QUARTERLY OF APPLIED MATHEMATICS

Vol. VI JANUARY, 1949 No. 4

AERODYNAMIC SYMMETRY OF PROJECTILES[1]

BY

C. G. MAPLE AND J. L. SYNGE

Carnegie Institute of Technology

1. Introduction. The theory of the motion of a projectile is usually based on the assumption that the projectile possesses complete rotational symmetry, in the sense that rotation through any angle about the axis of symmetry leaves the projectile as a whole unchanged in outward form and in internal mass distribution. This assumption of symmetry is suitable in the case of an ordinary shell, but is not suitable in the case of a finned projectile which possesses symmetry of a more restricted type. The purpose of the present paper is to investigate the effect of various types of rotational and reflectional symmetry on the aerodynamic force system acting on a projectile. According to the basic aerodynamic hypothesis[2] the aerodynamic force system depends on the instantaneous motion of the projectile. The form of this functional dependence is restricted by the symmetry of the projectile. Our purpose is to investigate this restriction mathematically.

The instantaneous motion of a projectile is described by choosing some base-point O, fixed in the projectile, and specifying the velocity ($\mathbf{u}$) of O and the angular velocity ($\boldsymbol{\omega}$) of the projectile. The aerodynamic force system, exerted by the air on the projectile, is described by the equipollent system consisting of a force ($\mathbf{F}$) acting at O, together with a couple ($\mathbf{G}$). Then the vectors $\mathbf{F}$ and $\mathbf{G}$ are functions of the vectors $\mathbf{u}$ and $\boldsymbol{\omega}$. They are also functions of the air density (ρ) and the local sound velocity (c), but these functional dependences will be suppressed as a matter of notational convenience. Choosing any rectangular coordinate axes $Ox_1x_2x_3$, we may express the functional dependence of force system on motion by the six equations

$$
\begin{aligned}
F_m &= f_m(u_1, u_2, u_3, \omega_1, \omega_2, \omega_3), \\
G_m &= g_m(u_1, u_2, u_3, \omega_1, \omega_2, \omega_3), \qquad (m = 1, 2, 3),
\end{aligned}
\tag{1.1}
$$

u_m, ω_m, F_m, G_m being components along Ox_m.

2. Types of symmetry. Let us consider projectiles possessing one or both of the following types of symmetry (Fig. 1):

(*i*) n-gonal rotational symmetry about an axis A;

(*ii*) reflectional symmetry in a plane P.

Symmetries may be described in terms of "covering operations" which leave the projectile as a whole unchanged. We shall be concerned only with the aerodynamic force

[1]Received March 15, 1948. The theory contained in this paper was developed under contract with Aberdeen Proving Ground, and is now published with permission.

[2]K. L. Nielsen and J. L. Synge, Q. Appl. Math. 4, 201-226 (1946).

system and not with the dynamics of the projectile. This means that we are interested in its external form, but not in its distribution of mass. Consequently all remarks about symmetry are to be interpreted relative to the external form only. The covering operation for n-gonal rotational symmetry about an axis A is a rotation about A through an angle α where $\alpha = 2\pi/n$, n being an integer greater than unity. (An ordinary bomb with four fins possesses tetragonal symmetry, $n = 4$.) For certain formal purposes, it is convenient to include the case $n = 1$, although this actually implies no symmetry. The covering operation for reflectional symmetry in a plane P is a reflection in P in the ordinary geometrical sense, i.e. each particle of the projectile is replaced by another particle at a position such that the line joining the two positions is bisected orthogonally by P. The covering operation for n-gonal symmetry may be carried out physically; that for reflectional symmetry cannot.

However, we may view the question of symmetry in a different, but equivalent, way. We may apply the covering operations, not to the projectile, but to a set of axes. Consider first the case of n-gonal symmetry about an axis A. Choose a set of rectangular axes $Ox_1x_2x_3$ with origin on A. Apply the covering operation to these axes, leaving the body unmoved. Let the new axes be $Ox_1'x_2'x_3'$. The two sets of axes are *equivalent* in the following sense: Any general statement about the connection between aerodynamic force system and motion must have the same *form* no matter which axes are used. Otherwise put, it is impossible for an experimenter to determine, by aerodynamic measurements, which of the two equivalent sets of axes he is using. This means that the six aerodynamic functions in (1.1) are the same for two sets of equivalent axes. If primes indicate components along $Ox_1'x_2'x_3'$, we must have

$$\begin{aligned} F_m' &= f_m(u_1', u_2', u_3', \omega_1', \omega_2', \omega_3'), \\ G_m' &= g_m(u_1', u_2', u_3', \omega_1', \omega_2', \omega_3'), \qquad (m = 1, 2, 3), \end{aligned} \tag{2.1}$$

f_m and g_m being *the same functions* in (1.1) and (2.1).

The same remarks apply in the case of reflectional symmetry in a plane P. In this case, we choose the origin O on P, and reflect $Ox_1x_2x_3$ in P to obtain $Ox_1'x_2'x_3'$. However some care must be exercised in this case because the orientation of the axes is changed. This introduces no difficulty in the case of the polar vectors $\mathbf{u}$ and $\mathbf{F}$, but must be taken into consideration in the case of the axial vectors $\boldsymbol{\omega}$ and $\mathbf{G}$. This point will be discussed below in the appropriate place.

On account of the ease with which the notation of complex variable lends itself to the treatment of rotations in a plane, we find it convenient to use the following notation:

$$u = u_1 + iu_2, \qquad \omega = \omega_1 + i\omega_2, \qquad F = F_1 + iF_2, \qquad G = G_1 + iG_2, \tag{2.2}$$

with similar definitions for the primed quantities. Let us rewrite (1.1) and (2.1) in the form

$$\begin{aligned} F &= f(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3), \qquad F_3 = f_3(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3), \\ G &= g(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3), \qquad G_3 = g_3(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3), \end{aligned} \tag{2.3}$$

$$\begin{aligned} F' &= f(u', \bar{u}', u_3', \omega', \bar{\omega}', \omega_3'), \qquad F_3' = f_3(u', \bar{u}', u_3', \omega', \bar{\omega}', \omega_3') \\ G' &= g(u', \bar{u}', u_3', \omega', \bar{\omega}', \omega_3'), \qquad G_3' = g_3(u', \bar{u}', u_3', \omega', \bar{\omega}', \omega_3'), \end{aligned} \tag{2.4}$$

where the bars denote complex conjugates. Here the functions f and g are complex functions of complex arguments, and f_3 and g_3 are real functions of complex arguments. Since we have changed the arguments in passing from (1.1) and (2.1) to (2.3) and (2.4), the functional forms f_3 and g_3 are different in the two sets of equations. No confusion will arise from this, because we shall not refer to (1.1) or (2.1) again.

3. Consequences of n-gonal rotational symmetry. We suppose that the projectile has n-gonal rotational symmetry about an axis A. We choose the origin O on A, the axis Ox_3 along A, and the other two axes in the plane perpendicular to A, but otherwise arbitrary except for the condition that they must of course be at right angles. We consider some state of motion and the corresponding aerodynamic force system. These are described respectively by the complex and real components $(u, u_3, \omega, \omega_3)$ and (F, F_3, G, G_3) on the axes $Ox_1x_2x_3$. Now apply to the axes the covering operation, i.e. a rotation about Ox_3 through an angle $\alpha = 2\pi/n$. *The same motion and the corresponding force system* may also be described by the complex and real components $(u', u_3', \omega', \omega_3')$, (F', F_3', G', G_3') on the new axes $Ox_1'x_2'x_3'$. The transformation from one set of components to the other is easily seen to be as follows:

$$\begin{aligned} &u' = ue^{-i\alpha}, \qquad u_3' = u_3, \qquad \omega' = \omega e^{-i\alpha}, \qquad \omega_3' = \omega_3, \\ &F' = Fe^{-i\alpha}, \qquad F_3' = F_3, \qquad G' = Ge^{-i\alpha}, \qquad G_3' = G_3. \end{aligned} \tag{3.1}$$

If we substitute these in (2.4) and then compare with (2.3), we get the following equations, which must be satisfied for arbitrary values of $(u, u_3, \omega, \omega_3)$:

$$\begin{aligned} e^{i\alpha}f(ue^{-i\alpha}, \bar{u}\, e^{i\alpha}, u_3, \omega e^{-i\alpha}, \bar{\omega}e^{i\alpha}, \omega_3) &= f(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3), \\ e^{i\alpha}g(ue^{-i\alpha}, \bar{u}\, e^{i\alpha}, u_3, \omega e^{-i\alpha}, \bar{\omega}e^{i\alpha}, \omega_3) &= g(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3), \\ f_3(ue^{-i\alpha}, \bar{u}\, e^{i\alpha}, u_3, \omega e^{-i\alpha}, \bar{\omega}e^{i\alpha}, \omega_3) &= f_3(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3), \\ g_3(ue^{-i\alpha}, \bar{u}\, e^{i\alpha}, u_3, \omega e^{-i\alpha}, \bar{\omega}e^{i\alpha}, \omega_3) &= g_3(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3). \end{aligned} \tag{3.2}$$

