HOMOGENEOUS HARMONIC FUNCTIONS*

BY

H. PORITSKY

General Electric Company

1. Introduction. We consider, in the following, solutions of the Laplace equation

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0 \tag{1.1}$$

which are homogeneous of degree n in the coordinates, that is, such that

$$\varphi(Cx, Cy, Cz) = C^n \varphi(x, y, z) \tag{1.2}$$

for any constant C. The case $n = 0$ is of special interest and is considered at great length. As will be explained in the following paper, with proper change of variables, this case finds an important application in the supersonic conical flows.

The relation (1.2) prescribes the variation of φ along lines through the origin; this relation may therefore be used to reduce the number of independent variables in (1.1), converting it to a differential equation in *two* variables. Transforming (1.1) to spherical coordinates, we shall use (1.2) first to eliminate

$$R = (x^2 + y^2 + z^2)^{1/2}, \tag{1.3}$$

thus converting (1.1) to a proper differential equation over the unit sphere $R = 1$; and then to eliminate z, converting (1.1) into a differential equation in the plane $z = 1$.

For $n = 0$ it is shown that by introducing *isothermal coordinates* along the unit sphere we may transform (1.1) into a two-dimensional Laplace equation. In this case the solutions of (1.1) can therefore be expressed in terms of analytic functions of a single complex variable.

For positive or negative integer n the solutions of (1.1), (1.2) may be expressed similarly in terms of analytic functions of a complex variable by either one of two methods: the first consists in differentiating the solutions for $n = 0$ with respect to x, y, z and using inversion; the second, for positive integer n, utilizes Euler's identity for homogeneous functions of degree n:

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\varphi = n\varphi. \tag{1.4}$$

The case $n = 1$, also of interest from the point of view of conical flows, is discussed at length, and will be applied in the following paper.

2. Harmonic functions of degree zero. In spherical coordinates the Laplace operator may be written in the form

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial R^2} + \frac{2}{R}\frac{\partial\varphi}{\partial R} + B_2(\varphi); \tag{2.1}$$

where B_2 is the second differential operator of Beltrami along the sphere $R = \text{const.}$

*Received April 26, 1948.

through the point in question. It will be recalled[1] that for any surface S, the operator B_2 can be defined by means of

$$B_2(\varphi) = \lim_{A\to 0} \int \frac{\partial\varphi}{\partial n} ds/A \tag{2.2}$$

where, as shown on Fig. 1, the integration of the normal derivative is carried out over

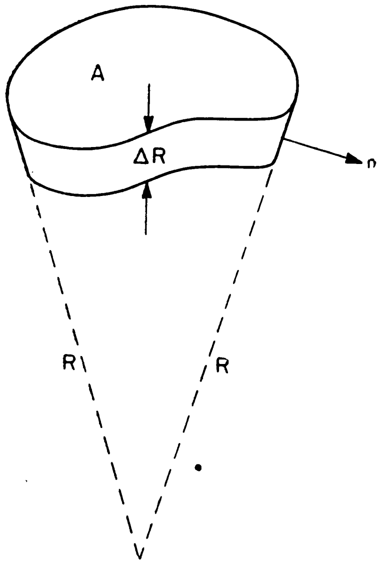

FIG. 1.

the boundary curve C of an area A on S, the normal to the curve being tangent to S, and the limit is taken as the area A shrinks to a point. The form (2.1) is obtained by applying Green's theorem to a volume bounded by two spheres of radii R, $R + \Delta R$ and a conical surface through C, dividing by ΔR and allowing ΔR to approach zero.

If φ is homogeneous of degree n, then along the unit sphere $R = 1$, Eq. (2.1) reduces to

$$B_2(\varphi) + n(n + 1)\varphi = 0. \tag{2.3}$$

In particular, for $n = 0$, we are led to

$$B_2(\varphi) = 0 \tag{2.4}$$

over the unit sphere.

