we consider only steady irrotational axially symmetric flows of a perfect incompressible fluid. The axis of symmetry will be taken as the x-axis. Let x, ρ be the coordinates in a meridian plane. The flow is completely determined if the velocity distribution is known in the half plane $-\infty < x < +\infty$, $\rho \geqq 0$. The motion being irrotational, there exists a velocity potential ϕ which is an harmonic function satisfying the Laplace equation in three-space. For axially symmetric flows, ϕ is a function only of x and ρ, so that the Laplace equation in cylindrical coordinates for $\phi(x, \rho)$ reduces to

$$\frac{\partial}{\partial x}\left(\rho \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial \rho}\left(\rho \frac{\partial \phi}{\partial \rho}\right) = 0. \tag{2.1}$$

This equation can be interpreted as an integrability condition showing the existence of a function $\psi(x, \rho)$ which is called the *Stokes stream function* and which is defined by the equations

$$\frac{\partial \psi}{\partial x} = -\rho \frac{\partial \phi}{\partial \rho}; \qquad \frac{\partial \psi}{\partial \rho} = \rho \frac{\partial \phi}{\partial x}. \tag{2.2}$$

It is obvious, by (2.1), that $d\psi = (\partial \psi/\partial x)dx + (\partial \psi/\partial \rho)d\rho$ is a total differential and that the elimination of ϕ in (2.2) gives for $\psi(x, \rho)$ the differential equation

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho} \frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial \rho}\left(\frac{1}{\rho} \frac{\partial \psi}{\partial \rho}\right) = 0. \tag{2.3}$$

Unlike ϕ, which is defined in three-space, the stream function $\psi(x, \rho)$ is defined only in the upper half plane $\rho \geqq 0$ by its total differential, i.e., by the differential equations (2.2). It is therefore to be expected that ψ will as a rule be a many-valued function. The hydrodynamical significance of the stream function ψ is well known: ψ remains constant along each streamline in the meridian plane, while $2\pi\psi$ represents the flow of a fluid of unit density between the given streamline and the streamline $\psi = 0$.

The velocity components u and v in the x and ρ direction are given by

$$\begin{aligned} u &= \frac{\partial \phi}{\partial x} = \frac{1}{\rho} \frac{\partial \psi}{\partial \rho}, \\ v &= \frac{\partial \phi}{\partial \rho} = -\frac{1}{\rho} \frac{\partial \psi}{\partial x}. \end{aligned} \tag{2.4}$$

By the convention adopted in this paper, the velocity vector is the gradient of the potential.

For a uniform flow parallel to the x-axis we have

$$\phi = Ux, \qquad \psi = \tfrac{1}{2}U\rho^2. \tag{2.5}$$

The flow $2\pi\psi$ is given by $\pi\rho^2 U$.

Beltrami calls ψ the associated function of ϕ. He shows, by a process of repeated differentiation and integration, that every potential or associated function generates a descending and an ascending sequence of pairs of functions of the same kind. This discovery is an important extension of the ordinary theory of analytic functions, and can be applied in a similar way to other cases.[2] However, our main interest will be focused on another part of Beltrami's work, in which he investigates fundamental solutions of (2.1) possessing basic singularities. The role played by such solutions in the theory of partial differential equations is too well known to be emphasized here, and was clearly recognized by Beltrami.

3. Sources and sinks on the axis. By our definition of the potential, the elementary potential in three-space is $-1/r$. In cylindrical coordinates x, ρ the potential ϕ of a source of strength m at the origin is given by

$$\phi_0 = \frac{-m}{\sqrt{x^2+\rho^2}}\,. \tag{3.1}$$

The corresponding stream function ψ is given by

$$\psi_0 = -m\left(1 + \frac{x}{\sqrt{x^2+\rho^2}}\right). \tag{3.2}$$

The additive constant is chosen in such a way that ψ_0 vanishes on the streamline $x<0$, $\rho=0$. We note that the stream function is single-valued in the upper half plane x, ρ.

4. The principle of superposition. The potential and the stream function of two or several combined flows is the sum of the respective functions corresponding to the separate flows. The method of sources and sinks uses combinations of flows due to sources and sinks and of a uniform parallel flow. A suitable choice leads to a flow with a streamline $\psi=0$ separating the domain of regular flow from the sources. This *dividing line* plays the part of a *rigid profile*, the interior of which can be replaced by a solid body. The principle of superposition for potentials is less important: the method of sources and sinks being based on the existence of the stream function. Closed dividing lines can be obtained only by taking a distribution of sources and sinks of total strength zero. For a positive total strength, the corresponding profile is open downstream and extends to infinity. Nevertheless, the corresponding surfaces of revolution, called *half-bodies*, are of great importance because in reality the potential flow is disturbed by viscosity everywhere except in a certain vicinity of the nose, which is at the same time the region in which big velocities and a danger of cavitation may be expected. The simplest case of a half-body will be discussed in the next paragraph.

