ON THE CHARACTERIZATION OF FIELDS OF DIABATIC FLOW*

BY

B. L. HICKS

Ballistic Research Laboratories, Aberdeen Proving Ground

INTRODUCTION

In an earlier paper (Reference 1) the steady flow of a compressible fluid containing distributed heat sources was described in terms of the velocity vector **V** and alternately two other vectors **W** and **M**. For rotational flow the **W** language was found to lead to more simple differential equations than the other two languages. The physical content of the equations expressed in each of the three languages was, of course, the same. When each of the vector fields **V**, **W**, **M**, was in turn assumed to be irrotational, however, the corresponding fields of diabatic flow were radically different in their physical characteristics.

These results suggest several questions concerning the characterization of fields of steady diabatic flow:

(*i*) is **W** the most convenient vector to use in formulating the equations for rotational flow?

(*ii*) how do the restrictions upon the heat source function Q depend upon the character of the vector function **N** chosen to represent an irrotational field of flow?

(*iii*) can a relation be established between the character of a vector representation **N** of an irrotational flow field and the form of the partial differential equation for the potential function φ_N ?

We answer these questions by introducing into the flow equations a vector **N** proportional to **V**, the scalar proportionality factor (V/N) not at first being explicitly restricted. From the form alone of the derived equations in **N**, it is possible to answer the first question. The assumption of irrotationality of the **N** field, $\nabla \times \mathbf{N} = 0$, then leads to the required information about irrotational fields of flow.

Through proper choice of the **V**, **N** relation and subsequent inspection of the form of the **N** equations, a number of new types of irrotational diabatic flow can be discussed by giving their *character*, specified by the function $g(N) = V/(N^2RT)^{1/2}$ and its *form*, specified by an arbitrary function $F(\varphi_N)$ that enters the partial differential equation for the potential function φ_N . Four general types of flow are chosen for discussion to illustrate the wide variety of diabatic irrotational flows, as compared to adiabatic irrotational flows, that are possible even without explicit formulation of boundary conditions. We thereby emphasize the need for preliminary and simultaneous investigation of all steady diabatic flows at the formulational level before the attempt is made to construct special solutions.

---

*Received April 5, 1948. Summaries of this paper and of related studies of diabatic flow have been presented at meetings of the American Physical Society. [Cf. Phys. Rev. **71**, 476(A), **72**, 179(A) (1947), **73**, 636 (L), **74**, 1230 (A) (1948).]

BASIC EQUATIONS

1. Derivation of N equations. We use essentially the same fundamental equations as in Reference 1, namely those governing the steady motion of an inviscid compressible fluid that contains heat sources.

$$\frac{1}{\rho}\nabla p + \mathbf{V}\cdot\nabla\mathbf{V} = 0, \tag{1.1}$$

$$\nabla\cdot\rho\mathbf{V} = 0, \tag{1.2}$$

$$p - R\rho T = 0, \tag{1.3}$$

$$c_p\mathbf{V}\cdot\nabla T_t = T\mathbf{V}\cdot\nabla S = Tc_p\mathbf{V}\cdot\nabla \log (Tp^{-(\gamma-1)/\gamma}) = Q. \tag{1.4}$$

The quantity Q is the heat added locally per unit mass of fluid and unit time. In Reference 1 these equations were studied in the **V**, **W**, and **M** languages, **W** and **M** being related to **V** by the equations

$$\mathbf{V} = V_t\mathbf{W} = a\mathbf{M} \tag{1.5}$$

in which $V_t = (2c_pT_t)^{1/2}$ is the limiting velocity corresponding to local stagnation temperature T_t and $a = (\gamma RT)^{1/2}$ will be called the local velocity of sound.* In the present paper we make the more general substitution

$$\mathbf{V} = (gRT)^{1/2}\mathbf{N} \tag{1.6}$$

which includes the **W** and **M** transformations when g is appropriately specialized. For the present g can be any function of the coordinates and other variables and will not be restricted until later. Both g and $N = |\mathbf{N}|$ are dimensionless.

Substitution of (1.6) in (1.1, 2, 4) and elimination of V and ρ gives

$$\nabla \log p + g\mathbf{N}\cdot\nabla\mathbf{N} + \frac{1}{2}g\mathbf{N}\mathbf{N}\cdot\nabla \log gT = 0, \tag{1.7}$$

$$\nabla\cdot\mathbf{N} + \mathbf{N}\cdot\nabla \log p + \frac{1}{2}\mathbf{N}\cdot\nabla \log (g/T) = 0, \tag{1.8}$$

$$-\frac{\gamma-1}{\gamma}\mathbf{N}\cdot\nabla \log p + \mathbf{N}\cdot\nabla \log T = Q/[c_pT(gRT)^{1/2}] = 2q_N . \tag{1.9}$$