To solve (2.4) (along *any* surface S), introduce along S "isothermal" coordinates p, q, that is, coordinates such that

$$ds^2 = \lambda(p, q)[(dp)^2 + (dq)^2]. \tag{2.5}$$

It is known[2] that then the operator B_2 reduces to

$$B_2(\varphi) = (\varphi_{pp} + \varphi_{qq})/\lambda. \tag{2.6}$$

---

[1] W. Blaschke, *Vorlesungen ueber Differentialgeometrie*, vol. 1, 2nd ed., J. Springer, Berlin, 1924, pp. 114-116.

[2] *Ibid.*, pp. 121–122.

Thus Eq. (1.1) is reduced to

$$\varphi_{pp} + \varphi_{qq} = 0, \tag{2.7}$$

whose general solution is

$$\varphi = f(p + iq) + g(p - iq) \tag{2.8}$$

or, if φ is real,

$$\varphi = Re\,[f(p + iq)] \tag{2.9}$$

where f, g are general analytic functions of their complex arguments.

It will further be recalled that if one system of isothermal coordinates p, q is available, the general system of such coordinates is given by

$$p' \pm iq' = h(p + iq) \tag{2.10}$$

where h is an arbitrary analytic function of the complex variable $p + iq$; this is in complete agreement with the fact that $B_2 = 0$ reduces to (2.7), and that under the change of variables (2.10), Eq. (2.7) is transformed into a similar Laplace equation in the coordinates p', q'.

Geometrically (2.5) can be interpreted to mean that the surface S is mapped conformally on a plane in which p, q are rectangular Cartesian coordinates. The term "isothermal" is tied up with the fact that Eq. (2.4) is satisfied by the temperature in steady state conduction problems over a thin conducting shell of uniform thickness and thermal conductivity, provided no heat is lost over the faces. In particular, in each isothermal coordinate system, the curves $p = \text{const}$, $q = \text{const}$ may be viewed as a family of isothermals and their normal trajectories.

To render the solution obtained non-vacuous a particular set of isothermal coordinates is required. It will now be shown[3] that a set of isothermal coordinates on the sphere

$$S\colon x^2 + y^2 + z^2 = 1 \tag{2.11}$$

is obtained by projecting a point P on S from the point Q: $(0, 0, -1)$ of S on the negative z-axis onto P' on the equatorial plane $z = 0$; the x, y coordinates of P' in $z = 0$ constitute the parameters p, q of P (see Fig. 2). A projection of P on the tangent plane $z = 1$ at Q': $(0, 0, 1)$ is, of course, also possible, leading to values of p, q which are double those obtained by using the equatorial plane $z = 0$.

Introducing in $z = 0$ the polar coordinates $r = OP' = \tan\psi$ and $\omega = \tan^{-1}y/x$, we have for the element of length $d\sigma$ in that plane

$$\begin{aligned} d\sigma^2 &= dx^2 + dy^2 = dp^2 + dq^2 \\ &= dr^2 + r^2\,d\omega^2 \\ &= \sec^4\psi\,d\psi^2 + \tan^2\psi\,d\omega^2. \end{aligned} \tag{2.12}$$

On the sphere S, using the angle $Q'OP = 2\psi = \theta$ and ω as spherical coordinates, we obtain for the element of length

[3] *Ibid.*, p. 168.