5. Blasius-Fuhrmann's half-body. By superposition of a source m at the origin

[2] L. Bers and A. Gelbart, Trans. Amer. Math. Soc., **56**, 67–93 (1944).

and of a uniform stream U in the x direction, we obtain a flow with the potential and stream functions:

$$\phi = Ux - \frac{m}{\sqrt{x^2 + \rho^2}} = Ux - (m/r), \tag{5.1}$$

$$\psi = \tfrac{1}{2}U\rho^2 - m\left(1 + \frac{x}{\sqrt{x^2 + \rho^2}}\right) = \tfrac{1}{2}U\rho^2 - m(1 + \cos\theta), \tag{5.2}$$

where r and θ denote polar coordinates in the meridian plane. There will be a stagnation point

$$x_N = -\sqrt{\frac{m}{U}} \tag{5.3}$$

on the negative x-axis, which is obtained by putting $u(x, 0)$ equal to zero. We see from (5.2) that the streamline $\psi = 0$ consists of the negative x-axis and of the dividing line

$$\rho^2 = \frac{2m}{U}(1 + \cos\theta), \qquad \left(\frac{m}{U} = x_N^2\right), \tag{5.4}$$

which contains the stagnation point x_N. For $x = +\infty$ we have $\theta = 0$ and $\rho = \rho_\infty = 2\sqrt{m/U}$. The radius ρ_0 of the main parallel, as we shall call it, is obtained by putting $x = 0$ and $\theta = \pi/2$. We find

$$\rho_0 = \sqrt{\frac{2m}{U}} = \sqrt{2}\,|\,x_N\,| = \frac{1}{\sqrt{2}}\rho_\infty. \tag{5.5}$$

A somewhat laborious computation shows that the curvature K of the profile is steadily decreasing from the value $(9U/16m)^{1/2}$ at the stagnation point to zero at infinity.

6. Distributed sources along the axis. Various shapes of bodies and half-bodies have been obtained by Fuhrmann,[3] Kármán[4] and their followers by using continuous distributions of sources and sinks along the x axis.

Denoting by $q(x)$ the density of the source distribution of an interval $0 \leqq x \leqq l$ and by U the velocity of the parallel flow and using (3.2), we obtain the following expression for the stream function ψ of the resulting flow:

$$\psi(x, \rho) = \tfrac{1}{2}U\rho^2 - \int_0^l q(\xi)\left\{1 + \frac{x - \xi}{\sqrt{(x - \xi)^2 + \rho^2}}\right\} d\xi. \tag{6.1}$$

The integral $\int_0^l q(\xi)d\xi$ represents the total strength m of all sources and sinks and is zero in the case of a closed profile. Point sources and sinks may be included in the formula without difficulty. The profile is given by the equation $\psi = 0$. By taking a constant positive density $q(x) = m/l$, we obtain a half-body corresponding to the stream function

$$\psi_l = \tfrac{1}{2}U\rho^2 - m\left(1 + \frac{r_0 + r_l}{l}\right), \tag{6.2}$$

where r_0 and r_l denote the distances of (x, ρ) from $x = 0$ and $x = l$.

[3] G. Fuhrmann, Dissertation, Goettingen, 1912.

[4] Th. von Kármán, Abhandlungen aus dem aerodynamischen Institut, Aachen, 1927.

Fuhrmann computed several profiles by taking for $q(x)$ a step function or a piecewise linear function of x. He used also some additional point sources. It is obvious that the influence of sources on the axis is predominantly felt only on nearby portions of the profile.

For a *given* profile an approximation can be obtained, according to Kármán, by subdividing an interval $0 \leqq x \leqq l$ in the interior of the profile into a finite number n of not necessarily adjacent subintervals. Let l_k denote the length of the kth subinterval and let $q_k = m/l_k$ be the constant density of sources assigned to this segment. Let us choose n points $x_1, \rho_1, \cdots, x_n, \rho_n$ on the given profile. In order that the profile $\psi = 0$ given by our sources and the parallel flow shall pass through the n prescribed points, we must have [see (6.2)]

$$\sum_{k=1}^{n} m_k\left(1 + \frac{r_{ik}' - r_{ik}''}{l_k}\right) = \tfrac{1}{2} U \rho_i^2; \qquad i = 1, 2, \cdots, n, \tag{6.3}$$

where r' and r'' denote the distance from the endpoints of the kth subinterval to the point x_i, ρ_i on the profile. In this way, n linear non-homogeneous equations for $m_1, \cdots, m_n$ are obtained, which can be solved provided that the determinant of the coefficients

$$c_{ik} = 1 + \frac{r_{ik}' - r_{ik}''}{l_k}$$

is different from zero. In practical cases all l_k are taken equal in order to reduce the already very considerable computational work.

For a given profile, the ordinate $\rho = \rho(x)$ is a given function of x. Substituting this function in (6.1) and putting $\psi[x, \rho(x)] = 0$, we obtain an integral equation for the determination of the unknown density $q(x)$. Unfortunately, the resulting integral equation is a Fredholm equation of the first kind and, up to now, has proved to be useless. In fact, it is nearly obvious that a distribution along the axis can give only a limited number of different types of profiles.

The restriction imposed by the exclusive use of sources and sinks distributed on the axis has prevented so far any further development of the method. In the next paragraphs we shall remove this restriction by considering symmetrical distributions located outside of the x-axis.

7. The potential of a homogeneous circumference. As already mentioned in the Introduction our investigations will be based to a certain extent on Beltrami's results which we shall present here with the revisions, corrections and extensions required for our purposes.