We now make an important assumption, namely, that the functions f, f_3, g, g_3 may be expanded in power series in the transverse components $u_1, u_2, \omega_1, \omega_2$, or, equivalently, in power series in the complex quantities $u, \bar{u}, \omega, \bar{\omega}$. We shall write these power series in the form

$$\begin{aligned} F &= f(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3) = \sum_{pqrs} F_{pqrs}(u_3, \omega_3)u^p\bar{u}^q\omega^r\bar{\omega}^s, \\ G &= g(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3) = \sum_{pqrs} G_{pqrs}(u_3, \omega_3)u^p\bar{u}^q\omega^r\bar{\omega}^s, \\ F_3 &= f_3(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3) = \sum_{pqrs} F^{(3)}_{pqrs}(u_3, \omega_3)u^p\bar{u}^q\omega^r\bar{\omega}^s, \\ G_3 &= g_3(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3) = \sum_{pqrs} G^{(3)}_{pqrs}(u_3, \omega_3)u^p\bar{u}^q\omega^r\bar{\omega}^s, \end{aligned} \tag{3.3}$$

the summations covering the ranges $p, q, r, s = 0, 1, 2, \cdots$. Here the functions

$$F_{pqrs}(u_3, \omega_3),\ G_{pqrs}(u_3, \omega_3),\ F^{(3)}_{pqrs}(u_3, \omega_3),\ G^{(3)}_{pqrs}(u_3, \omega_3)$$

are the same functions for equivalent axes.

We note that the coefficients $F^{(3)}_{pqrs}$ and $G^{(3)}_{pqrs}$ are in general complex, and certain relations must be satisfied by them in order that f_3 and g_3 may be real. We have $f_3 = \bar{f}_3$, and so

$$\begin{aligned}\sum_{pqrs} F^{(3)}_{pqrs}(u_3, \omega_3)u^p\bar{u}^q\omega^r\bar{\omega}^s &= \sum_{pqrs} \bar{F}^{(3)}_{pqrs}(u_3, \omega_3)\bar{u}^p u^q\bar{\omega}^r\omega^s \\ &= \sum_{pqrs} \bar{F}^{(3)}_{qpsr}(u_3, \omega_3)u^p\bar{u}^q\omega^r\bar{\omega}^s.\end{aligned} \tag{3.4}$$

This holds for arbitrary values of $u_1, u_2, \omega_1, \omega_2$, and may be treated formally as an identity in $u, \bar{u}, \omega, \bar{\omega}$. A similar identity arises from g_3, and so we have

$$F^{(3)}_{pqrs}(u_3, \omega_3) = \bar{F}^{(3)}_{qpsr}(u_3, \omega_3), \quad G^{(3)}_{pqrs}(u_3, \omega_3) = \bar{G}^{(3)}_{qpsr}(u_3, \omega_3), \qquad (p, q, r, s = 0, 1, 2, \cdots). \tag{3.5}$$

We see that $F^{(3)}_{pprr}(u_3, \omega_3)$ and $G^{(3)}_{pprr}(u_3, \omega_3)$ are real; this result is important later in the paper. (The summation convention for repeated suffixes does not operate, here or elsewhere.)

We now substitute the series (3.3) in (3.2), and obtain the following result from the first of (3.2):

$$\sum_{pqrs} F_{pqrs}(u_3, \omega_3)u^p\bar{u}^q\omega^r\bar{\omega}^s = \sum_{pqrs} \exp\,[i\alpha(1 - p + q - r + s)]F_{pqrs}(u_3, \omega_3)u^p\bar{u}^q\omega^r\bar{\omega}^s. \tag{3.6}$$

This may be treated formally as an identity in $u, \bar{u}, \omega, \bar{\omega}$, and we obtain

$$F_{pqrs}(u_3, \omega_3) \sin\frac{1}{2}\alpha(1 - p + q - r + s) = 0, \qquad (p, q, r, s = 0, 1, 2, \cdots). \tag{3.7}$$

Thus all coefficients $F_{pqrs}(u_3, \omega_3)$ must vanish except those for which p, q, r, s are such that

$$\sin\frac{1}{2}\alpha(1 - p + q - r + s) = 0. \tag{3.8}$$

Since $\alpha = 2\pi/n$, we see that *the only surviving coefficients $F_{pqrs}(u_3, \omega_3)$ in the case of n-gonal rotational symmetry are those for which p, q, r, s satisfy one of the equations*

$$p - q + r - s - 1 = mn \qquad (m = 0, \pm 1, \pm 2, \cdots) \tag{3.9}$$

(We recall that p, q, r, s are of course non-negative integers.) Similarly, *the only surviving coefficients $G_{pqrs}(u_3, \omega_3)$ in the case of n-gonal rotational symmetry are those for which p, q, r, s satisfy one of the equations* (3.9).

A similar argument applied to the third and fourth equations of (3.2) shows that *the only surviving coefficients $F^{(3)}_{pqrs}(u_3, \omega_3)$ and $G^{(3)}_{pqrs}(u_3, \omega_3)$ in the case of n-gonal rotational symmetry are those for which p, q, r, s satisfy one of the equations*

$$p - q + r - s = mn \qquad (m = 0, \pm 1, \pm 2, \cdots). \tag{3.10}$$

The results are most easily interpreted if we introduce P and Q defined by

$$P = p + r, \qquad Q = q + s. \tag{3.11}$$

We note that as p, q, r, s cover the range $0, 1, 2, \cdots$, P and Q cover the same range. For a given value of P, there will occur in the power series (3.3) the combinations

$$u^P, \quad u^{P-1}\omega, \quad u^{P-2}\omega^2, \quad \cdots \quad u\omega^{P-1}, \quad \omega^P, \tag{3.12}$$

and for a given value of Q the combinations

$$\bar{u}^Q, \quad \bar{u}^{Q-1}\bar{\omega}, \quad \bar{u}^{Q-2}\bar{\omega}^2, \quad \cdots \quad \bar{u}\bar{\omega}^{Q-1}, \quad \bar{\omega}^Q. \tag{3.13}$$

The result regarding surviving terms may be displayed as follows:

$$\begin{array}{cc} F_{pqrs}(u_3\,,\,\omega_3) & F^{(3)}_{pqrs}(u_3\,,\,\omega_3) \\ G_{pqrs}(u_3\,,\,\omega_3) & G^{(3)}_{pqrs}(u_3\,,\,\omega_3) \\ \text{survive only if} & \text{survive only if} \\ P - Q = mn + 1 & P - Q = mn \\ \multicolumn{2}{c}{m = 0, \pm 1, \pm 2, \cdots} \\ \multicolumn{2}{c}{(P = p + r, \quad Q = q + s)} \end{array} \tag{3.14}$$

Since, for small cross velocity and cross spin, the significance of the terms in the series (3.3) decreases with increasing degree, it appears advisable to arrange the terms in order of increasing degree. The degree of a term with coefficient having subscripts p, q, r, s is

$$D = p + q + r + s = P + Q. \tag{3.15}$$

To get a surviving coefficient F_{pqrs} or G_{pqrs} for a term of degree D, we must satisfy the first of the following equations and one of the second:

$$P + Q = D, \qquad P - Q = mn + 1 \qquad (m = 0, \pm 1, \pm 2, \cdots). \tag{3.16}$$

Equivalently,

$$P = \frac{1}{2}(D + 1 + mn), \qquad Q = \frac{1}{2}(D - 1 - mn). \tag{3.17}$$

To get a surviving coefficient $F^{(3)}_{pqrs}$ or $G^{(3)}_{pqrs}$ for a term of degree D, we must satisfy the first of the following equations and one of the second:

$$P + Q = D, \qquad P - Q = mn \qquad (m = 0, \pm 1, \pm 2, \cdots). \tag{3.18}$$

Equivalently,

$$P = \frac{1}{2}(D + mn), \qquad Q = \frac{1}{2}(D - mn). \tag{3.19}$$

The equations (3.16) and (3.18) form a sieve by means of which we reject certain terms from the series (3.3), the existence of the rejected terms being in fact inconsistent with the assumed symmetry. It is difficult to describe in words the detailed consequences of the operation of this sieve. Some statements will be made below, but the most rapid way to see the results is to consult the diagrams (Figs. 2a–n). These show by heavy dots the surviving terms for symmetries ranging from digonal to octagonal.

To interpret these diagrams, let us take Fig. 2e as an example. This refers to the cross force F and cross torque G in the case of tetragonal symmetry. P and Q are plotted as rectangular Cartesian coordinates; since neither P nor Q can be negative, only the first quadrant is shown. The two families of lines are those given by (3.16) for $D = 0, 1, 2, \cdots$ and $m = 0, \pm 1, \pm 2; \cdots$. The dots are those intersections of these lines which occur at integer or zero values of P and Q. Lines, such as $P + Q = 2$, which give no such intersections, are not shown.