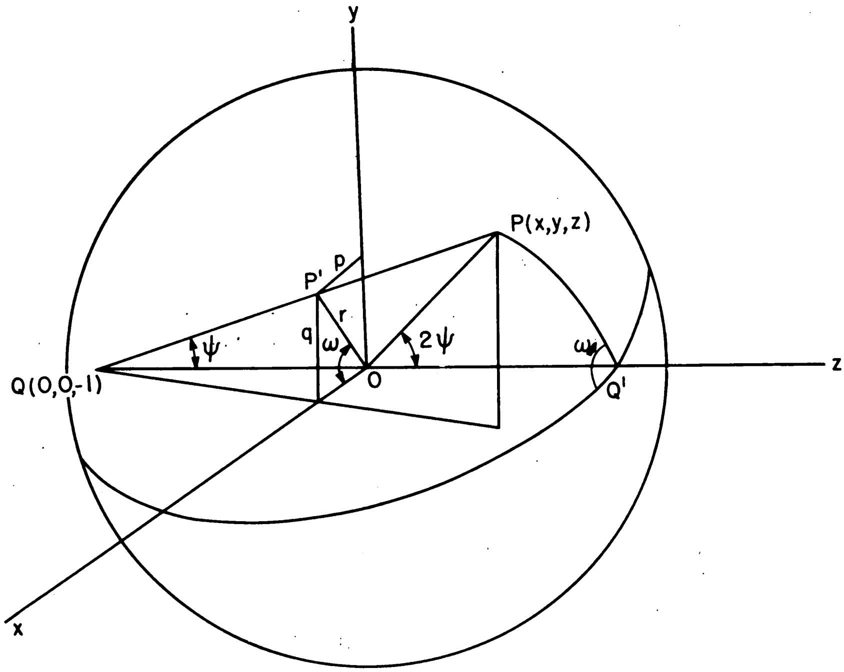

FIG. 2.

$$\begin{aligned} ds^2 &= d(2\psi)^2 + \sin^2 2\psi\, d\omega^2 \\ &= 4(d\psi^2 + \sin^2\psi \cos^2\psi\, d\omega^2). \end{aligned} \tag{2.13}$$

Comparing (2.13) with (2.12) we obtain

$$ds^2 = 4\cos^4\psi(dp^2 + dq^2); \tag{2.14}$$

and since

$$\tan^2\psi = r^2 = p^2 + q^2, \tag{2.15}$$

it will be concluded that (2.14) is of the form (2.5), hence that p, q are isothermal coördinates on the unit sphere.

The components of the vector QP on Fig. 2 are x, y, $1 + z$ where x, y, z are the coördinates of P; hence the components of the vector QP' are

$$\frac{x}{1+z}, \qquad \frac{y}{1+z}, \qquad 1. \tag{2.16}$$

Thus the coordinates p, q of P' are given by

$$p = \frac{x}{1+z}, \qquad q = \frac{y}{1+z}, \tag{2.17}$$

while the complex argument of (2.9) becomes

$$+ iq = \frac{x+iy}{1+z}. \tag{2.18}$$

From the above it now follows that the general (real) harmonic function which is homogeneous of degree 0 in x, y, z takes on over the unit sphere the values which form the real (or imaginary) part of

$$f(p + iq) = f\left(\frac{x + iy}{1 + z}\right). \tag{2.19}$$

In order to obtain a representation for this function at arbitrary points of space, we replace x, y, z above by their ratios to R, thus obtaining

$$f\left(\frac{x/R + iy/R}{1 + z/R}\right) = f\left(\frac{x + iy}{R + z}\right) = f(Z) \tag{2.20}$$

where

$$Z = \frac{x + iy}{R + z}. \tag{2.21}$$

It has thus been shown that any real harmonic function which is homogeneous of degree 0 is the real part of an analytic function of Z.

As further examples of isothermal coordinates it is evident that an equally good set of such coordinates on the unit sphere can be obtained by projecting it from *any* one of its points onto the equatorial plane normal to the radius to that point. Thus if we project (2.11) from the point $(0, -1, 0)$ onto the plane $y = 0$ we are led to the complex variable

$$Y = \frac{z + ix}{R + y} \tag{2.22}$$

in place of (2.17), as another possible variable in terms of which the harmonic functions in question may be expressed. The quantity Y must be expressible in terms of the variable Z in accordance with (2.10). Indeed, we may verify that

$$Y = \frac{i - Z}{i + Z}, \qquad Z = (-i)\,\frac{Y - 1}{Y + 1} \tag{2.23}$$

and this agrees with the fact that the only conformal mapping of a plane on itself (including the "point at infinity" as a "point") is given by a linear fractional transformation. Similarly we obtain from a projection from $(-1, 0, 0)$ on $x = 0$ the variable

$$X = \frac{y + iz}{R + x} = \frac{i - Y}{i + Y} = i\,\frac{1 - Z}{1 + Z}. \tag{2.24}$$