Let us consider an axially symmetric distribution of sources and sinks, not necessarily located on the x-axis. It is clear that in the present case the potential ϕ_c of a homogeneous circumference with its axis on the x-axis will play the same role as the potential of a point source plays in the general case of an arbitrary three dimensional flow. Beltrami[5] gives two expressions for this elementary potential with axial symmetry about the axis. For our purposes we shall use the expression involving Bessel functions as more suitable for computation of the associated stream function.

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[5] E. Beltrami, *Opere matematiche*, vol. 3, U. Hoepli, Milano, 1911 (especially pp. 349–382).

The alternative formula for ϕ_c and its relation to Laplace's expression for a potential has been recently investigated by Bateman and Rice.[6]

Beltrami's original proof of his formula for ϕ_c is complicated and will be replaced here by a more elementary proof covering several similar cases. This proof, which in its original form, was criticized by Watson[7] will be presented here with a slight addition which makes the reasoning convincing. (See also H. Bateman[8] who apparently accepts Watson's criticism.)

Bessel's Functions and Potential Theory. It is well known that the elementary method of separation of variables shows immediately that $e^{\pm xs} J_0(\rho s)$ is, for every value of the real parameter s, a solution of the equation (2.1) for a symmetric potential. ($J_0(z)$ denotes the Bessel function of index 0.) Throughout this paper we shall use Watson's notations. The function $J_0(z)$ is regular for every value of z and takes the value 1 for $z=0$.

Consider the potential function represented by the definite integral

$$\int_0^\infty e^{-|x|s} J_0(\rho s) F(s) ds, \qquad (x \neq 0) \tag{7.1}$$

in which the function $F(s)$ is supposed to be one which ensures uniform convergence and makes the limit of (7.1), as ρ tends to zero, equal to the result of making $\rho=0$ under the integral sign. When $x\neq 0$, this function takes the value

$$f(x) = \int_0^\infty e^{-|x|s} F(s) ds \tag{7.2}$$

on the x axis and may often be identified from the form of $f(x)$. In fact, we know from the elements of the theory of developments of potential functions in series, that an axially symmetric harmonic function is *uniquely* determined by the values it takes on a segment of the axis of symmetry.

Taking $F(s)=1$, we have $f(x)=|x|^{-1}$ by (7.2). The corresponding harmonic function is r^{-1}. By (7.1) we have, therefore, following formula

$$\int_0^\infty e^{-|x|s} J_0(\rho s) ds = \frac{1}{\sqrt{x^2+\rho^2}} \tag{7.3}$$

which will be used immediately. This formula is due to Lipschitz (Watson,[7] p. 384).

Let us now consider a homogeneous circumference C of radius b and of unit density with its center at the origin and its axis coinciding with the x-axis. The total mass (or total strength of sources) of C is $M=2\pi b$. By adding the elementary potentials due to the elements of C, we see at once that the potential of C at a point x of the axis is equal to

$$f(x) = -\frac{2\pi b}{\sqrt{x^2+b^2}}, \tag{7.4}$$

where $\sqrt{x^2+b^2}$ is the distance of x from any point of C. By using (7.2), (7.3), and (7.1) we immediately obtain the following expression for the potential ϕ_c of C:

[6] H. Bateman and S. O. Rice, American Journal of Mathematics, **60**, 297–308 (1938).

[7] G. N. Watson, *Theory of Bessel functions*, Cambridge University Press, 1922, p. 388.

[8] H. Bateman, *Partial differential equations*, Cambridge University Press, 1932, §7.32.

$$\phi_c(x, \rho) = -2\pi b \int_0^\infty e^{-|x|s} J_0(\rho s) J_0(bs) ds. \tag{7.5}$$

8. The stream function for a homogeneous circumference. In order to obtain the stream function ψ_c for our circumference C, let us consider the function

$$\psi_c^- = -2\pi b\rho \int_0^\infty e^{-|x|s} J_1(\rho s) J_0(bs) ds \tag{8.1}$$

for $x \leqq 0$. The Bessel function $J_1(z)$ vanishes for $z=0$ and satisfies the equation $J_1(z) = -J_0'(z)$. It is easily checked by (8.1) and (2.2) that ψ_c^- is the associated function to the potential ϕ_c as defined by (7.5), and it is obvious that $\psi_c^-(x, 0)$ vanishes for $x \leqq 0$. Similarly,

$$\psi_c^+(x, \rho) = 2\pi b\rho \int_0^\infty e^{-|x|s} J_1(\rho s) J_0(bs) ds \tag{8.2}$$

is for $x \geqq 0$, an associated function of (7.5). However, ψ_c^- and ψ_c^+ do not coincide on the positive ρ axis. In fact, we have as a special case of the discontinuous integral of Weber and Schafheitlin (Watson,[7] p. 406)

$$\int_0^\infty J_1(\rho s) J_0(bs) ds = \begin{cases} \dfrac{1}{\rho} & \text{for } \rho > b \geqq 0, \\ 0 & \text{for } b > \rho \geqq 0. \end{cases} \tag{8.3}$$

It follows that

$$\psi_c^-(-0, \rho) = \begin{cases} -2\pi b & \text{for } \rho > b \geqq 0, \\ 0 & \text{for } b > \rho \geqq 0, \end{cases} \tag{8.4}$$

and

$$\psi_c^+(+0, \rho) = \begin{cases} 2\pi b & \text{for } \rho > b \geqq 0, \\ 0 & \text{for } b > \rho \geqq 0. \end{cases} \tag{8.5}$$