We may analyze the diagram by proceeding through increasing values of the degree D. There is no term of zero degree, nor indeed of any even degree; this is consistent with the general Theorem I, given later.

For terms of the first degree, we have $D = 1$, and an intersection occurs at $P = 1$ and $Q = 0$. Thus, by (3.11), we have surviving terms with

$$p = 1, \qquad q = 0, \qquad r = 0, \qquad s = 0,$$

and

$$p = 0, \qquad q = 0, \qquad r = 1, \qquad s = 0.$$

Consequently there are linear terms in the expansions of F and G of the forms

$$\begin{aligned} &F_{1000}(u_3\,,\,\omega_3)u + F_{0010}(u_3\,,\,\omega_3)\omega, \\ &G_{1000}(u_3\,,\,\omega_3)u + G_{0010}(u_3\,,\,\omega_3)\omega, \end{aligned} \tag{3.20}$$

respectively, and no other linear terms.

For terms of the third degree, we have $D = 3$, and intersections occur at $(P = 2, Q = 1)$ and $(P = 0, Q = 3)$. Thus we have surviving terms for the following values of p, q, r, s:

$$\begin{array}{cccc} p & q & r & s \\[4pt] 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{array}\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}}\right\} P = 2, \qquad Q = 1$$

$$\begin{array}{cccc} 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 3 \end{array}\left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} P = 0, \qquad Q = 3$$

Thus the following terms of the third degree occur in the expansion (3.3) of F, with similar terms in the expansion of G:

$$
\begin{aligned}
&F_{2100}(u_3, \omega_3)u^2\bar{u} + F_{1110}(u_3, \omega_3)u\bar{u}\omega + F_{0120}(u_3, \omega_3)\bar{u}\omega^2 \\
&\quad + F_{2001}(u_3, \omega_3)u^2\bar{\omega} + F_{1011}(u_3, \omega_3)u\omega\bar{\omega} + F_{0021}(u_3, \omega_3)\omega^2\bar{\omega} \\
&\quad + F_{0300}(u_3, \omega_3)\bar{u}^3 + F_{0201}(u_3, \omega_3)\bar{u}^2\bar{\omega} + F_{0102}(u_3, \omega_3)\bar{u}\bar{\omega}^2 \\
&\quad + F_{0003}(u_3, \omega_3)\bar{\omega}^3.
\end{aligned} \tag{3.21}
$$

There are no other surviving terms of the third degree in these expansions.

In the case of the diagrams for the axial components F_3, G_3, the lines shown have the equations (3.18). The reader should have no difficulty in interpreting the other details of the diagrams. Let us now make some general statements with regard to the consequences of n-gonal rotational symmetry.

First, let us consider consequences of evenness or oddness in the order n of the n-gonal rotational symmetry. The following results follow easily from (3.17) and (3.19):

I. *In the case of n-gonal rotational symmetry of EVEN order (digonal, tetragonal, hexagonal, etc.), the series* (3.3) *for F and G contain only terms of ODD degree, and the series* (3.3) *for F_3 and G_3 contain only terms of EVEN degree.*

II. *In the case of n-gonal rotational symmetry of ODD degree (trigonal, pentagonal, etc.), the lowest EVEN degree occurring in the series* (3.3) *for F and G is* $(n - 1)$, *and the lowest ODD degree occurring in the series* (3.3) *for F_3 and G_3 is n.*

Secondly, let us consider the survival of terms of low degrees ($D = 0, 1, 2$).

If $D = 0$, then $P = Q = 0$. In the case of F and G, we have to satisfy the second of (3.16) in the form $0 = mn + 1$. If $n \neq 1$, this cannot be satisfied by any m in the range $0, \pm 1, \pm 2, \cdots$. If $n = 1$, it is satisfied by $m = -1$. Thus we have the result:

III. *The absolute terms in the series* (3.3) *for F and G survive only in the degenerate case* $n = 1$, *which corresponds to no real symmetry at all.*

In the case of F_3 and G_3 we have to satisfy the second of (3.18) in the form $0 = mn$. This can always be done by taking $m = 0$. Hence we have the result:

IV. *The series* (3.3) *for F_3 and G_3 always have absolute terms.*

Now let us consider terms of the first degree, $D = 1$. Then either

$$P = 1, \qquad Q = 0, \tag{3.22}$$

or

$$P = 0, \qquad Q = 1. \tag{3.23}$$

In the case of F and G, we have to satisfy the second of (3.16). Corresponding to (3.22) and (3.23) respectively, this gives

$$mn = 0, \tag{3.24}$$

and

$$mn = -2. \tag{3.25}$$

Equation (3.24) is satisfied for any n by taking $m = 0$. Equation (3.25) can be satisfied only if $n = 1$ or $n = 2$, in which cases it has solutions $m = -2$, $m = -1$, respectively. Noting that (3.22) gives

$$
\begin{aligned}
p &= 1, \qquad q = 0, \qquad r = 0, \qquad s = 0, \\
p &= 0, \qquad q = 0, \qquad r = 1, \qquad s = 0,
\end{aligned}
$$

and (3.23) gives

$$p = 0, \quad q = 1, \quad r = 0, \quad s = 0,$$
$$p = 0, \quad q = 0, \quad r = 0, \quad s = 1,$$

we may state the following result:

V. *For any n, the series* (3.3) *for F and G contain linear terms of the form*

$$\begin{aligned} &F_{1000}(u_3\,,\,\omega_3)u + F_{0010}(u_3\,,\,\omega_3)\omega,\\ &G_{1000}(u_3\,,\,\omega_3)u + G_{0010}(u_3\,,\,\omega_3)\omega. \end{aligned} \tag{3.26}$$

If $n > 2$, these are the only terms of the first degree; if $n = 1$ or $n = 2$, there are additional linear terms of the form

$$\begin{aligned} &F_{0100}(u_3\,,\,\omega_3)\bar{u} + F_{0001}(u_3\,,\,\omega_3)\bar{\omega},\\ &G_{0100}(u_3\,,\,\omega_3)\bar{u} + G_{0001}(u_3\,,\,\omega_3)\bar{\omega}, \end{aligned} \tag{3.27}$$

respectively.

To discuss terms of the first degree in the series for F_3 and G_3, we go back to (3.22) and (3.23), and combine them with the second of (3.18). We get respectively,

$$mn = 1, \tag{3.28}$$

or

$$mn = -1. \tag{3.29}$$

These can be satisfied if and only if $n = 1$. Thus we have the result:

VI. *In the series* (3.3) *for F_3 and G_3, linear terms occur only in the degenerate case $n = 1$.*

Finally let us consider terms of the second degree, $D = 2$. Now we have three alternatives:

$$P = 2, \qquad Q = 0, \tag{3.30}$$

or

$$P = 1, \qquad Q = 1, \tag{3.31}$$

or

$$P = 0, \qquad Q = 2. \tag{3.32}$$

In the case of F and G we have correspondingly from the second of (3.16):

$$mn = 1, \tag{3.33}$$

or

$$mn = -1, \tag{3.34}$$

or

$$mn = -3. \tag{3.35}$$

The first two can be satisfied if and only if $n = 1$. Equation (3.35) can be satisfied if and only if $n = 1$ or $n = 3$. Thus we have the result:

VII. *The series* (3.3) *for F and G contain terms of the second degree if and only if $n = 1$ or $n = 3$ (no symmetry or trigonal symmetry). If $n \neq 1$, all terms of the second degree involve $\bar{u}^2$, $\overline{u\omega}$, $\bar{\omega}^2$ only.*

To discuss terms of the second degree in the series for F_3 and G_3, we use (3.30), (3.31), (3.32) with the second of (3.18). We get the following corresponding equations:

$$mn = 2, \tag{3.36}$$

or

$$mn = 0, \tag{3.37}$$

or

$$mn = -2. \tag{3.38}$$

Equation (3.37) can be satisfied for any n; the other two equations can be satisfied if and only if $n = 1$ or $n = 2$. Hence we have this result:

VIII. *If $n > 2$, the series* (3.3) *for F_3 and G_3 possess terms of the second degree of the following forms only*:

$$\begin{aligned} &F^{(3)}_{1100}(u_3, \omega_3)u\bar{u} + F^{(3)}_{1001}(u_3, \omega_3)u\bar{\omega} + F^{(3)}_{0110}(u_3, \omega_3)\bar{u}\omega + F^{(3)}_{0011}(u_3, \omega_3)\omega\bar{\omega}, \\ &G^{(3)}_{1100}(u_3, \omega_3)u\bar{u} + G^{(3)}_{1001}(u_3, \omega_3)u\bar{\omega} + G^{(3)}_{0110}(u_3, \omega_3)\bar{u}\omega + G^{(3)}_{0011}(u_3, \omega_3)\omega\bar{\omega}. \end{aligned} \tag{3.39}$$

If $n = 1$ or $n = 2$ (no symmetry or digonal), all terms of the second degree occur.