As a further example consider (see Fig. 2):

$$p' + iq' = \log\,(p + iq) = \log r + i\omega = \log \tan\,(\theta/2) + i\omega \tag{2.25}$$

where (θ, φ) are spherical coordinates on S. This isothermal coordinate system consists of small circles (circles of latitude) with Q as pole and meridians through Q. The function

$$\varphi = \log \tan \theta/2 \tag{2.26}$$

corresponds to the temperature solution of a point source at Q' and a point sink at Q. In space the harmonic function (2.26) corresponds to the potential of a uniform positive charge distribution over the positive Z-axis, and a similar negative distribution over the negative z-axis.

The (p, q)-net given by (2.17) consists of circles tangent to the directions of the x or y-axes at Q. The isothermal curves correspond to temperature flow due to a doublet at Q.

3. Alternative derivation of the above results. Let φ be a homogeneous harmonic function of degree zero. Then Euler's identity (1.4) yields

$$x\varphi_x + y\varphi_y + z\varphi_z = 0, \tag{3.1}$$

and when applied to φ_x, φ_y, φ_z which are of degree -1,

$$\begin{aligned} x\varphi_{xx} + y\varphi_{xy} + z\varphi_{xz} &= -\varphi_x , \\ x\varphi_{xy} + y\varphi_{yy} + z\varphi_{yz} &= -\varphi_y , \\ x\varphi_{xz} + y\varphi_{yz} + z\varphi_{zz} &= -\varphi_z . \end{aligned} \tag{3.2}$$

Solving the first two Eqs. (3.2) for φ_{xz}, φ_{yz} and substituting into the first equation, we can express φ_{zz} in terms of x- and y-derivatives of φ as well as in terms of φ_z. Elimination of the latter by means of (3.1) and utilization of

$$x\partial/\partial x + y\partial/\partial y = r\partial/\partial r \tag{3.3}$$

leads to

$$z^2\varphi_{zz} = r^2\varphi_{rr} + 2r\varphi_r ; \tag{3.4}$$

substitution into (1.1) and introduction of cylindrical coordinates r, ω, then yields

$$\varphi_{xx} + \varphi_{yy} + (2r\varphi_r + r^2\varphi_{rr})/z^2 = \left(1 + \frac{r^2}{z^2}\right)\varphi_{rr} + \left(\frac{1}{r} + \frac{2r}{z^2}\right)\varphi_r + \frac{1}{r^2}\varphi_{\omega\omega} = 0. \tag{3.5}$$

For $z = 1$ this reduces to

$$(r^4 + r^2)\varphi_{rr} + (2r^3 + r)\varphi_r + \varphi_{\omega\omega} = 0 \tag{3.6}$$

or in a modified form,

$$\left[\left\{r(1 + r^2)^{1/2}\frac{\partial}{\partial r}\right\}^2 + \frac{\partial^2}{\partial\omega^2}\right]\varphi = 0. \tag{3.7}$$

By introducing the variable s

$$s = \int\frac{dr}{r(1 + r^2)^{1/2}} = \log\frac{r}{1 + (1 + r^2)^{1/2}} \tag{3.8}$$

we may reduce (3.7) to

$$\varphi_{ss} + \varphi_{\omega\omega} = 0, \tag{3.9}$$

leading to the introduction of the analytic functions

$$f(s + i\omega). \tag{3.10}$$

For z other than $z = 1$, Eq. (3.8) is replaced by

$$s = \log\frac{r}{z + (z^2 + R^2)^{1/2}} = \log | Z |, \tag{3.11}$$

leading to

$$s + i\omega = \log Z. \tag{3.12}$$

As a further alternative procedure[4] we may start with spherical coordinates R, θ, ω, writing (2.1) in the form

$$\varphi_{RR} + \frac{2}{R}\varphi_R + \frac{1}{R^2}\left[\left(\frac{1}{\sin\theta}\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\varphi}{\partial\omega^2}\right] = 0, \tag{3.13}$$

and for φ of degree zero, convert to (3.9) by introducing

$$s = \log\tan\theta/2; \tag{3.14}$$

its general solution is exprssible either in the form

$$\varphi = Re[f(se^{i\omega})], \tag{3.15}$$

or in the form (3.10).