Let us consider a simple closed curve L in the half plane $\rho \geqq 0$ containing the critical point $x=0$, $\rho=b$ (i.e., the trace of the circumference C) in its interior. The function ψ_c^+ is obviously the analytic continuation of ψ_c^- across the segment $0 \leqq \rho < b$. However, if we describe L counterclockwise and come back to a point $+0, \rho$, with $\rho > b$, we obtain a final value which is by $4\pi b$ greater than the initial value of ψ_c^-. It follows that *the stream function for a homogeneous circumference C is a many-valued function in the half-plane $\rho \geqq 0$ with the period $4\pi b$.* Since $2\pi b$ is the total strength M of the distributed sources, we can say that *the period is equal to $2M$.*

By rotation about the x axis the closed line L generates an axially symmetric torus, enclosing the sources on C. By the definition of the stream function, the outward flow across this torus is equal to 2π times the period of the stream function, i.e., this flow is equal to $4\pi M$, in perfect agreement with Gauss' theorem.

A single valued branch of the stream function ψ_c can be obtained in the domain D_b bounded by the x-axis and by the segment $x=0$, $0 \leqq \rho \leqq b$ by putting

$$\begin{aligned} \psi_c &= \overset{-}{\psi_c} && \text{for} \quad x \leqq 0, \\ \psi_c &= \overset{+}{\psi_c} - 4\pi b && \text{for} \quad x \geqq 0. \end{aligned} \tag{8.6}$$

This function is continuous in D_b, vanishes on the negative x-axis and, by (8.4), takes the constant value $-2\pi b$ on the streamline $x=0$, $\rho>b$. For a circumference of total strength m and of uniform density $m/2\pi b$, we would have

$$\psi_c(0, \rho) = -m, \quad \text{for} \quad \rho > b. \tag{8.7}$$

For $\rho>b$ the constant value of $\psi_c(0, \rho)$ is equal to $-m$, independently of the radius b of C.

Comparing our result with Beltrami's formulae for the same stream function, we see that on p. 355 of the paper quoted in Footnote 5 Beltrami, failing to notice that ψ_c is a many-valued function, puts the wrong branches together and obtains a discontinuity along the ρ-axis. The same error occurs in A. G. Webster.[9] The formula given by Bateman[8] (p. 417; example 1) is also inaccurate.

The velocity u on the x axis due to the sources on C can be easily obtained by differentiating the potential (7.4) with respect to x. Denoting by m the total strength of C, we find

$$u(x, 0) = \frac{mx}{(x^2 + b^2)^{3/2}}, \tag{8.8}$$

so that u is zero for $x=0$. The maximum of $|u|$ is attained for $x^2 = \frac{1}{2}b^2$, where $|u| = 2\cdot{}^{-2/3}mb^{-1/3}$.

9. The discontinuous integral of Weber-Schafheitlin. We give in this paragraph a new and simple proof for the evaluation of the integral of Weber-Schafheitlin [see (8.3)]. This proof is based on the general principles of Potential Theory and does not require any extensive technical knowledge of Bessel function.

As in Sec. 8 let us consider a homogeneous circumference C of radius b and unit total strength, so that $2\pi b=1$. Let S be a sphere with center at the origin and with radius $R>b$. By Gauss' fundamental theorem, the outward flow across S is 4π. The surface of S cuts the half-plane $\rho>0$ in a half-circle H. For $x\leqq 0$, the stream function $\psi_c(x, \rho)$ is given by (8.1). It takes the value zero along the negative x axis and up to the factor 2π, its value for a point P of H is given by the inward flow passing through the spherical cap generated by the rotation of the arc P, $-R$. The flow across a hemisphere being -2π, we have $\psi_c(0, R) = -1$. We see by (8.1) that

$$\lim_{x=0} \int_0^\infty e^{-|x|s} J_1(Rs) J_0(bs)\, ds = R^{-1}, \quad \text{for} \quad R > b. \tag{9.1}$$

On the other hand, it is known [and used in the proofs of (8.3)] that the left hand side of (8.3) can be obtained by putting $x=0$ in the integral (9.1). In this way we obtain

$$\int_0^\infty J_1(Rs) J_0(bs)\, ds = R^{-1}, \quad \text{for} \quad R > b.$$

In order to obtain the second part of (8.3) we have only to take a sphere with radius $R<b$.

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[9] A. G. Webster, *Partial differential equations*, New York, Hafner Publishing Co., 1947 p. 368 ff.

10. The potential and the stream function for a disc. Let us consider a disc of sources of radius b with its center at the origin and its axis coinciding with the x-axis. Let $q(\rho)$ denote the areal density of the sources; the total strength m being given by

$$m = 2\pi \int_0^b \rho q(\rho) d\rho. \tag{10.1}$$

By the principle of superposition, the potential ϕ_d of the disc is obtained by integration of (7.5). Replacing b in this formula by a variable of integration η, we find, after multiplication of $q(\eta)d\eta$ and integration from 0 to b,

$$\phi_d(x, \rho) = -2\pi \int_0^b \eta q(\eta) d\eta \int_0^\infty e^{-|x|s} J_0(\rho s) J_0(\eta s) ds. \tag{10.2}$$

Putting

$$\chi(s) = \int_0^b \eta q(\eta) J_0(\eta s) d\eta \tag{10.3}$$

and interchanging the order of integration in (10.2), we obtain

$$\phi_d(x, \rho) = -2\pi \int_0^\infty e^{-|x|s} \chi(s) J_0(\rho s) ds. \tag{10.4}$$