The following result may be added as an immediate deduction from the second of (3.16):

IX. *Except in the degenerate case* ($n = 1$), *there occur in the series* (3.3) *for F and G no terms making*

$$p + r = q + s.$$

This means that, in the diagrams of Fig. 2 which refer to F and G, we find no dots on the main diagonal, $P = Q$.

4. Consequences of reflectional symmetry. Let us now suppose that the projectile has reflectional symmetry in a plane P. If there is also n-gonal rotational symmetry about an axis A, we shall suppose that A is contained in P.

Let us choose the origin O in P (and on A if there is n-gonal rotational symmetry). Let us choose the axis Ox_3 in P (and along A if there is n-gonal rotational symmetry). Finally, let us choose the axis Ox_1 in P, so that Ox_2 is perpendicular to P.

The covering operation is a reflection in P. Applied to the axes, this operation changes $Ox_1x_2x_3$ into $Ox_1'x_2'x_3'$, where Ox_1', Ox_3' coincide with Ox_1, Ox_3 respectively, and Ox_2' is Ox_2 reversed.

Consider any state of motion of the projectile. This may be described by the components $(u_1, u_2, u_3, \omega_1, \omega_2, \omega_3)$ along $Ox_1x_2x_3$ or by the components $(u_1', u_2', u_3', \omega_1', \omega_2', \omega_3')$ along $Ox_1'x_2'x_3'$. It is understood that a component of angular velocity in a given direction is positive if it corresponds to a positive rotation about that direction. Although we may be accustomed to defining a positive rotation as a right-handed rotation, that definition must not be used here, because we are considering two sets of axes with different orientations. We accept the definition that a rotation is positive if it corresponds to a cyclical rotation of the axes. Thus ω_3 is positive if it tends to turn Ox_1 into Ox_2, and ω_3' is positive if it tends to turn Ox_1' into Ox_2'. It is then easy to see that when we reflect the axes in accordance with the covering operation described above, and describe one single state of motion by components along the two sets of axes, the two sets of components are related by the following formulae of transformation:

$$\begin{aligned} u_1' &= u_1 , & u_2' &= -u_2 , & u_3' &= u_3 , \\ \omega_1' &= -\omega_1 , & \omega_2' &= \omega_2 , & \omega_3' &= -\omega_3 . \end{aligned} \tag{4.1}$$

To this state of motion there corresponds an aerodynamic force system, which may be described by the components $(F_1 , F_2 , F_3 , G_1 , G_2 , G_3)$ along $Ox_1x_2x_3$ or by the components $(F_1' , F_2' , F_3' , G_1' , G_2' , G_3')$ along $Ox_1'x_2'x_3'$. As in the case of angular velocities, a component of torque is positive if it tends to produce a cyclical rotation of the axes. Thus G_3 is positive if it tends to turn Ox_1 into Ox_2 , and G_3' is positive if it tends to turn Ox_1' into Ox_2' . We have then the following formulae of transformation analogous to (4.1):

$$\begin{aligned} F_1' &= F_1 , & F_2' &= -F_2 , & F_3' &= F_3 , \\ G_1' &= -G_1 , & G_2' &= G_2 , & G_3' &= -G_3 . \end{aligned} \tag{4.2}$$

The formulae (4.1) and (4.2) are most easily remembered by the following rule: When the axes are reflected, polar vectors (velocity and force) undergo ordinary reflection, but axial vectors (angular velocity and torque) undergo ordinary reflection followed by a reversal of sense.

In the complex notation of (2.2), the transformations (4.1) and (4.2) read

$$\begin{aligned} u' &= \bar{u}, & u_3' &= u_3 , & \omega' &= -\bar{\omega}, & \omega_3' &= -\omega_3 , \\ F' &= \bar{F}, & F_3' &= F_3 , & G' &= -\bar{G}, & G_3' &= -G_3 . \end{aligned} \tag{4.3}$$

The above formulae refer to the effects of reflection of axes; the idea that the projectile is symmetric is not involved. This idea we now introduce, and substitute from (4.3) in (2.4). We see that reflectional symmetry in the plane Ox_1x_3 implies the following identities:

$$\begin{aligned} f(\bar{u}, u, u_3 , -\bar{\omega}, -\omega, -\omega_3) &= \bar{f}(u, \bar{u}, u_3 , \omega, \bar{\omega}, \omega_3), \\ g(\bar{u}, u, u_3 , -\bar{\omega}, -\omega, -\omega_3) &= -\bar{g}(u, \bar{u}, u_3 , \omega, \bar{\omega}, \omega_3), \\ f_3(\bar{u}, u, u_3 , -\bar{\omega}, -\omega, -\omega_3) &= \bar{f}_3(u, \bar{u}, u_3 , \omega, \bar{\omega}, \omega_3), \\ g_3(\bar{u}, u, u_3 , -\bar{\omega}, -\omega, -\omega_3) &= -\bar{g}_3(u, \bar{u}, u_3 , \omega, \bar{\omega}, \omega_3). \end{aligned} \tag{4.4}$$

Here a bar over a functional symbol implies that the sign of i is changed throughout, i.e. both in the arguments and in any complex coefficients which occur in the function. Since f_3 and g_3 are real, it is of course a matter of indifference whether we use bars for them or not.

We now assume that the functions f, g, f_3 , g_3 admit series expansions as in (3.3). Substitution in (4.4) gives

$$\begin{aligned} \sum_{pqrs} (-1)^{r+s} F_{pqrs}(u_3 , -\omega_3)\bar{u}^p u^q \bar{\omega}^r \omega^s &= \sum_{pqrs} \bar{F}_{pqrs}(u_3 , \omega_3)\bar{u}^p u^q \bar{\omega}^r \omega^s, \\ \sum_{pqrs} (-1)^{r+s} G_{pqrs}(u_3 , -\omega_3)\bar{u}^p u^q \bar{\omega}^r \omega^s &= -\sum_{pqrs} \bar{G}_{pqrs}(u_3 , \omega_3)\bar{u}^p u^q \bar{\omega}^r \omega^s, \\ \sum_{pqrs} (-1)^{r+s} F^{(3)}_{pqrs}(u_3 , -\omega_3)\bar{u}^p u^q \bar{\omega}^r \omega^s &= \sum_{pqrs} \bar{F}^{(3)}_{pqrs}(u_3 , \omega_3)\bar{u}^p u^q \bar{\omega}^r \omega^s, \\ \sum_{pqrs} (-1)^{r+s} G^{(3)}_{pqrs}(u_3 , -\omega_3)\bar{u}^p u^q \bar{\omega}^r \omega^s &= -\sum_{pqrs} \bar{G}^{(3)}_{pqrs}(u_3 , \omega_3)\bar{u}^p u^q \bar{\omega}^r \omega^s. \end{aligned} \tag{4.5}$$

Since these may be treated as identities in u, $\bar{u}$, u_3, ω, $\bar{\omega}$, ω_3, we deduce

$$\begin{aligned} F_{pqrs}(u_3, -\omega_3) &= \quad (-1)^{r+s}\bar{F}_{pqrs}(u_3, \omega_3), \\ G_{pqrs}(u_3, -\omega_3) &= -(-1)^{r+s}\bar{G}_{pqrs}(u_3, \omega_3), \\ F^{(3)}_{pqrs}(u_3, -\omega_3) &= \quad (-1)^{r+s}\bar{F}^{(3)}_{pqrs}(u_3, \omega_3), \\ G^{(3)}_{pqrs}(u_3, -\omega_3) &= -(-1)^{r+s}\bar{G}^{(3)}_{pqrs}(u_3, \omega_3), \\ &(p, q, r, s = 0, 1, 2, \cdots). \end{aligned} \tag{4.6}$$

We note that the exponent $(r + s)$ is the degree in cross spin of a term in the expansions (3.3).

To show the implications of (4.6), we introduce the symbols E and O to represent real functions of u_3 and ω_3 which are respectively *even* amd *odd* in ω_3. We may then show the results as follows:

			$(r+s)$ even	$(r+s)$ odd	
F_{pqrs}	and	$F^{(3)}_{pqrs}$	$E + iO$	$O + iE$	(4.7)
G_{pqrs}	and	$G^{(3)}_{pqrs}$	$O + iE$	$E + iO$	

In making the above deductions from the assumption of reflectional symmetry, we have made no use of the existence or non-existence of n-gonal rotational symmetry. In fact, the consequences of the two types of symmetry are essentially different in nature. As we saw in Section 3, n-gonal rotational symmetry eliminates terms in the series (3.3). Reflectional symmetry does not eliminate any terms; it merely gives us information with regard to odd or even functional dependence on axial spin ω_3 in the coefficients in (3.3). But in the special case discussed below in Secs. 6 and 7, reflectional symmetry does enable us to eliminate further terms from the expansion G_3.