In (1.9) we have defined a quantity q_N which represents in dimensionless form the local heating of the fluid. We adjoin the equation obtained from (1.7) by scalar multiplication of **N**

$$\mathbf{N}\cdot\nabla \log p + \frac{1}{2}N^2g\mathbf{N}\cdot\nabla \log T = -\frac{1}{2}N^2g\mathbf{N}\cdot\nabla \log g - g\mathbf{N}\cdot\nabla\frac{1}{2}N^2 \tag{1.10}$$

---

*In a gas undergoing dissociation or chemical reactions, the velocity of sound is a function of the local thermochemical properties of the gas.

and solve (1.9, 10) for $\mathbf{N}\cdot\nabla \log p$, $\mathbf{N}\cdot\nabla \log T$ in terms of $\mathbf{N}\cdot\nabla\,(N^2/2)$, $\mathbf{N}\cdot\nabla \log g$ and q_N :

$$\mathbf{N}\cdot\nabla \log p = -g\left(\mathbf{N}\cdot\nabla \frac{1}{2}N^2 + \frac{1}{2}N^2\mathbf{N}\cdot\nabla \log g + N^2 q_N\right)\Big/\left(1 + \frac{\gamma-1}{2\gamma}N^2 g\right), \tag{1.11}$$

$$\mathbf{N}\cdot\nabla \log T$$
$$= \left[2q_N - \frac{\gamma-1}{\gamma}\, g\left(\mathbf{N}\cdot\nabla\frac{1}{2}N^2 + \frac{1}{2}N^2\mathbf{N}\cdot\nabla \log g\right)\right]\Big/\left(1 + \frac{\gamma-1}{2\gamma}N^2 g\right). \tag{1.12}$$

Substitution in (1.7,8) gives the equations of motion and continuity in $\mathbf{N}$ language.

$$\begin{aligned}\nabla \log p = g\left(1 + \frac{\gamma-1}{2\gamma}N^2 g\right)^{-1}&\left(\frac{\gamma-1}{2\gamma}\, g\mathbf{N}\cdot\nabla\frac{1}{2}N^2 - \frac{1}{2}\mathbf{N}\cdot\nabla \log g - q_N\right)\mathbf{N}\\ &- g\nabla\frac{1}{2}N^2 + g\mathbf{N}\times(\nabla\times\mathbf{N}),\end{aligned} \tag{1.13}$$

$$\begin{aligned}\nabla\cdot\mathbf{N} = \left(1 + \frac{\gamma-1}{2\gamma}N^2 g\right)^{-1}&\left[-\frac{1}{2}\left(1 - \frac{N^2 g}{\gamma}\right)\mathbf{N}\cdot\nabla \log g\right.\\ &\left.+ \frac{\gamma+1}{2\gamma}\, g\mathbf{N}\cdot\nabla\frac{1}{2}N^2 + (1 + N^2 g)q_N\right].\end{aligned} \tag{1.14}$$

For completeness we list the expressions for $\mathbf{N}\cdot\nabla \log T_t$, $\mathbf{N}\cdot\nabla \log p_t$, quantities proportional to the rate of variation of stagnation temperature and of stagnation pressure in the direction of flow.

$$\left(1 + \frac{\gamma-1}{2\gamma}N^2 g\right)\mathbf{N}\cdot\nabla \log T_t = \frac{1}{c_p}\mathbf{N}\cdot\nabla S = 2q_N\,, \tag{1.15}$$

$$\mathbf{N}\cdot\nabla \log p_t = -N^2 g q_N\Big/\left(1 + \frac{\gamma-1}{2\gamma}N^2 g\right). \tag{1.16}$$

As we might expect, these variations are proportional to q_N and vanish for $q_N = 0$. Furthermore T_t increases and p_t decreases in the direction of flow for positive q_N . The decrease of p_t owing to heat addition is known as "momentum pressure drop" in one-dimensional theory (compare Eqs. (15) and (60) in Reference 2 and references cited there).

2. Significance of choice of the function g. We can now observe how the character of the function g affects the appearance of the $\mathbf{N}$ equation. The factor $[1 + (\gamma - 1)N^2 g/2\gamma]$ appears in both Eqs. (1.13, 14) together with g, ∇g and q_N . If either g or q_N is an explicit function of T (cf. the special case of $\mathbf{V}$ language described in Sec. 3) then it may not be especially profitable to reduce the equations to the $\mathbf{N}$ form because T cannot be determined and eliminated from (1.13, 14) prior to complete solution of the set (1.1, 2, 3, 4) or the set (1.12, 13, 14, 15). If, on the other hand, g and q_N are functions of N, p and the coordinates x_i or even of the individual components N_i of $\mathbf{N}$ together with p and the coordinates, one can regard (1.13, 14) as a set of four simultaneous non-linear partial differential equations in the four dependent variables $\mathbf{N}$, p.