Putting $\rho = 0$ and using Lipschitz' integral (7.3), we have

$$\phi_d(x, 0) = -2\pi \int_0^\infty e^{-|x|s} \chi(s) ds = -2\pi \int_0^b \frac{\eta q(\eta) d\eta}{\sqrt{x^2 + \eta^2}} \tag{10.5}$$

on the x axis, a result easily verified by direct integration of (7.4). By differentiation of $\phi_d(x, 0)$ we obtain the velocity $u_d(x, 0)$ due to the sources on the disc:

$$u_d(x, 0) = 2\pi x \int_0^b \frac{\eta q(\eta) d\eta}{(x^2 + \eta^2)^{3/2}} \cdot \tag{10.6}$$

An alternative formula for u_d can be obtained in the following way. Assuming that the density $q(\rho)$ is differentiable, we have, by integration by parts,

$$\phi_d(x, 0) = -2\pi[q(b)\sqrt{x^2 + b^2} - q(0) | x |] + 2\pi \int_0^b q'(\eta) \sqrt{x^2 + \eta^2} d\eta. \tag{10.7}$$

Differentiating, we obtain

$$u_d(x, 0) = 2\pi \left[\mp q(0) - \frac{q(b) x}{\sqrt{x^2 + b^2}} + x \int_0^b \frac{q'(\eta) d\eta}{\sqrt{x^2 + \eta^2}} \right] \cdot \tag{10.8}$$

The upper sign holds for $x < 0$, the lower for $x > 0$. Furthermore,

$$\frac{du_d}{dx} = 2\pi \left[-\frac{q(b) b^2}{(x^2 + b^2)^{3/2}} + \int_0^b \frac{q'(\eta) d\eta}{(x^2 + \eta^2)^{3/2}} \right]. \tag{10.9}$$

Assuming that the density q is a positive and non-increasing function of ρ, we see that $|u_d(x, 0)|$ is a decreasing function of $|x|$.

We now turn to the stream function. Replacing b in (8.1), and (8.2) and (8.6) by a variable of integration η, multiplying by $q(\eta)d\eta$ and integrating from 0 to b, we obtain

$$\psi_d(x, \rho) = \mp 2\pi\rho \int_0^\infty e^{-|x|s}\chi(s)J_1(\rho s)ds \begin{cases} +0 & \text{for} \quad x \leqq 0, \\ -2m & \text{for} \quad x \geqq 0, \end{cases} \tag{10.10}$$

where $\chi(s)$ denotes the function defined by (10.3), while m is the total strength of the disc, given by (10.1). The stream function (10.10) is single-valued and continuous in the domain D_b, defined in Sec. 8. The value of ψ_d on the streamline $x=0$, $\rho>b$ is obtained by multiplying the corresponding value $-2\pi\eta$ for a circumference of radius η by $q(\eta)d\eta$ and integrating the product from 0 to b. Since the integral in question is by (10.10) equal to the total strength of the disc, we have

$$\psi_d(0, \rho) = -m \quad \text{for} \quad \rho > b. \tag{10.11}$$

The value of the stream function for a disc on the streamline $x=0$, $\rho>b$, is equal to $-m$, independently of its radius b and of the density function $q(\rho)$.

This value being given by the right hand side of (10.10) for $x \to -0$, we have the identity

$$\rho \int_0^\infty \chi(s)J_1(\rho s)ds = \int_0^b \eta q(\eta)d\eta \quad \text{for} \quad \rho > b \geqq 0 \tag{10.12}$$

which could have been obtained directly by integrating the formula of Weber-Schafheitlin.

The results of this paragraph could be easily generalized by taking discontinuous density functions $q(\rho)$, possessing a finite or even an infinite number of jumps. As a particular case, we would obtain the potential and the stream function for one or several concentric rings of finite width.

11. Single-valued and many-valued Stokes' stream functions. As we have already mentioned, the domain D of definition of any Stokes' stream function is a subdomain of the upper half-plane $-\infty < x < +\infty$, $\rho \geqq 0$. It consists of all points in which the flow under consideration is regular. Sources, sinks and other singularities, as well as the x axis, are on the boundary of D. The Stokes' function has been defined by its total differential. It is therefore to be expected that this function will be many-valued when D is not simply connected. This is, for instance, the case for a circumference of sources, the corresponding domain D being bounded by the x-axis and by the point $x=0$, $\rho=b$, representing the trace of the circumference in the upper half-plane. On the other hand, the domain $D=D_b$ (see Sec. 8) corresponding to a disc of sources is simply connected. In fact, its boundary is a single continuum consisting of the x axis and of the vertical segment $0 \leqq \rho \leqq b$. For this reason the stream function for a disc is necessarily single-valued, which is in agreement with the results of Sec. 10. For the same reason, any distribution of sources on the x axis will generate a single-valued stream function. The exclusive use of such sources has up to now prevented the recognition of an essential property of the Stokes' function.

In diagrams as usually given, the streamlines of an axially symmetric flow are drawn in the entire meridian plane, the lower half being the reflection of the upper one. In this representation, the domain corresponding to the simplest case of a point source at the origin appears to be doubly connected, as is the case for a point source in plane flow. However, the corresponding stream function (3.2) is single-valued, while the stream function of the plane flow is many-valued, as is to be expected. Our explanation concerning the domain of definition, removes the paradox and our remarks apply to all cases when associated functions are defined by their total differential.