4. Harmonic functions of integer degrees. The Euler identity (1.4), when applied to a homogeneous harmonic function φ_n of positive integral degree n, yields

$$\varphi = \frac{1}{n}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\varphi, \tag{4.1}$$

thus expressing φ in terms of the harnonic functions φ_x, φ_y, φ_z of degree $n - 1$. By successive applications of (4.1) we are led to the identity:

$$\varphi_n = \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)^n\varphi_n, \tag{4.2}$$

where the subscript denotes the degree; this shows that φ_n is a polynomial in x, y, z with coefficients which are harmonic functions of zero degree, so that each one is expressible as the real part of an analytic function, say of Z, as in (2.20). These analytic functions, however, are *not unrelated* to each other. We shall illustrate this for $n = 1$.

Suppose that φ is a real harmonic function of degree 1, then φ_x, φ_y, φ_z are harmonic of degree zero, and hence

$$\varphi_x = Re(U), \qquad \varphi_y = Re(V), \qquad \varphi_z = Re(W) \tag{4.3}$$

where U, V, W are analytic functions of Z. These functions must satisfy the three relations obtained by equating the cross derivatives:

$$\frac{\partial(\varphi_x)}{\partial y} = \frac{\partial(\varphi_y)}{\partial x}, \qquad \frac{\partial(\varphi_y)}{\partial z} = \frac{\partial(\varphi_z)}{\partial y}, \qquad \frac{\partial(\varphi_z)}{\partial x} = \frac{\partial(\varphi_x)}{\partial z}. \tag{4.4}$$

Thus the last equation yields

$$Re\left(\frac{\partial W}{\partial x}\right) = Re\left(\frac{\partial U}{\partial z}\right). \tag{4.5}$$

[4]H. Bateman, *Partial differential equations of mathematical physics*, Cambridge University Press, 1932, pp. 356-357.

Considering U as a function of Z, we obtain

$$Re\left(\frac{\partial U}{\partial z}\right) = Re\left(\frac{dU}{dZ}\frac{\partial Z}{\partial z}\right); \tag{4.6}$$

now from Eq. (2.21)

$$\frac{\partial Z}{\partial z} = -\frac{x + iy}{(R + z)^2}\left(\frac{z}{R} + 1\right) = -\frac{Z}{R}; \tag{4.7}$$

hence there follows

$$Re\left(\frac{\partial U}{\partial z}\right) = -Re\left(\frac{dU}{dZ}\frac{Z}{R}\right). \tag{4.8}$$

Similarly, considering W as a function of X and noting that

$$\frac{\partial X}{\partial x} = -\frac{X}{R}, \qquad \frac{\partial Y}{\partial y} = -\frac{Y}{R} \tag{4.9}$$

we obtain

$$Re\left(\frac{\partial W}{\partial x}\right) = -Re\left(\frac{dW}{dX}\frac{X}{R}\right). \tag{4.10}$$