5. Case of vanishing cross spin ($\omega = 0$). Let us now consider the case where the projectile has no cross spin. This may be either an instantaneous condition in a general motion, or it may be a permanent condition enforced by constraints on the projectile. Thus, if the projectile is mounted in a wind tunnel in such a way that its axis is fixed in direction, then the cross spin vanishes. The projectile may be completely fixed, or it may be free to rotate about its axis. We note that, for wind tunnel discussions, the free air stream is to be used as frame of reference.

For vanishing cross spin, we have $\omega_1 = \omega_2 = 0$, and so $\omega = \bar{\omega} = 0$. Thus all terms disappear from the series (3.3) except those for which $r = s = 0$. In fact, the series read

$$\begin{aligned} f(u, \bar{u}, u_3, 0, 0, \omega_3) &= \sum_{pq} F_{pq00}(u_3, \omega_3)u^p\bar{u}^q, \\ g(u, \bar{u}, u_3, 0, 0, \omega_3) &= \sum_{pq} G_{pq00}(u_3, \omega_3)u^p\bar{u}^q, \\ f_3(u, \bar{u}, u_3, 0, 0, \omega_3) &= \sum_{pq} F^{(3)}_{pq00}(u_3, \omega_3)u^p\bar{u}^q, \\ g_3(u, \bar{u}, u_3, 0, 0, \omega_3) &= \sum_{pq} G^{(3)}_{pq00}(u_3, \omega_3)u^p\bar{u}^q. \end{aligned} \tag{5.1}$$

We shall suppose that the projectile has reflectional symmetry in the plane Ox_1x_3 ang also n-gonal rotational symmetry about the axis Ox_3 .

Since $r + s$ is even, it follows from (4.7) as a deduction from reflectional symmetry

$$\begin{aligned} F_{pq00} &= E + iO, & F^{(3)}_{pq00} &= E + iO, \\ G_{pq00} &= O + iE, & G^{(3)}_{pq00} &= O + iE. \end{aligned} \tag{5.2}$$

Turning to (3.11), we note that $P = p$ and $Q = q$. Figure 2 shows the coefficients which survive in the case of n-gonal rotational symmetry, but now we may read p instead of P and q instead of Q on the axes of the diagrams. As already noted in Theorem IX, on the main diagonal ($p = q$) there are no surviving coefficients in the series for F and G (except in the degenerate case of no symmetry ($n = 1$), which is not shown in the diagrams). But in the series for F_3 and G_3 there are terms on the main diagonal in all cases. The coefficients are of the forms

$$F^{(3)}_{pp00}(u_3\ , \omega_3), \qquad G^{(3)}_{pp00}(u_3\ , \omega_3). \tag{5.3}$$

From the reality of F_3 and G_3 , it follows (as seen after (3.5)) that the above coefficients are *real*. When this information is compared with (5.2), the truth of the following statement is obvious:

X. *For a projectile possessing reflectional symmetry and n-gonal rotational symmetry, the coefficients* (5.3) *are respectively real EVEN and real ODD functions of* ω_3 .

6. Case of vanishing axial spin ($\omega_3 = 0$). We next consider the case where the projectile has no axial spin ($\omega_3 = 0$), but may have cross spin. This may be an instantaneous state of motion for a fin-stabilized projectile in free flight, or a permanent state enforced by constraints to prevent axial spin for a projectile mounted in a wind tunnel.

Since an odd function of ω_3 must vanish with ω_3 , it follows from (4.7) that, when $\omega_3 = 0$, the coefficients take the forms:

$$\begin{array}{lllll} & & & (r+s)\ \text{even} & (r+s)\ \text{odd} \\ F_{pqrs} & \text{and} & F^{(3)}_{pqrs} & k(u_3) & ik(u_3) \\ G_{pqrs} & \text{and} & G^{(3)}_{pqrs} & ik(u_3) & k(u_3) \end{array} \tag{6.1}$$

where k is used to denote real functions of u_3 ; k does not, of course, represent a single function. These results are consequences of reflectional symmetry, which assigns odd or even dependence on ω_3 , and of the condition of vanishing axial spin, which eliminates either the real or the imaginary part for any given coefficient.

Since the coefficients in the expansions of F and G are initially to be regarded as unrestricted complex numbers, the terms which survive n-gonal rotational symmetry also survive after application of the above two assumptions. The character of the restriction placed on the coefficients in the series for F and G due to vanishing axial spin may be summed up in the following statement:

XI. *For a projectile possessing reflectional symmetry and n-gonal rotational symmetry, and having no axial spin, the coefficients* F_{pqrs} , G_{pqrs} *are respectively pure real and pure imaginary functions of* u_3 *for* $(r + s)$ *an even integer. If* $(r + s)$ *is odd, the situation is reversed.*

Further, it is easily seen that the assumption of vanishing axial spin does not eliminate any terms in the expansion of F_3. From (6.1) it is seen that these coefficients $F^{(3)}_{pqrs}$ are pure real or pure imaginary functions of u_3 depending on whether $(r + s)$ is an even or odd integer.

The fact that a projectile possesses reflectional symmetry and no axial spin, when combined with the result (3.5), *does* eliminate additional terms in the expansion of the axial torque G_3 for the projectile. By (3.5), $G^{(3)}_{pprr}(u_3, \omega_3)$ are real, and so, in particular, $G^{(3)}_{pprr}(u_3, 0)$ are real. But, by (6.1), $G^{(3)}_{pprr}(u_3, 0)$ are pure imaginary. Hence

$$G^{(3)}_{pprr}(u_3, 0) = 0. \tag{6.2}$$

Thus we have this result:

XII. *For a projectile possessing reflectional symmetry and n-gonal rotational symmetry, and having no axial spin, the series* (3.3) *for* G_3 *contains no terms for which* $p = q$ and $r = s$.

Let us examine this result, proceeding by increasing degree D.

If $D = 0$, we have the coefficient $G^{(3)}_{0000}(u_3, 0)$, and it vanishes by (6.2).

If $D = 1$, we have the coefficients

$$G^{(3)}_{1000}(u_3, 0), \qquad G^{(3)}_{0100}(u_3, 0), \qquad G^{(3)}_{0010}(u_3, 0), \qquad G^{(3)}_{0001}(u_3, 0).$$

Obviously, (6.2) tells us nothing about these coefficients. However, as we saw in Theorem VI, these coefficients occur only in the degenerate case $n = 1$.

If $D = 2$, the subscripts of $G^{(3)}_{pqrs}(u_3, 0)$ may be written as follows:

$$\begin{array}{c} 2000 \\ 0020 \\ {}^*1100 \\ 1001 \\ 0110 \\ {}^*0011 \\ 0200 \\ 0002. \end{array} \tag{6.3}$$

Of these, those marked with a star vanish by (6.2). We are left with 6 terms. Combining this with Theorem VIII and (6.1), we have the following result:

XIII. *For a projectile possessing reflectional symmetry and n-gonal rotational symmetry, and having no axial spin, the terms in the series* (3.3) *for the axial torque* G_3 *may be described as follows*:

a. *There is no absolute term.*
b. *There are linear terms only if* $n = 1$ (*degenerate case*).
c. *If* $n = 1$ *or* $n = 2$, *there are six terms of the second degree, with subscripts as shown by the unstarred entries in* (6.3); *if* $n > 2$, *there are only two terms of the second degree, which may be written*

$$k(u_3)(u\bar{\omega} + \bar{u}\omega) \tag{6.4}$$

where k is a real function.

The last deduction follows from (3.37), since, by (3.5),

$$G^{(3)}_{1001}(u_3\,,\,0) = \bar{G}^{(3)}_{0110}(u_3\,,\,0), \tag{6.5}$$

and, by (6.1), $G^{(3)}_{1001}(u_3\,,\,0)$ is real.

We note, that for $n > 2$, the expansion of G_3 reads

$$G_3 = k(u_3)(u\bar{\omega} + \bar{u}\omega) + (\text{terms of higher degree}). \tag{6.6}$$

7. Case where cross spin and axial spin both vanish ($\omega = 0$, $\omega_3 = 0$). We shall now bring together the situations discussed in Secs. 5 and 6 by supposing that the axial spin ω_3 and the cross spin ω vanish simultaneously. We discuss here only the axial torque G_3.

As a particular case of (6.2), we have

$$G^{(3)}_{pp00}(u_3\,,\,0) = 0 \qquad (p = 0,\,1,\,\cdots). \tag{7.1}$$

This means that when $\omega = \omega_3 = 0$, for a projectile possessing reflectional and n-gonal rotational symmetry, *we are to delete the main diagonals in the charts of* Fig. 2 *for* G_3 ; *these coefficients no longer survive.* Under these circumstances, let us seek the surviving coefficients of lowest degree in the series (3.3) for G_3. It is then a question of finding the smallest non-negative integer D to satisfy (3.18) with the additional condition $P \neq Q$, since the main diagonal has been eliminated.