We choose to study a less general function g, namely

$$g = g(N) \tag{2.1}$$

corresponding to the form for q_N

$$q_N = q_N(N_i\ , x_i). \tag{2.2}$$

When g and q_N are thus restricted, the $\mathbf{N}$ equation of motion (1.13) contains as dependent variables only $\mathbf{N}$ and p and the $\mathbf{N}$ equation of continuity (1.14) *only* $\mathbf{N}$. Equation (1.14) can then be rewritten in a more compact form which exhibits the "sources" proportional to $q\mathbf{N}$ in the $g\mathbf{N}/H_1$ field, letting $g' = dg/dN$,

$$\nabla \cdot H_1\mathbf{N} = (1 + N^2 g)\left(1 + \frac{\gamma - 1}{2\gamma} N^2 g\right)^{-1} H_1 q_N\ , \tag{2.3}$$

where

$$\log H_1(N) = -\frac{1}{2}\int \left[\frac{\gamma + 1}{\gamma}\, g - \left(1 - \frac{N^2 g}{\gamma}\right)\frac{g'}{Ng}\right]\left(1 + \frac{\gamma - 1}{2\gamma} N^2 g\right)^{-1} d\,\frac{1}{2} N^2. \tag{2.4}$$

Equation (2.1) includes both $\mathbf{W}$ and $\mathbf{M}$ languages for which g has the forms

$$g_W = \frac{2\gamma}{\gamma - 1}(1 - W^2)^{-1}, \qquad (N = W), \tag{2.5}$$

$$g_M = \gamma, \qquad (N = M). \tag{2.6}$$

Since $N^2 g = V^2/RT$ has the same value for all $g(N)$, the relations among V, N, W and M are

$$(V^2/RT) = N^2 g = \frac{2\gamma}{\gamma - 1} W^2(1 - W^2)^{-1} = \gamma M^2 \tag{2.7}$$

These equations are often used in the form

$$\left(1 + \frac{\gamma - 1}{2\gamma} N^2 g\right) = (1 - W^2)^{-1} = \left(1 + \frac{\gamma - 1}{2} M^2\right). \tag{2.8}$$

It is only $\mathbf{W}$ language that a term in $\mathbf{N} \cdot \nabla\, N$(or $\nabla \cdot \mathbf{N}$) does not appear in the equation of motion. The $\mathbf{W}$ language therefore yields more simple equations than any other $\mathbf{N}$ (or $\mathbf{V}$) language, and $\mathbf{W}$ is thus the most convenient vector to use in formulating the equations for rotational flow.

PROPERTIES OF IRROTATIONAL N FIELDS

3. Integrability of the equation of motion. We now suppose that the $\mathbf{N}$ field is continuous and irrotational and therefore admits a potential function φ_N

$$\mathbf{N} = \nabla \varphi_N\ . \tag{3.1}$$

The consequences of the assumption of irrotationality for an $\mathbf{N}$ field were first examined in References 3a, b for the adiabatic **M** and **W** cases. In Reference 1 we saw that irrotational *diabatic* **M** and **W** flows were quite different from one another. We may therefore now expect a still wider variety of flows because of flexibility in choice of the function g.

With $\nabla \times \mathbf{N} = 0$, the equation of motion becomes

$$\nabla \log p + \eta \nabla \varphi_N + g \nabla \frac{1}{2} N^2 = 0 \tag{3.2}$$

in which

$$\begin{aligned} \eta &= \frac{1}{2} g \mathbf{N} \cdot \nabla \log (gT) \\ &= g\left(1 + \frac{\gamma - 1}{2\gamma} N^2 g\right)^{-1}\left[q_N - \frac{\gamma - 1}{2\gamma} g\mathbf{N} \cdot \nabla \frac{1}{2} N^2 + \frac{1}{2} \mathbf{N} \cdot \nabla \log g\right]. \end{aligned} \tag{3.3}$$

We distinguish two cases, $\eta = 0$ and $\eta \neq 0$. For the degenerate case $\eta = 0$ we suppose that g is not restricted to be a function of N alone. Then from (3.2, 3)

$$\nabla \log p + g \nabla \frac{1}{2} N^2 = 0, \tag{3.4}$$

$$\frac{\gamma - 1}{2\gamma} g\mathbf{N} \cdot \nabla \frac{1}{2} N^2 - \frac{1}{2} \mathbf{N} \cdot \nabla \log g = q_N . \tag{3.5}$$

Equation (3.4) implies that p and g are functions of N alone. Also Eq. (3.3) shows that (gT) and therefore, by (1.6), that (V/N) are constant on each streamline. This type of flow therefore corresponds to irrotational **V** flow (see Reference 1) that has been generalized to allow variation of (gT) between streamlines.

