in actually obtaining u_3, u_5, $\cdots$, etc. are so great that we do not feel justified in pursuing this point here.

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TRANSONIC DRAG OF AN ACCELERATED BODY*

By M. A. BIOT (*Brown University*)

It is known from the linear theory that the steady state drag of a body at the speed of sound is infinite. The occurrence of this infinite value may be interpreted as due to a resonance phenomenon and the accumulation of disturbances over an infinite interval of time. In non-steady motion, however, this resonance does not occur, and a finite value must be expected for the drag, which becomes smaller as the acceleration increases. The investigation of this phenomenon is the object of the present paper. An investigation of the drag of an accelerated body was made by F. J. Frankl.[1] His method however is approximate and does not apply at the speed of sound.

We consider a two-dimensional symmetric wedge of vertex angle 2α moving along the x-axis. The wedge is uniformly accelerated with an acceleration γ. The coordinate of the vertex O as a function of time t is (Fig. 1).

$$x = 1/2\, \gamma t^2. \tag{1}$$

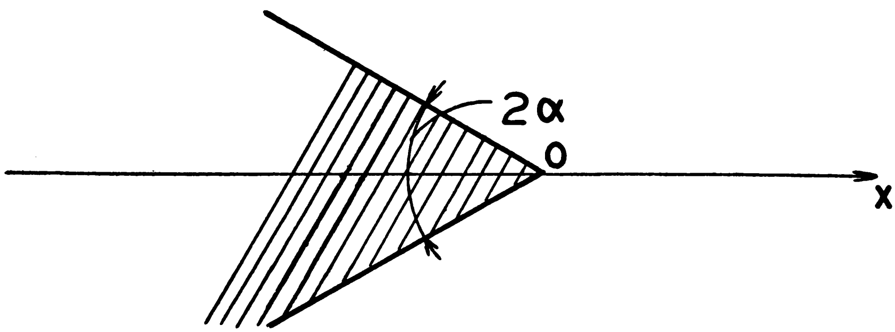

FIG. 1.

We shall simulate the motion of the solid wedge by distributing variable sources along the x-axis in such a way that the velocity component normal to x is the same as that

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[1]F. J. Frankl, *Influence of the acceleration of oblong bodies of revolution upon the resistance of the gas,* Inst. of Mech., Acad. Sci. USSR Appl. Math. Mech., Vol. X, 1946.

due to the wedge motion. We assume that the wedge angle is small and introduce the usual assumption of the linearized perturbation theory. The velocity potential φ of the perturbation satisfies the acoustic equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} \tag{2}$$

where c is the sound velocity. The basic solution from which all following expressions are derived is

$$\varphi = \frac{v}{\pi}\log\left[\frac{ct}{r} + \frac{(c^2t^2 - r^2)^{1/2}}{r}\right], \tag{3}$$

$$r = (x^2 + y^2)^{1/2},$$

which represents a source appearing suddenly at the origin at $t = 0$. The velocity field becomes identical with that of a source in an incompressible fluid for sufficiently small values of r. Therefore, if we distribute sources along the x-axis with intensities per unit length corresponding to expressions (3), the normal velocity generated by these sources will be a constant v appearing suddenly at $t = 0$. What we are interested in is the pressure associated with the potential φ. This pressure is

$$p = \rho\frac{\partial\varphi}{\partial t} = \frac{\rho vc}{\pi}\frac{1}{(c^2t^2 - r^2)^{1/2}}. \tag{4}$$

This may be generalized to cover the case of the accelerated wedge. The normal velocity at any point along x due to the wedge motion acquires a sudden value v_O when the nose of the wedge hits that point, and then continues to increase linearly following $v = v_O + \alpha\gamma t$. The pressure field of such a source is derived from (4) by superposition. Assuming this source to be located at the origin and starting at $t = 0$ the pressure is

$$p = \frac{\rho v_0 c}{\pi}\frac{1}{(c^2t^2 - r^2)^{1/2}} + \frac{\rho c}{\pi}\int_0^t \frac{1}{[c^2(t-\tau)^2 - r^2]^{1/2}}\frac{dv}{d\tau}\,d\tau \tag{5}$$

with $dv/d\tau = \alpha\gamma$.