The equations to be solved read

$$\begin{gathered} P + Q = D, \qquad P - Q = mn \qquad (m = 0,\,\pm1,\,\cdots), \\ P \neq Q. \end{gathered} \tag{7.2}$$

These imply

$$D^2 = (P + Q)^2 = (P - Q)^2 + 4PQ = m^2n^2 + 4PQ. \tag{7.3}$$

The possibility $m = 0$ is excluded, since by (7.2) it would give the forbidden relation $P = Q$. Hence the smallest D is given by taking $m^2 = 1$ and either $P = 0$ or $Q = 0$. This smallest value of D is n, and the solutions giving the smallest D are

$$P = n, \qquad Q = 0, \qquad m = 1; \qquad P = 0, \qquad Q = n, \qquad m = -1. \tag{7.4}$$

Let us summarize as follows:

XIV. *A projectile possesses reflectional symmetry and n-gonal rotational symmetry. The cross spin and the axial spin both vanish. Under these conditions the series* (5.1) *for the axial torque* G_3 *starts with terms of degree n, and the series is of the form*

$$G_3 = -ik(u_3)(u^n - \bar{u}^n) + \text{terms of degree } n + 2. \tag{7.5}$$

To complete the proof of this statement, it follows from (7.4) that

$$G_3 = G^{(3)}_{n000}(u_3\,,\,0)u^n + G^{(3)}_{0n00}(u_3\,,\,0)\bar{u}^n + \text{terms of degree } n + 2. \tag{7.6}$$

The coefficients are pure imaginaries by (4.7) and complex conjugates by (3.5), since G_3 is real. Thus, in terms of a real function $k(u_3)$, we may write (7.6) in the form (7.5).

The angle of yaw (δ) is the angle between the velocity of the point O and the axis of the projectile. Thus

$$\tan \delta = | u |/u_3 . \tag{7.7}$$

Let θ denote the angle between the cross velocity vector u and the axis Ox_1, which it will be remembered lies in a plane of reflectional symmetry of the projectile. We shall count θ positive when the direction of u is generated from Ox_1 by a positive rotation (Fig. 3). Then

$$u = | u | e^{i\theta}. \tag{7.8}$$

Combining (7.7) and (7.8), we have

$$u = u_3 \tan \delta \cdot e^{i\theta}. \tag{7.9}$$

Substitution in (7.6) gives

$$G_3 = 2u_3^n k(u_3) \tan^n \delta \sin n\theta + \text{terms of order } \tan^{n+2} \delta. \tag{7.10}$$

To summarize:

XV. *The formula* (7.10) *shows explicitly the dependence of the principal part of the axial torque* G_3 *on the yaw* δ *and the angle* θ *for a projectile possessing reflectional symmetry and n-gonal rotational symmetry, in the case where the projectile has no cross spin or axial spin.*

As was to be expected, G_3 is periodic in θ with period $\alpha = 2\pi/n$, the angle of the covering operation.

The formula (7.10) holds even in the degenerate case $n = 1$, when we have no rotational symmetry, only reflectional symmetry.

8. Stability. An interesting question concerning the generation of axial spin arises in respect of (7.10). Suppose that a projectile possessing reflectional symmetry and n-gonal rotational symmetry has no cross spin or axial spin ($\omega = \omega_3 = 0$). Then G_3 is of the form (7.10), from which we see that $G_3 = 0$ if either (a) the cross velocity lies in the plane of reflectional symmetry $Ox_1x_3(\theta = 0)$, or if (b) there in no yaw ($\delta = 0$).

Let us consider the case in which the projectile is yawing ($\delta \neq 0$) so that the vanishing or non-vanishing of the axial torque depends only on θ. We shall then have $G_3 = 0$ for $\theta = m\pi/n$, where $m = 0, \pm 1, \pm 2, \cdots$. For these values of θ, the cross velocity vector lies in a plane of symmetry. If the projectile has n-gonal rotational symmetry and one plane of reflectional symmetry (containing the axis of rotational symmetry), then it necessarily has n planes of reflectional symmetry. The angles between these planes are π/n.

We must observe that, for a given projectile, the axes $Ox_1x_2x_3$ are not uniquely determined. Choosing Ox_3 along the axis of the projectile in the general sense of flight, and deciding to use right-handed axes, we have still a choice of $2n$ consecutive directions differing by an angle π/n for the Ox_1 axis. For a given motion, a change from one such choice of Ox_1 to the next alters θ by π/n, and changes the sign of $\sin n\theta$ in (7.10). Since G_3 is not altered, such a change reverses the sign of the function $k(u_3)$. We must therefore be careful not to attach importance to the sign of $k(u_3)$ until we are sure what axes we are using.

Suppose then that we have made some definite choice of Ox_1, as in Fig. 3. Consider various directions of the cross velocity u. As we rotate u, we get successive positions of vanishing axial torque, namely, whenever u crosses a plane of symmetry. These will be alternately stable and unstable, in the sense that if the vector u is displaced slightly from one of these positions, the axial torque consequent on this displacement will tend to decrease or to increase the angular displacement between u and the plane of symmetry originally coincident with u. The criterion for stability is that the generated axial torque should rotate the projectile so as to bring the plane of symmetry from which u was displaced back into coincidence with u. This requires a positive G_3 for a positive increment in θ; in other words *the criterion for stability is*

$$dG_3/d\theta > 0. \tag{8.1}$$

Referring to (7.10), we see that we may make the following statement:

XVI. *If* $k(u_3) > 0$, *the projectile is stable with respect to axial spin when yawed in the planes of symmetry* $\theta = 2m\pi/n$ $(m = 0, 1, 2, \cdots)$, *and unstable when yawed in the planes of symmetry* $\theta = (2m + 1)\pi/n$ $(m = 0, 1, 2, \cdots)$. *If* $k(u_3) < 0$, *the situation is reversed.*

We have no a priori knowledge about the sign of $k(u_3)$. Suppose the projectile is to consist of a body of revolution and fins as shown in Fig. 1, the broken lines being included so that the projectile has reflectional symmetry. Let us for the sake of definiteness take Ox_1 in a fin.

In the degenerate case $(n = 1)$, where there is no symmetry except reflectional symmetry (Fig. 4a), we expect stability when the fin is downstream. We note that u represents the cross velocity of the projectile relative to the air, so that $-u$ represents the cross wind relative to the projectile; we may expect that stability occurs when $\theta = \pi$, as shown by the vector marked S in Fig. 4a. Thus, according to the above criterion for stability, in order that (8.1) holds for $\theta = \pi$ we must have $k(u_3) < 0$ when $n = 1$. Figure 4a also shows the relation between the angle θ and stability. The shaded portion of the plane is a region of stability in the sense that if the cross velocity vector u once enters the shaded half-plane it can never leave and tends to settle on the position S. If u is initially in the unstable region (right half-plane) the projectile will rotate so that u leaves the unstable region by the shorter route. These statements are not mathematical deductions; they represent what, from general experience, we would expect to happen in the case of a projectile with a single fin.

In the case $n = 2$ (Fig. 4b), it would generally be agreed that the projectile would turn so as to set the face of the fins at right angles to the cross wind. That is, taking Ox_1 in one of the fins, the two positions for u given by $\theta = \pi/2, 3\pi/2$ are positions of stability and the positions $\theta = 0, \pi$ are positions of instability. In order that this situation may result from the above criterion of stability, it is necessary that $k(u_3) < 0$ when $n = 2$.

In both the above cases, it is interesting to note that the positions of stability are those for which the cross velocity u is coincident with the bisector of the angle between adjacent fins. The question then arises as to whether this holds true for projectiles having more than two fins. This of course could be answered if there were an a priori method for the determination of the algebraic sign of $k(u_3)$. In the absence of such a method, experiment alone can decide. To explain the point at issue, let us discuss the case of a bomb with three fins $(n = 3)$, as shown in Figs. 4c, d.

Let us choose the axis Ox_1 in one of the fins. Two cases can arise, $k(u_3) > 0$ and $k(u_3) < 0$, if we omit from consideration the particular case $k(u_3) = 0$.