12. Blunt-nosed profiles. We turn now to the investigation of flows obtained by superposition of a parallel flow of constant velocity U in the direction of the positive x-axis and of a flow produced by an axially symmetric distribution of sources and sinks, around the axis.

As a first significant case, let us consider a single disc of (positive) sources in a parallel flow. The potential ϕ and the stream function ψ of the combined flow is given by the principle of superposition. Using (2.5), (10.2) and (10.10) we have

$$\begin{aligned} \phi &= Ux + \phi_d, \\ \psi &= \tfrac{1}{2}U\rho^2 + \psi_d. \end{aligned} \tag{12.1}$$

Let us now consider a *family of discs of variable* radius b with their centers at the origin and their axes coinciding with the x axis, all these discs having the *same total strength* m, independently of b. For $b=0$, when the total strength m is concentrated in a point-source at the origin, the dividing profile $\psi=0$ will be the classical Blasius-Fuhrmann half-body. As b increases from zero, we will have for sufficiently small values of this parameter a new family of profiles of half-bodies intersecting the ρ-axis above the edge of the corresponding disc.

In order to obtain this point of intersection which will give us the radius of the *main parallel*, $x=0$, of the half-body, we have to solve the equation

$$\psi(0, \rho) = 0. \tag{12.2}$$

Since its solution ρ is, by assumption, greater than b, we obtain, from (12.1) and (10.11),

$$\tfrac{1}{2}U\rho^2 - m = 0.$$

This equation yields the following result: *The radius $\rho=\rho_0$ of the main parallel is given by*

$$\rho_0 = \sqrt{\frac{2m}{U}}\,. \tag{12.3}$$

We see that ρ_0 *is independent of the radius b of the disc, which cannot exceed* ρ_0.

The velocity components u and v are obtained from (2.1) by (2.4). On the negative x axis we have $v=0$ and

$$u(x, 0) = U + u_d(x, 0), \tag{12.4}$$

the term $u_d=\partial\phi_d/\partial x$ being negative. A general theorem on the normal derivative of a surface distribution (Kellogg)[10] yields for $x=-0$ the result

$$u(-0, 0) = U - 2\pi q(0). \tag{12.5}$$

[10] O. Kellogg, *Potential theory*, J. Springer, Berlin, 1929, p. 164, theorem VI.

(More generally, we have $u(-0, \rho)=U-2\pi q(\rho)$ for $0\leqq\rho<b$. Observe that the integral in Kellogg's theorem VI vanishes identically for a disc.) Let us now assume that the (positive) density $q(\rho)$ is a non-increasing function of ρ. According to Sec. 10, the velocity u is then negative on the x axis and increases steadily in absolute value as x increases from $-\infty$ to 0. At the same time the total velocity $u=U+u_d$ decreases steadily from the value U to $U-2\pi q(0)$ and remains therefore different from zero if $U-2\pi q(0)>0$. We have the following result: *the necessary and sufficient condition for the existence of a single stagnation point on the negative x axis is given by the inequality*

$$U \leqq 2\pi q(0). \tag{12.6}$$

Since $q(\rho)$ is non-increasing, the value $q(0)$ is not less than $m/\pi b^2$ which represents the constant density of a homogeneous disc with the same radius b and the same total strength m. Therefore, there will certainly be a single stagnation point for

$$U \leqq 2\pi m/\pi b^2 = 2m/b^2. \tag{12.7}$$

Considering U and m as given, we find that *a stagnation point x_N exists for $b\leqq(2m/U)^{1/2}$*. On the other hand, we have seen that a dividing line, $\psi=0$, cannot exist for $b>(2m/U)^{1/2}$. *To every disc of radius $b\leqq(2m/U)^{1/2}$ corresponds a profile of a half-body*, the radius of the main parallel being $\rho_0=(2m/U)^{1/2}$ independently of b.

Let $q_1(\rho)$ denote the density for a disc of radius one. Let us define the density for a disc of radius b by the formula

$$q(\rho) = \frac{1}{b^2} q_1\left(\frac{\rho}{b}\right), \qquad 0 \leqq \rho \leqq b. \tag{12.8}$$

The total strength is obviously independent of b. According to (12.4) and (10.6), the velocity on the x axis is given by

$$u = U + 2\pi x \int_0^b \frac{\eta q(\eta) d\eta}{(x^2+\eta^2)^{3/2}} = U + 2\pi x \int_0^1 \frac{\xi q_1(\xi) d\xi}{(x^2+b^2\xi^2)^{3/2}} \cdot \tag{12.9}$$

This formula shows that for any fixed negative value of x, the velocity u is steadily increasing with b. We see that *the distance $|x_N|$ of the stagnation point from the origin is* steadily decreasing with increasing b. As $b\to\rho_0$ its abscissa x_N tends to a limit value, which will be negative for $U-2\pi q(0)>0$, and zero for $U-2\pi q(0)=0$. (q denotes here the density for the disc of radius ρ_0). For a non-increasing density, the second case occurs only for a disc of uniform density $m/\pi\rho_0^2$.