When $\eta \neq 0$, it is sufficient to consider only functions of N alone as was shown to be appropriate in Sec. 2. If the integrability condition with $g' = dg/dN$

$$\eta = \frac{1}{2} g\left[2q_N + N\mathbf{N} \cdot \nabla N\left(\frac{g'}{Ng} - \frac{\gamma - 1}{\gamma} g\right)\right]\left(1 + \frac{\gamma - 1}{2\gamma} N^2 g\right)^{-1} = F(\varphi_N) \tag{3.6}$$

for the equation of motion (3.2) is satisfied then the latter integrates to

$$\log P_N + \int F(\varphi_N)\, d\varphi_N = \text{constant}, \tag{3.7}$$

where

$$P_N = pH_2(N) = p \exp \int g\, d\, \tfrac{1}{2} N^2. \tag{3.8}$$

(Integration of (3.3) also yields a relation among N, p and T along each streamline.) Equation (3.7) is the analogue of the barotropic condition $p = f(\rho)$ for irrotational **V** fields of flow. It states that P_N is constant on potential surfaces and expresses the variation of P_N along streamlines, i.e. $\partial \log P_N / \partial s = NF(\varphi_N)$. The function P_N becomes equal to the stagnation pressure $P_W = p_t$ in **W** language. In general the behavior of P_N is similar to p_t in that as N (or W) increases, p decreases for constant P_N or p_t. It is noted that in terms of H_2, the continuity equation (1.14) becomes (compare (2.3))

$$\nabla \cdot (g\mathbf{N}/H_2) = H_2^{-1}(1 + N^2 g) F(\varphi_N) \tag{3.9}$$

showing that "sources" in the $(g\mathbf{N}/H_2)$ field are proportional to $F(\varphi_N)$.

The integrability condition can be rewritten in two other useful forms by combining it with the continuity equation (1.14). Then

$$g^{-1}\left[\frac{\gamma+1}{\gamma}g - \left(1 - \frac{N^2 g}{\gamma}\right)\frac{g'}{Ng}\right]F(\varphi_N) = \left(\frac{g'}{Ng} - \frac{\gamma-1}{\gamma}g\right)\nabla^2\varphi_N + 2\left(g - \frac{g'}{Ng}\right)q_N \tag{3.10}$$

and

$$g^{-1}\left(1 - \frac{N^2 g}{\gamma}\right)\left(1 + \frac{1}{2}\frac{Ng'}{g}\right)F(\varphi_N) = -\frac{1}{2}N^3\left(\frac{g'}{Ng} - \frac{\gamma-1}{\gamma}g\right)\nabla\cdot\mathbf{s} + \left[1 - N^2\left(g - \frac{g'}{Ng}\right)\right]q_N \tag{3.11}$$

in which $\mathbf{s} = \mathbf{N}/N$ is a unit vector in the direction of flow. According to Reference 3a,

$$\nabla\cdot\mathbf{s} = \lim_{\Delta A\to 0}\frac{1}{\Delta A}\frac{\partial\Delta A}{\partial s}$$

can be interpreted as the fractional rate of change of stream tube area ΔA with respect to arc length ds along the stream tube. Equations (3.6, 10, 11) are alternative statements of the relation between the character of the function $g(N)$ and the mode of variation of $\nabla\cdot\mathbf{s}$ and q_N throughout the field of flow. Since the coefficients in these equations depend only upon N and $g(N)$, through special choice of the function $g(N)$ we will later be enabled to study flows characterized by special relations among the functions q_N, $\nabla\cdot\mathbf{s}$ and $F(\varphi_N)$.

4. The partial differential equation satisfied by the potential function φ_N. If $\mathbf{N} = \nabla\varphi_N$ is substituted directly into (1.14), the resulting equation contains q_N as well as derivatives of φ_N. Because in general the function q_N is not open to arbitrary specification in an irrotational $\mathbf{N}$ field, it is desirable to consider $F(\varphi_N)$ that appears in the integrability condition as the basic function describing the form of the flow. For a given $g(N)$ and $F(\varphi_N)$ the behavior of q_N follows from the integrability condition. It is therefore best to use Eq. (3.9) which does not contain q_N and which can be expanded to read

$$\nabla\cdot\mathbf{N} = g^{-1}(1 + N^2 g)F(\varphi_N) + \left(g - \frac{g'}{Ng}\right)\mathbf{N}\cdot\nabla\frac{1}{2}N^2. \tag{4.1}$$

Substitute

$$\nabla\cdot\mathbf{N} = \sum_i\frac{\partial^2\varphi_N}{\partial x_i^2}, \qquad \mathbf{N}\cdot\nabla\frac{1}{2}N^2 = \sum_i\left(\frac{\partial\varphi_N}{\partial x_i}\right)^2\frac{\partial^2\varphi_N}{\partial x_i^2} + 2\sum_{i>j}\frac{\partial\varphi_N}{\partial x_i}\frac{\partial\varphi_N}{\partial x_j}\frac{\partial^2\varphi_N}{\partial x_i\,\partial x_j}.$$