Let us consider first the case $k(u_3) > 0$. From Theorem XVI we infer that the projectile is in a position of stable equilibrium when $\theta = 0, 2\pi/3, 4\pi/3$, i.e. when the cross velocity vector u lies in one of the fins (Fig. 4c). Also we see that there is unstable equilibrium when $\theta = \pi/3, \pi, 5\pi/3$, i.e. when the cross velocity vector u bisects the angle between two fins. The shaded regions in Fig. 4c indicate regions of stability, in

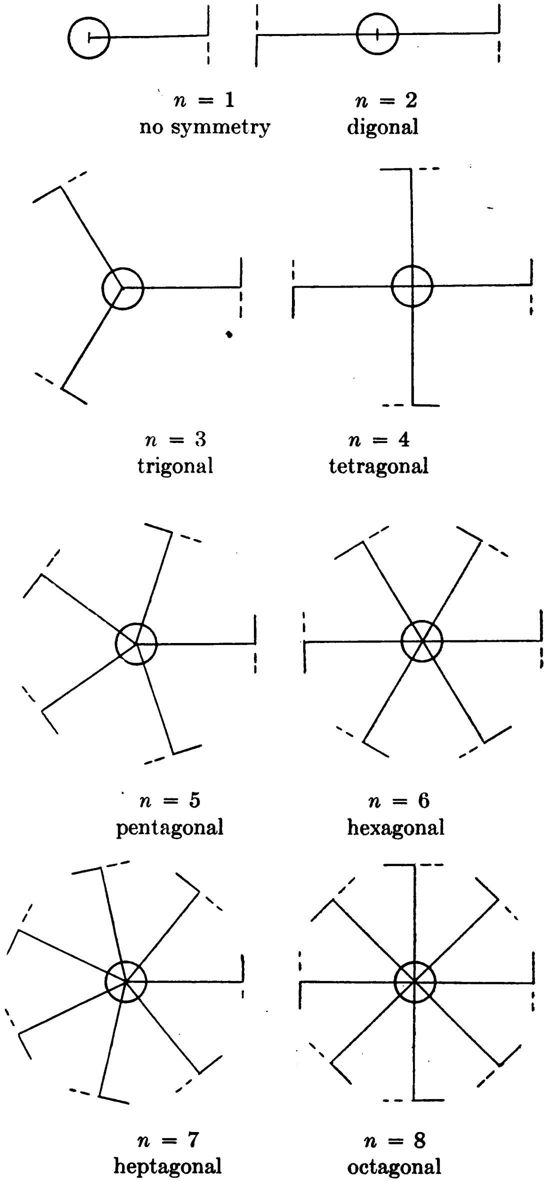

Fig. 1

Diagram showing cross-section with n-gonal rotational symmetry, with and without reflectional symmetry. (If dotted portions are included, we have reflectional symmetry; if they are omitted, we have no reflectional symmetry).

the sense that, if u lies in one of them, the projectile will rotate so as to bring u into coincidence with one of the fins, and there will be an oscillation of u (in the shaded region) about that fin.

On the other hand, if $k(u_3) < 0$, the situation is reversed as shown in Fig. 4d. Now the stable positions of equilibrium are those for which u bisects an angle between fins.

We have then the following interesting question, capable of answer by wind tunnel experiments: If a three finned projectile is mounted in a wind tunnel so that it is free to turn around its axis, are the stable positions of equilibrium those in which a fin lies in the plane of yaw (Fig. 4c), or are they those in which the plane of yaw bisects the angle between two fins (Fig. 4d)? The same question may of course be asked for the case of four fins, or indeed for the case of any number of fins. It would indeed be interesting to know whether the situation discussed for $n = 1$ and $n = 2$ continues for higher values of n. It may be that for higher values of n the screening effect of the body of the projectile becomes of significance, and no general statements can be made covering all projectiles which have the same value of n.

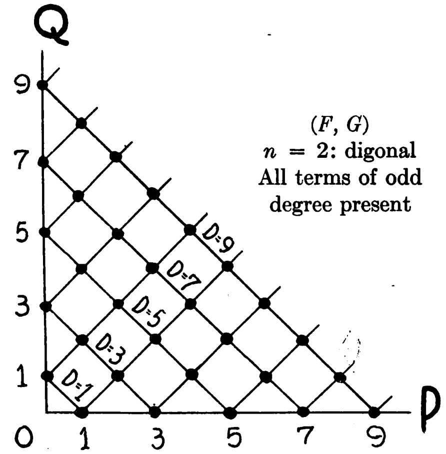

Fig. 2a

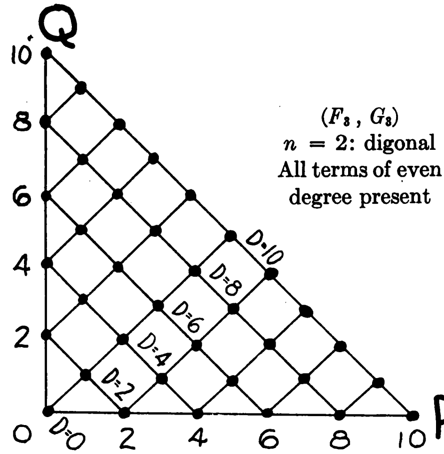

Fig. 2b.

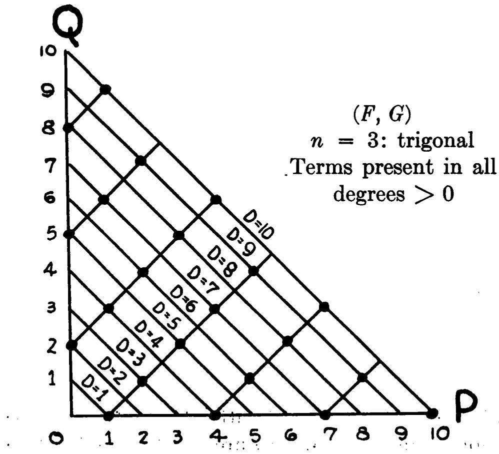

Fig. 2c.

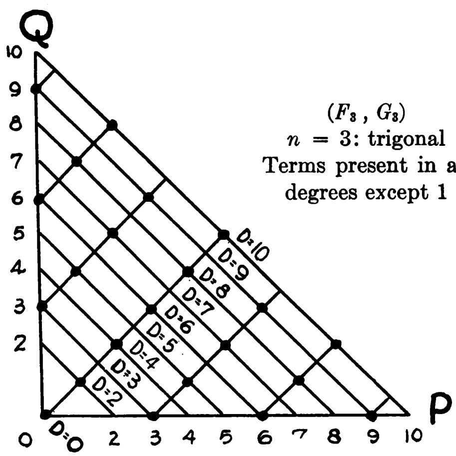

Fig. 2d.

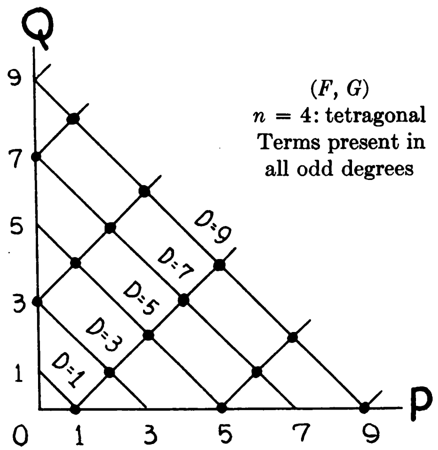

FIG. 2e.

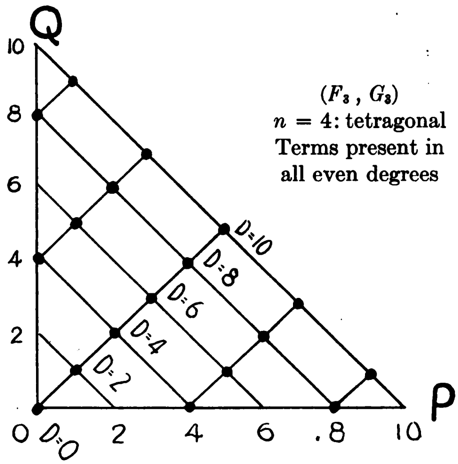

FIG. 2f.

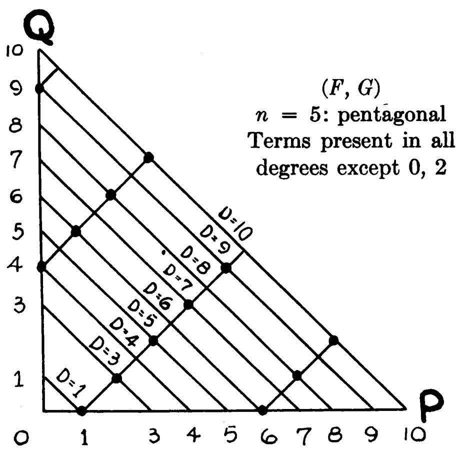

FIG. 2g.

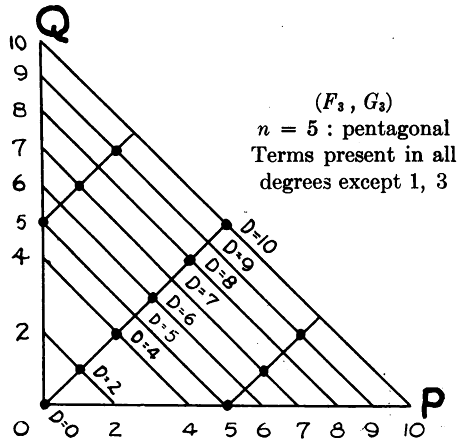

FIG. 2h.

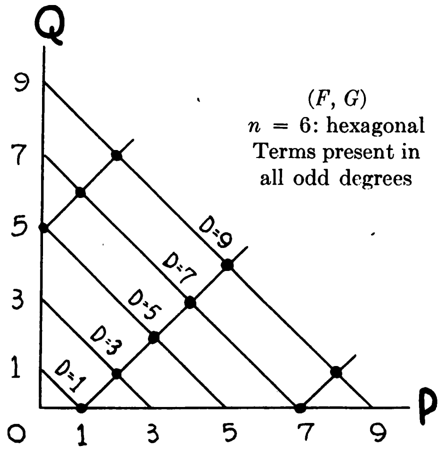

FIG. 2i.