A distribution of sources over a finite region acts at infinity as the so-called equivalent point source at the origin with the same total strength m. For this reason, all our profiles are half-bodies with the same asymptotic radius

$$\rho_\infty = \sqrt{2}\,\rho_0 \tag{12.10}$$

which has been computed in (5.5), for the limiting case of the Blasius-Fuhrmann half-body.

We summarize now our results, which, in their entirety, hold for a non-increasing positive density given by the distribution law (12.8). We have obtained a *family of half-bodies depending on a parameter b. The only member of this family mentioned in the existing literature, is the Blasius-Fuhrmann half-body corresponding to the limiting* *case $b=0$. As b increases from 0 to $\rho_0=(2m/U)^{1/2}$, we obtain the other members of the family which are (increasingly) blunt-nosed and tend, for $b\to(2m/U)^{1/2}$, to a limiting profile. All these profiles have the same main parallel of radius ρ_0 and the same asymptotic radius at infinity.* Similar results will be given later for other symmetric flows in three as well as in two dimensions.

In the following section, we shall turn to the investigation of some particular cases which we shall call the *integrable cases*. A computer would doubtless like to be given some examples, at least, in which the formula (10.10) for the stream function could be simplified by a convenient choice of the function $\chi(s)$. Unfortunately, $\chi(s)$ is connected with the density q by the integral equation of the first kind (10.3), and cannot be taken arbitrarily. So we have to take the density as the arbitrary function and discuss the *integrable cases*, in which the formula (10.3) can be simplified. Let us also point out that the distribution of sources is actually more important than the function $\chi(s)$, since it gives us at least a qualitative idea of the shape of the profile.

13. Discs of uniform density. In this case we have to take $q=m/\pi b^2$. By (10.3) we have

$$\chi(s) = \frac{m}{\pi b^2}\int_0^b \eta J_0(\eta s)d\eta.$$

Setting $s\eta=\xi$ and using the classical formula

$$\xi J_0(\xi) = \frac{d}{d\xi}\{\xi J_1(\xi)\}, \tag{13.1}$$

we obtain

$$\chi(s) = \frac{m}{\pi bs}J_1(bs). \tag{13.2}$$

By (12.1), (10.2) and (10.10), the explicit formulae for ϕ and ψ are therefore given by

$$\phi(x,\rho) = Ux - \frac{2m}{b}\int_0^\infty e^{-|x|s}J_0(\rho s)J_1(bs)\frac{ds}{s}, \tag{13.3}$$

$$\psi(x,\rho) = \tfrac{1}{2}U\rho^2 \mp \frac{2m\rho}{b}\int_0^\infty e^{-|x|s}J_1(\rho s)J_1(bs)\frac{ds}{s}\begin{cases}+\ 0 & \text{for } x\leqq 0\\ -\ 2m & \text{for } x\geqq 0.\end{cases} \tag{13.4}$$

The profile is given by the equation $\psi=0$. The terms containing the integrals have been denoted in Sec. 10 by ϕ_d and ψ_d, respectively. It is obvious that these functions could have been obtained directly by integration of the potential and stream functions for a circumference. Using the general equation (10.11), we obtain the formula

$$\int_0^\infty J_1(\rho s)J_1(bs)\frac{ds}{s} = \frac{1}{2}\,\frac{b}{\rho} \quad \text{for} \quad \rho > b \tag{13.5}$$

which is a special case of the Weber-Schafheitlin discontinuous integral [Watson,[7] p. 405 (1)] but which has been proved here directly from the basic principles of Potential Theory.

The velocity components are given by

$$u(x, \rho) = U \mp \frac{2m}{b}\int_0^\infty e^{-|x|s}J_0(\rho s)J_1(bs)ds, \tag{13.6}$$

$$v(x, \rho) = \frac{2m}{b}\int_0^\infty e^{-|x|s}J_1(\rho s)J_1(bs)ds. \tag{13.7}$$

The upper sign holds for $x \leqq 0$, the lower, for $x \geqq 0$. The stagnation point is obtained by putting $u(x, 0) = 0$ in the first Eq. (13.6). According to (12.9), this gives

$$U + \frac{2mx}{b^2}\int_0^b \frac{\eta d\eta}{(\eta^2 + x^2)^{3/2}} = 0; \qquad (x \leqq 0).$$

The integral is this formula can be easily computed. In this way we obtain, for $x = x_N$,

$$U - \frac{2m}{b^2}\left(1 + \frac{x_N}{\sqrt{x_N^2 + b^2}}\right) = 0.$$

Observing, that $2m/U = \rho_0^2$, by (12.3), we see that

$$x_N^2 = \frac{b^2(\rho_0^2 - b^2)}{\rho_0^4 - (\rho_0^2 - b^2)^2} \cdot \tag{13.8}$$

According to the general results of Sec. 12, the distance $|x_N|$ of the stagnation point from the disc takes, for $b = 0$, the value $|x_N| = \rho_0/\sqrt{2}$, corresponding to the Blasius-Fuhrmann half-body [*cf.* (15.5)], and decreases steadily to zero, as $b \to \rho_0$. This fact can be verified in an elementary way by using (13.8). The radius of the main parallel is $\rho_0 = \sqrt{2m/U}$; the asymptotic radius is $\rho_\infty = \rho_0\sqrt{2}$. The nose of the profile becomes blunter with increasing b. The value $b = \rho_0$ corresponds to a limiting singular case of an *ultra-flat profile** with a nose coinciding with the disc. The velocity in this case becomes infinite at the edge of the disc.