The desired differential equation for the potential function is then

$$\sum_i\frac{\partial^2\varphi_N}{\partial x_i^2}\left[1 - G\left(\frac{\partial\varphi_N}{\partial x_i}\right)^2\right] - 2G\sum_{i>j}\frac{\partial\varphi_N}{\partial x_i}\frac{\partial\varphi_N}{\partial x_j}\frac{\partial^2\varphi_N}{\partial x_i\,\partial x_j} = \left(N^2 + \frac{1}{g}\right)F(\varphi_N) \tag{4.2}$$

in which

$$G = g - \frac{g'}{Ng}. \tag{4.3}$$

The type of this partial differential equation is determined by the sign of the determinant $D = \det |\alpha_{ij}|$ where α_{ij} is the coefficient of $\partial^2\varphi_N/\partial x_i \partial x_j$ in the partial differential equation. Calculation of D gives

$$D = 1 - N^2 G = 1 + N\frac{g'}{g} - N^2 g. \tag{4.4}$$

The differential equation (4.2) is therefore of

$$\left.\begin{array}{l}\text{elliptic}\\[2ex] \text{parabolic}\\[2ex] \text{hyperbolic}\end{array}\right\} \quad \text{type for} \quad \left\{\begin{array}{l} G < N^{-2},\\[2ex] G = N^{-2},\\[2ex] G > N^{-2}.\end{array}\right. \tag{4.5}$$

Since G is a function of N alone, the equation can change type within the field of flow where N attains the values specified in (4.5). This behavior is similar to that of irrotational adiabatic flow in the **V** field which is of elliptic, parabolic or hyperbolic type for M <, =, >1. There is however, a major difference. The character of the function $g(N)$ alone determines the values of N, if there be any, for which each of the expressions in (4.5) is satisfied. Appropriate choice of $g(N)$ can then lead to flows whose type is fixed (e.g. elliptic for all values of N) as well as to flows which change type for specified values of N. Because the character of g likewise determines the mutual behavior of $\nabla \cdot \mathbf{s}$, q_N and $F(\varphi_N)$ (cf. Eq. (3.11)), the required connection between character of g and behavior of q_N has thereby been demonstrated. Whenever, for example, $D = 0$, the relation between $\nabla \cdot \mathbf{s}$ and $F(\varphi_N)$ does not depend upon q_N (cf. Eq. 3.11). In a later section, we shall study some flows illustrating these remarks.

There may be a question at first sight as to whether $F(\varphi_N)$ and q_N can be found such that φ_N will satisfy both (3.10) and (4.2). We can analyze the situation as follows. Let $\mathbf{N} = [N_i]$, $i = 1, 2, 3$. For rotational flow there are five dependent variables N_i, p and q_N that satisfy the four equations (1.13, 14) giving an underdetermined system. The variation of one dependent variable, e.g. p, q_N (or one N_i) can then be specified arbitrarily. Now when $N_i = \partial\varphi_N/\partial x_i$ we have seven equations in the six variables N_i, p, q_N and φ_N and the system is overdetermined. The equation of motion (3.2) however reduces to the two equations (3.6, 7) which contain the new (arbitrary) function $F(\varphi_N)$. As this function is not determinable from the basic equation we regard its specification as part of the formulation of our problem and can consider then (3.7) to be an integral of the motion which eliminates p from further consideration. We now take (3.10) and (1.14 or 4.2) to form the basic system that must be solved. This system contains q_N, F_N, φ_N and its derivatives and is accordingly underdetermined. If we choose q_N to be the known function (1.14) with ($N = \nabla\varphi_N$) can presumably be solved for φ_N. There would then be, however, no reason to suppose that these functions φ_N and q_N would satisfy (3.10). We shall consequently regard (4.2) as the basic equation. Choice of the functions g and $F(\varphi_N)$ will specify the *character* and *form* of the flow respectively. Presumably specification of both form and character will work a corresponding restriction upon the nature of the boundary conditions that can be imposed on the function φ_N. For example we have already seen that the type and change of type (with corresponding adjustment of the nature of the appropriate boundary conditions) of the partial differential equation (4.2) is entirely governed by the character of the flow.

The solution of a problem in diabatic irrotational flow can then be visualized as consisting of the following steps. With character and form of the flow specified together with matching boundary conditions, Eq. (4.2) can be solved for φ_N. Substitution of φ_N thus determined in the integrability condition (3.6 or 10) and the "Bernoulli" type equation (3.7) yields the functions q_N and p. The system consisting of Eqs. (1.13, 14) can then be regarded as solved. Determination of the temperature follows by integration of (1.12) if T is known at one point on each streamline. The velocity $\mathbf{V}$ and heat function Q can also be calculated subsequently from (1.6, 9).

The indirect nature of this solution process must be noted. Although one might normally expect Q to be known rather than $F(\varphi_N)$, it is the latter that, at present at least, must be chosen first and Q thereafter determined. It is this peculiarity of irrotational diabatic flow which leads us to investigate in general fashion irrotational flows whose character only is specified, in order that we may derive more explicit equations containing q_N.