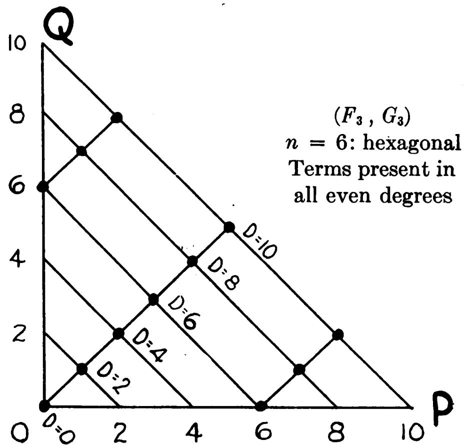

FIG. 2j.

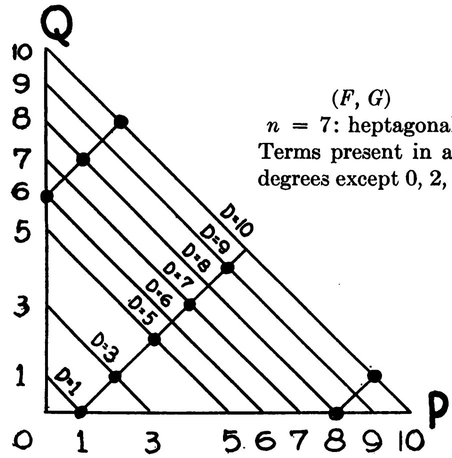

FIG. 2k.

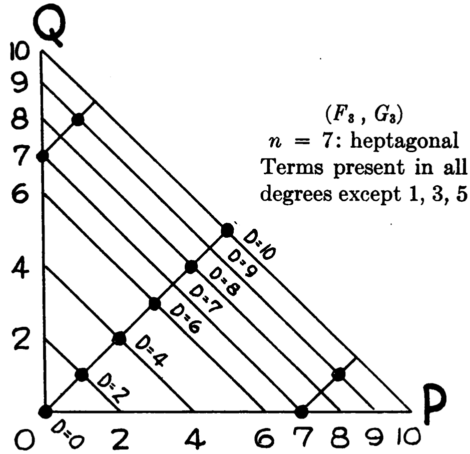

FIG. 2l.

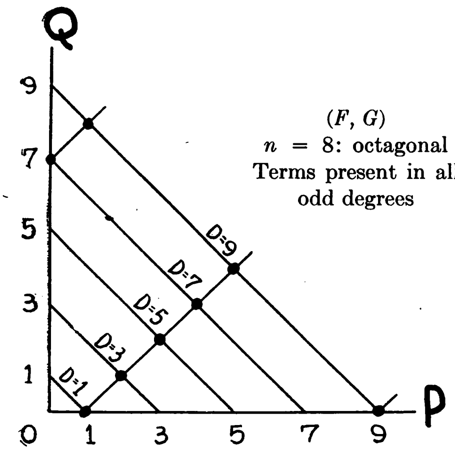

FIG. 2m.

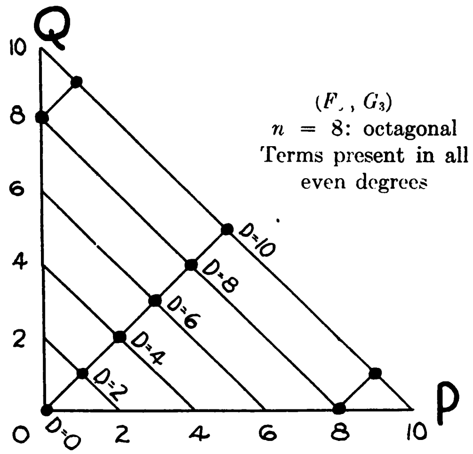

FIG. 2n.

FIG. 2

The heavy dots indicate surviving terms in the expansions (3.3) for various types of rotational symmetry. (F, G) are cross force and cross torque respectively. (F_3, G_3) are axial force and axial torque respectively. For P and Q, see (3.11). D is the degree of the term.

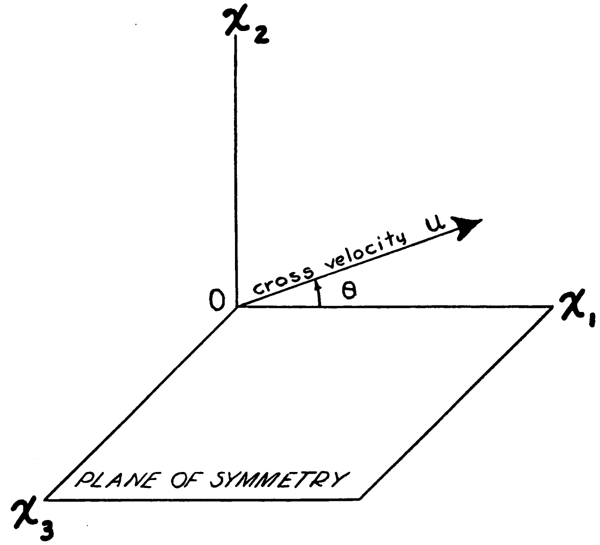

FIG. 3

Projectile with reflectional symmetry; definition of the angle θ.

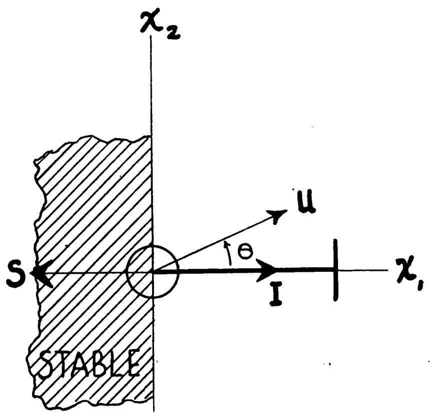

FIG. 4a. $n = 1$, single fin, $k(u_3) < 0$.

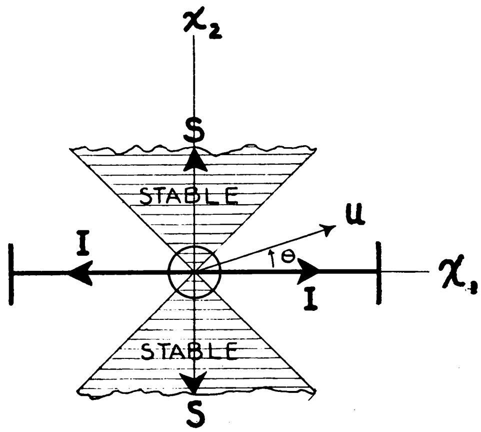

FIG. 4b. $n = 2$, two fins, $k(u_3) < 0$.

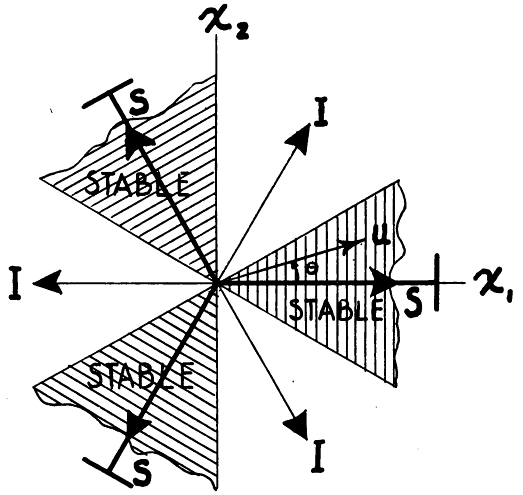

FIG. 4c. $n = 3$, three fins, $k(u_3) > 0$.

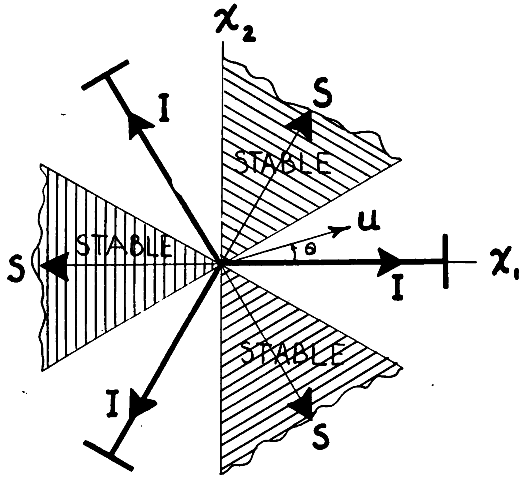

FIG. 4d. $n = 3$, three fins, $k(u_3) < 0$.

FIG. 4

Stability of a projectile with respect to axial spin. S indicates equilibrium directions of u (cross velocity of projectile) which probably correspond to stability. I indicates equilibrium directions of u which probably correspond to instability. Shaded regions are regions of stability.