14. Curvature of the profiles. We have seen in Sec. 5 that the curvature K of the Blasius-Fuhrmann half-body (corresponding to $b = 0$) is steadily decreasing along the profile. For the other limiting case ($b = \rho_0$) the curvature of the nose of the ultra-flat profile is identically zero along the disc, but jumps suddenly to infinity at its edge. Since changes take place gradually, we may venture the following *conjectures about the curvature K of the intermediate profiles.*

As b increases from zero, the curvature of the profile will decrease in the vicinity of its stagnation point and will increase in the vicinity of the edge b. There will be a moment in which we will have an *equalization of curvatures* on a certain arc adjacent to the stagnation point. At this moment the profile will have a *nearly-spherical cap,* while the adjacent infinite branch of the dividing line will have a curvature steadily decreasing to zero. This stage can last for awhile, but, as b continues to increase, the edge of the disc, loaded with sources, will act like a spearhead repulsing the parts of

* Professor G. Birkhoff has kindly drawn my attention to the fact that the *potential* of the flow around the ultra-flat profile has been computed (in the neighborhood of the nose) by T. L. Smith in 1943.

the profile in its immediate neighborhood. There will appear a maximum of K in the vicinity of this edge, while the nose continues to become blunter. As b increases still further, this maximum will become greater and its ordinate will tend to ρ_0. The final stage will be reached when b takes the value ρ_0 at which moment this maximum becomes infinite.

15. Discs with Bessel's distribution of sources. Let us first consider a disc of unit radius. In order to obtain another integrable case, we take as surface density the function

$$q_1(\rho) = C_1 J_0(j\rho), \tag{15.1}$$

where $j = 2.40483$ denotes the smallest positive root of $J_0(z)$, so that $q(1) = 0$. (This is the most interesting case. However, the following consideration would hold with slight modifications for other positive values of the constant j.) The total strength m is connected with the constant C_1 by the equation

$$2\pi C_1 \int_0^1 \eta J_0(j\eta)d\eta = m.$$

Using the classical formula (13.1), we obtain

$$C_1 = \frac{mj}{2\pi J_1(j)} \cdot \tag{15.2}$$

It is well known that $J_1(j)$ is different from zero. For $m>0$, the density $q_1(\rho)$ is positive and decreasing in the interval $0 \leqq \rho \leqq 1$.

Let us now consider a disc of radius b and of the same strength m. According to (12.8), we put

$$q(\rho) = \frac{mj}{2\pi b^2 J_1(j)} J_0\left(\frac{j\rho}{b}\right) \tag{15.3}$$

and call this formula *Bessel's distribution law.*

Let us introduce the abbreviations

$$C = \frac{mj}{2\pi b^2 J_1(j)}, \qquad \alpha = \frac{j}{b} \cdot \tag{15.4}$$

By (15.3), we have then

$$q(\rho) = CJ_0(\alpha\rho). \tag{15.5}$$

The integral in (10.3) can be computed by using the following fundamental formula of the theory of Bessel functions

$$(\alpha^2 - s^2)\int_0^b \eta J_0(\alpha\eta)J_0(s\eta)d\eta = \left[\eta\{sJ_0(\alpha\eta)J_0'(s\eta) - \alpha J_0(s\eta)J_0'(\alpha\eta)\}\right]_{\eta=1}^{\eta=b}, \tag{15.6}$$

which gives for (10.3)

$$\chi(s) = \frac{C\alpha b}{\alpha^2 - s^2} J_0(bs)J_1(\alpha b) = \frac{mj^2}{2\pi(j^2 - b^2s^2)} J_0(bs). \tag{15.7}$$

We remark that the function $\chi(s)$ remains finite and continuous for $s=j/b$, since $J_0(j)=0$.

Inserting the expression (15.7) in the general formulae of Sec. 10 for a disc and adding a uniform flow, we obtain the formulae for the resulting flow. From Sec. 12 we know that there is exactly one stagnation point x_N on the negative x axis for every value of $b<\rho_0$ and that x_N steadily tends to a negative limiting value with $b\to\rho_0$. All profiles pass through the point $x=0$, $\rho=\rho_0$ and all possess the same asymptotic radius $\rho_\infty=\rho_0\sqrt{2}$.

Let us consider the limiting profile (corresponding to $b=\rho_0$) which passes through the edge of the disc at the point $\rho=\rho_0$ of the ρ-axis. The density of the sources being zero at that point, we can extend the definition of $q(\rho)$ to a greater disc by putting $q(\rho)=0$ for $\rho>\rho_0$. The point $\rho=\rho_0$ is now an interior point of the greater disc in which the density is continuous and satisfies a Hoelder condition. It follows from well known theorems on tangential and normal derivatives of a surface distribution (Kellogg,[10] p. 162, theorem V and p. 164, theorem VI) that the velocity components u and v remain continuous for $x=0$, $\rho=\rho_0$, and that the limiting profile has, at the point in question, a continuous tangent. However, the curvature of the profile becomes infinite at the same point. (All these results can be easily checked by the formulae given in this paragraph, provided that certain precautions are taken in the use of discontinuous integrals.) Our limiting profile is non-analytic at the point ρ_0. We note that a *singularity on the profile* can be obtained only when the profile passes through the boundary of the region occupied by the sources.