THE RELATIONS AMONG CHARACTER, FORM, TYPE CHANGE AND HEAT SOURCES

5. Methods of investigation. We now propose to illustrate the remarks in the last section by examining a number of flows of given character. There are several paths we may follow in proceeding to this specialization. We could for example try to find all functions for which the flow remains of a given type. This procedure is simple only for cases for which D is constant throughout the flow. We might also return to Eq. (3.11), which connects the physically visualizable quantities $\nabla \cdot \mathbf{s}$ and q_N, and impose conditions on the coefficients appearing there in order to insure, for example, that $\nabla \cdot \mathbf{s}$ be proportional to $F(\varphi_N)$ throughout the field of flow. As a third possibility we could assume *a priori* convenient special forms for $g(N)$ (such as rational functions) and examine the corresponding characteristics of the flow. Any of these procedures will lead to conditions upon $g(N)$ and will supply partial answers to the questions implied by the other procedures.

For most of the special flows to be discussed now the first procedure is convenient. Let us suppose throughout the flow

$$D = D_0 \text{ constant.} \tag{5.1}$$

Accordingly as D_0 is $>$, $=$ or <0 we shall term the flow field elliptic, parabolic or hyperbolic. By (4.4) we are then led to the differential equation for $g(N)$

$$N \frac{g'}{g} - N^2 g + 1 - D_0 = 0, \tag{5.2}$$

whose solution for $D_0 \neq -1$ is

$$g = k^{-2}(1 + D_0)[(N/k)^{1-D_0} - (N/k)^2]^{-1} \tag{5.3}$$

and for $D_0 = -1$

$$g = N^{-2}(\log k - \log N)^{-1}. \tag{5.4}$$

In both cases, (and also for D not a constant)

$$G = \frac{1 - D_0}{N^2} \tag{5.5}$$

the partial differential equation for φ_N becomes

$$\begin{aligned}\sum_i \frac{\partial^2 \varphi_N}{\partial x_i^2}\left[N^2 - (1 - D_0)\left(\frac{\partial \varphi_N}{\partial x_i}\right)^2\right] - 2(1 - D_0)\sum_{i>j}\frac{\partial \varphi_N}{\partial x_i}\frac{\partial \varphi_N}{\partial x_j}\frac{\partial^2 \varphi_N}{\partial x_i\,\partial x_j}& \\ = N^2\left(N^2 + \frac{1}{g}\right)F(\varphi_N)&\end{aligned} \tag{5.6}$$

and the integrability condition in the form (3.11) simplifies to

$$g^{-1}\left(1 - \frac{N^2 g}{\gamma}\right)(1 + D_0 + N^2 g)F(\varphi_N) = 2D_0 q_N + N\left(1 - D_0 - \frac{N^2 g}{\gamma}\right)\nabla\cdot\dot{\mathbf{s}}. \tag{5.7}$$

The flows characterized by (5.6) do not resemble "incompressible" adiabatic irrotational flows for small N unless $D_0 = 1$ (cf. Sec. 6) because all terms in the coefficient of $\partial^2\varphi_N/\partial x_i \partial x_j$ are of order N^2. The Glauert-Prandtl type of approximation can be made, however, for suppose that throughout the flow

$$\frac{N_1}{N_0} \sim 1, \qquad \frac{N_2}{N_0} \sim \frac{N_3}{N_0} \sim 0,$$

where N is taken to differ but little from N_0 anywhere. Then (5.6) becomes

$$D_0 \frac{\partial^2 \varphi_N}{\partial x_1^2} + \frac{\partial^2 \varphi_N}{\partial x_2^2} + \frac{\partial^2 \varphi_N}{\partial x_3^2} = \left[N_0^2 + \frac{1}{g(N_0)}\right]F(\varphi_N). \tag{5.8}$$

That the value of D_0 determines the type of the differential equation is now quite clear. Equation (5.7) may be developed in similar approximate fashion once the value of D_0 is chosen, and the function H_2 (Eq. (3.8)) can be regarded as a function of N_0.

For some flows of interest (such as **W** and **M** flows) the type of the partial differential equation (4.2) may change in the flow field. A simple application of the second procedure is then appropriate. Let

$$g = g_0 + g_1 N + g_2 N^2 + \cdots. \tag{5.9}$$

Then for $N \ll 1$

$$G = g_0 + g_1 N - \frac{g_1 + 2g_2 N}{N(g_0 + g_1 N)} + \cdots. \tag{5.10}$$

Only for $g_0 \neq 0$ will Eq. (4.2) admit an "incompressible" approximation for $N \ll 1$,

$$\nabla^2 \varphi_N = g_0^{-1} F(\varphi_N). \tag{5.11}$$

The Glauert-Prandtl approximation can be written down for almost any choice of $g(N)$.