Upon equating of (4.8), (4.10) and cancellation of R we obtain

$$Re\left(\frac{dW}{dX}X\right) = Re\left(\frac{dU}{dZ}U\right), \tag{4.11}$$

whence

$$\frac{dW}{dX}X = \frac{dU}{dZ}Z + Ci \tag{4.12}$$

where C is a real constant. Dropping the latter, there results

$$U = \int \frac{X}{Z}\frac{dZ}{dX}\,dW. \tag{4.13}$$

Recalling (2.24) it will be noted that

$$\frac{dX}{X} = d(\log X) = -\frac{dZ}{1 - Z} - \frac{dZ}{1 + Z} = \frac{2dZ}{(Z^2 - 1)}; \tag{4.14}$$

hence (4.13) becomes

$$U = \frac{1}{2}\int \frac{Z^2 - 1}{Z}\,dW. \tag{4.15}$$

Similarly (2.23) yields

$$\frac{dY}{Y} = \frac{2idZ}{Z^2 + 1}, \tag{4.16}$$

and from the second equation (4.4) we now obtain

$$V = \frac{1}{2i}\int \frac{Z^2 + 1}{Z}\,dW. \tag{4.17}$$

The first relation (4.4) may now be verified from (4.14), (4.17).

Application of (4.1) for $n = 1$ now leads to

$$\begin{aligned}\varphi_1 &= Re\left[\frac{x}{2}\int\frac{Z^2-1}{Z}\,dW + \frac{y}{2i}\int\frac{Z^2+1}{Z}\,dW + zW\right]\\ &= Re\left[\frac{x-iy}{2}\int Z\,dW - \frac{x+iy}{2}\int\frac{dW}{Z} + zW\right].\end{aligned}\tag{4.18}$$

If the constant C in (4.12) and a similar constant of integration D occurring in the derivation of (4.17) had not been dropped, it would have been necessary to add terms $-C\omega$, $-D\omega$ to u, v. However, the first relation (4.4) shows that $C = D = 0$.

In a similar manner, by utilizing (4.1) for $n = 2$, expressing φ_{zx}, φ_{zy}, φ_{zz} in the form (4.18), then substituting in (4.4), it is possible to express φ_2 in terms of a single analytic function of Z and its proper double quadratures.

Another method of handling homogeneous harmonic functions of integral degree n is by differentiation of functions of degree zero, and spherical inversion. We shall illustrate this first for $n = 1$.

Inversion of (2.20) leads to

$$\varphi_{-1} = Re[f(Z)/R].\tag{4.19}$$

Upon differentiating (4.19) with respect to z, changing sign, and recalling (4.7), there results

$$\varphi_{-2} = Re\left[\frac{f'(Z)Z}{R^2} + \frac{zf(Z)}{R^3}\right].\tag{4.20}$$

Further inversion of (4.20) yields

$$\varphi_1 = Re[RZf'(Z) + zf(Z)]\tag{4.21}$$

for real harmonic functions of degree 1.* More generally, differentiating (4.19) n times with respect to z yields

$$\varphi_{-(n+1)} = Re\left\{\frac{\partial^n}{\partial z^n}\left[\frac{f(Z)}{R}\right]\right\}\tag{4.22}$$

for harmonics of negative integral degree, and by inversion of (4.22) we obtain a similar expression for harmonics of positive integral degree.

In particular, choosing

$$f(Z) = 1\tag{4.23}$$

the above procedure, in a familiar manner, leads to spherical harmonics with axial symmetry involving Legendre polynomials:

$$\varphi_{-(n+1)} = \frac{\partial^n}{\partial z^n}\left(\frac{1}{R}\right) = \text{const. } P_n(\cos\theta)/R^{n+1}.\tag{4.24}$$

*To put (4.21) in a form resembling (4.18) replace R by $z + (x - iy)Z$.

Similarly, by choosing

$$f(Z) = \log Z, \tag{4.25}$$

we obtain axially symmetric spherical harmonics involving Legendre functions of the second kind:

$$\varphi_{-(n+1)} = \frac{\partial}{\partial z^n}\left[\frac{1}{R}\log\frac{x+iy}{R+z}\right] = \text{const. } Q_n(\cos\theta)/R^{n+1}. \tag{4.26}$$

Again, by letting

$$f(Z) = Z^m, \tag{4.27}$$

$$f(Z) = Z^m \log Z \tag{4.28}$$

we are led to harmonics involving the associated Legendre functions.