6. Elliptic flow with $\mathbf{D}_0 = 1$. The general formula (5.3) for g corresponding to flow of constant type reduces for $D_0 = 1$ to $G = 0$ and

$$g = H_2 = 2/(k^2 - N^2). \tag{6.1}$$

The equation for φ_N (compare (2.3) and (3.10)) is

$$\nabla^2\varphi_N = \frac{1}{2}(k^2 + N^2)F(\varphi_N) \tag{6.2}$$

which in the Glauert-Prandtl approximation (cf. (5.8)) is

$$\nabla^2\varphi_N = \frac{1}{2}(k^2 + N_0^2)F(\varphi_N). \tag{6.3}$$

The heating parameter q_N is given by

$$q_N = \frac{1}{2}k^2\left(k^2 - \frac{\gamma+2}{\gamma}N^2\right)(k^2 - N^2)^{-1}F(\varphi_N) + \frac{1}{\gamma}N^3(k^2 - N^2)^{-1}\nabla\cdot\mathbf{s}. \tag{6.4}$$

The "Bernoulli" equation becomes

$$\log\left(\frac{p}{k^2 - N^2}\right) + \int F(\varphi_N)\,d\varphi_N = \text{constant}. \tag{6.5}$$

Since the relation between W and N is

$$W^2 = (\gamma - 1)N^2/(\gamma k^2 - N^2); \tag{6.6}$$

values of $W^2 = 0$, $(\gamma - 1)/(\gamma + 1)$, 1 correspond to 0, $\gamma/(\gamma + 2)$, 1 for $(N/k)^2$.

It is noted that flow discontinuities such as stationary shocks and combustion fronts cannot occur within this or any other field wholly of elliptic type.

7. Parabolic flow: $\mathbf{D}_0 = 0$. The only function g that gives wholly parabolic flow is

$$g = 1/N(k - N) \tag{7.1}$$

obtained from (5.3) where $D_0 = 0$. In this case $G = 1/N^2$, $H_2 = (k - N)^{-1} = Ng$. The partial differential equation for φ_N becomes

$$\sum_i \left[N^2 - \left(\frac{\partial\varphi_N}{\partial x_i}\right)^2\right]\frac{\partial^2\varphi_N}{\partial x_i^2} - 2\sum_{i>j}\frac{\partial\varphi_N}{\partial x_i}\frac{\partial\varphi_N}{\partial x_j}\frac{\partial^2\varphi_N}{\partial x_i\,\partial x_j} = kN^3F(\varphi_N) \tag{7.2}$$

of which the Glauert-Prandtl approximation is

$$\frac{\partial^2\varphi_N}{\partial x_2^2} + \frac{\partial^2\varphi_N}{\partial x_3^2} = kN_0F(\varphi_N). \tag{7.3}$$

When $D \equiv D_0 \equiv 0$ the coefficient of q_N in Eq. (3.11) vanishes and the equation expresses the proportionality of $\nabla\cdot\mathbf{s}$ and $F(\varphi_N)$ (cf. also 2.3)

$$\nabla\cdot\mathbf{s} = kF(\varphi_N). \tag{7.4}$$

The condition for vanishing of the coefficient of q_N is thus synonymous with the condition for wholly parabolic flow. Equation (7.4) expresses a simple geometrical property of parabolic flow, namely that the fractional rate of change of stream tube area in the direction of flow is a function of φ_N only. Accordingly, the fractional rate of change of area of every stream tube intersecting a given potential surface is the same at that surface.

Because (3.11) and (7.4) do not contain q_N we use (3.10) to express its variation

$$q_N = \frac{1}{2} N(k - N)^{-1}\left(\frac{\gamma + 1}{\gamma} N^2 - 2kN + k^2\right)F(\varphi_N) + \frac{1}{2}(k - N)^{-1}\left(k - \frac{\gamma + 1}{\gamma} N\right)\nabla^2\varphi_N . \tag{7.5}$$

The "Bernoulli" equation becomes

$$\log \frac{p}{k - N} + \int F(\varphi_N)\, d\varphi_N = \text{constant}, \tag{7.6}$$

and the relation between W and N

$$W^2 = \frac{\gamma - 1}{2\gamma} N\Big/\left(k - \frac{\gamma + 1}{2\gamma} N\right) \tag{7.7}$$

corresponding to $W^2 = 0, (\gamma - 1)/(\gamma + 1), 1$ for $(N/k) = 0, \gamma/(\gamma + 1), 1$.

8. Hyperbolic flow with $D_0 = -(\gamma + 1)$. The character of g which is sufficient if the coefficient of $F(\varphi_N)$ in (3.11) is to vanish is expressed as

$$g = \gamma/N^2. \tag{8.1}$$

This corresponds to a type of wholly hyperbolic flow specified by $D_0 = -(\gamma + 1)$, $k \to \infty$ and to $G = +(\gamma + 2)/N^2$, $H_2 = N^\gamma$. The equation for φ_N is

$$\sum_i \left[N^2 - (\gamma + 2)\left(\frac{\partial \varphi_N}{\partial x_i}\right)^2\right] \frac{\partial^2 \varphi_N}{\partial x_i^2} - 2(\gamma + 2) \sum_{i>j} \frac{\partial \varphi_N}{\partial x_i}\frac{\partial \varphi_N}{\partial x_j}\frac{\partial^2 \varphi_N}{\partial x_i\, \partial x_j} = \frac{\gamma + 1}{\gamma} N^4 F(\varphi_N) \tag{8.2}$$

or in the condensed form of (3.9)

$$\nabla\cdot(N^{-\gamma-2}\mathbf{N}) = \frac{\gamma + 1}{\gamma} N^{-\gamma} F(\varphi_N). \tag{8.3}$$

The Glauert-Prandtl approximation is

$$-(\gamma + 1)\frac{\partial^2 \varphi_N}{\partial x_1^2} + \frac{\partial^2 \varphi_N}{\partial x_2^2} + \frac{\partial^2 \varphi_N}{\partial x_3^2} = \frac{\gamma + 1}{\gamma} N_0^2 F(\varphi_N). \tag{8.4}$$

Since the coefficient of $F(\varphi_N)$ in (3.11) is zero, q_N is given by the simple expression

$$q_N = \frac{1}{2} N \nabla\cdot\mathbf{s}. \tag{8.5}$$

In this case the area change of stream tubes is determined solely by q_N and N. The Bernoulli equation is

$$\log(pN^\gamma) + \int F(\varphi_N)\, d\varphi_N = \text{constant}, \tag{8.6}$$

and the relation between W and N

$$W^2 = \frac{\gamma - 1}{\gamma + 1}. \tag{8.7}$$

This particular wholly hyperbolic flow is thus characterized by a constant value of W (or $M = 1$) throughout the field of flow.

9. Mixed flow. It has been seen how proper choice of the character of the flow, as defined by the function $g(N)$, can establish flow whose type is constant throughout the field of flow. That the type may change within the flow field is illustrated by reconsideration of the **W** and **M** fields of irrotational diabatic flow (Reference 1). (The functions g_W and g_M are expressible in the form of (5.9) and accordingly admit the incompressible as well as the Glauert-Prandtl approximation; see Reference 1.) The partial differential equation for φ_M is of elliptic, parabolic or hyperbolic type accordingly as $M(=N) <, =,$ or $> (1/\gamma)^{1/2}$. The partial differential equation for φ_W changes type similarly at $W = N = [(\gamma - 1)/(\gamma + 1)]^{1/2}$, $(M = 1)$. It is the latter field which completes the set of three flows determined by the vanishing in turn of the coefficients of q_N, $F(\varphi_N)$ and $\nabla \cdot \mathbf{s}$ in equation (3.11), for an irrotational **W** field (Reference 1) possesses the property that

$$q_W = \frac{\gamma - 1}{2\gamma} F(\varphi_N) \tag{9.1}$$

10. Concluding remarks. If, in the definition $\mathbf{N} = \mathbf{V}[g(N)RT]^{-1/2}$, g is taken to be a function of N alone, any **N** language yields a simpler set of fundamental differential equations for rotational diabatic flow than the **V** language. In particular the **W** language (derived from $g = 2\gamma/(1 - N^2)(\gamma - 1)$) corresponds to the most simple expression of the equation of motion and hence is best suited to describe a general rotational diabatic flow.

With proper choice of the function $g(N)$, which is taken to determine the *character* of the flow, the partial differential equation for the potential function φ_N of irrotational **N** flow can be made wholly of one type (e.g. elliptic) or of mixed type throughout the field of flow. The differential equation for φ_N always contains an arbitrary function $F(\varphi_N)$ which fixes the *form* of the flow and also occurs in the analogue of the Bernoulli equation. Choice of the flow *character* alone leads to the form of the relation among $F(\varphi_N)$, $\nabla \cdot \mathbf{s}$ (the fractional rate of change of stream tube area in the direction of flow) and q_N (proportional to the local rate of heat addition per unit mass of fluid). Because of the universal occurrence of $F(\varphi_N)$ in the differential equations for φ_N there is need for further study of such quasi-linear partial differential equations containing arbitrary functions of the dependent variable if irrotational diabatic flows are to be characterized more fully with any degree of generality.

References

1. B. L. Hicks, *Diabatic flow of a compressible fluid*, Q. Appl. Math. **6**, 221–237 (1948).
2. B. L. Hicks, D. J. Montgomery and R. H. Wasserman, *On the one-dimensional theory of steady compressible flow in ducts with friction and heat addition*, J. Appl. Phys. **18**, 891-902 (1947).
3. a. B. L. Hicks, P. E. Guenther and R. H. Wasserman, *New formulations of the equations for steady compressible flow*, Q. Appl. Math. **5**, 357-61 (1947).
   b. B. L. Hicks, *On explicit solutions of the equations of steady compressible flow*, Phys. Rev. **72**, 433 (1947).