By integration this may be written

$$p = \frac{\rho v_0 c}{\pi}\frac{1}{(c^2t^2 - r^2)^{1/2}} + \frac{\rho\alpha\gamma}{\pi}\log\left[\frac{ct}{r} + \frac{(c^2t^2 - r^2)^{1/2}}{r}\right]. \tag{6}$$

Actually this source is generated by the wedge at point ξ. We must distinguish between positive and negative values of ξ. For $\xi < 0$ all sources originate at an instant $t_1 = 0$ and the initial normal velocity is $v_O = 0$. For $\xi > 0$ the sources originate at the instant $t_1 = (2\xi/\gamma)^{1/2}$ when the nose of the wedge reaches that point and the initial normal velocity is $v_O = \gamma t_1 = (2\xi\gamma)^{1/2}$. With a step function $1(\xi)$

$$1(\xi) = 0 \text{ for } \xi < 0,$$

$$1(\xi) = 1 \text{ for } \xi > 0,$$

we may write the general expressions

$$t_1 = \left(\frac{2\xi}{\gamma}\right)^{1/2} 1(\xi),$$

$$v_0 = (2\xi\gamma)^{1/2} 1(\xi),$$

valid for the complete ξ-axis.

This particular source produces at point x and time t a pressure

$$p(x, t, \xi) = \frac{\rho c}{\pi} \frac{(2\xi\gamma)^{1/2} 1(\xi)}{\{c^2[t - (2\xi/\gamma)^{1/2} 1(\xi)]^2 - (x - \xi)^2\}^{1/2}} + \frac{\rho\alpha\gamma}{\pi} \log \frac{c[t - (2\xi/\gamma)^{1/2} 1(\xi)] + \{c^2[t - (2\xi/\gamma)^{1/2} 1(\xi)]^2 - (x - \xi)^2\}^{1/2}}{| x - \xi |}. \tag{7}$$

In order to obtain the pressure due to the motion of the wedge we must remember that expression (7) represents the pressure due to a source per unit length at point ξ. Hence the total pressure is obtained by multiplying expression (7) by $d\xi$ and integrating with respect to ξ.

$$p(x, t) = \int_{-\infty}^{+\infty} p(x, t, \xi)\, d\xi. \tag{8}$$

This represents the pressure distribution along the x-axis at any time t. In performing this integration we must remember that the integrand is to vanish whenever the radical becomes imaginary or whenever $t < (2\xi/\gamma)^{1/2} 1(\xi)$. The range of integration therefore only covers intervals of ξ such that

$$c^2\left[t - \left(\frac{2\xi}{\gamma}\right)^{1/2} 1(\xi)\right]^2 - (x - \xi)^2 \geq 0$$

and

$$t - \left(\frac{2\xi}{\gamma}\right)^{1/2} 1(\xi) \geq 0. \tag{9}$$

Consider the (x, ct) plane.

The parametric equation

$$ct = c\left(\frac{2\xi}{\gamma}\right)^{1/2} 1(\xi), \tag{10}$$

$$x = \xi$$

represents a curve AOB (Fig. 2) constituted by a straight line and a parabola.

Now consider the two straight lines of slope ± 1 which issue downward from a point P with coordinates (x, ct). They intersect the line AOB at points ξ_1, ξ_2, ξ_3, ξ_4. It may be verified that for this point P the expressions (9) are positive only for $\xi_1 < \xi < \xi_2$ and $\xi_3 < \xi < \xi_4$. Hence these intervals constitute the intervals of integration for (8). Depending on the location of P there may be two or four points of intersection.

The problem of determining the pressure distribution along the wedge at any particular instant t is thus solved.

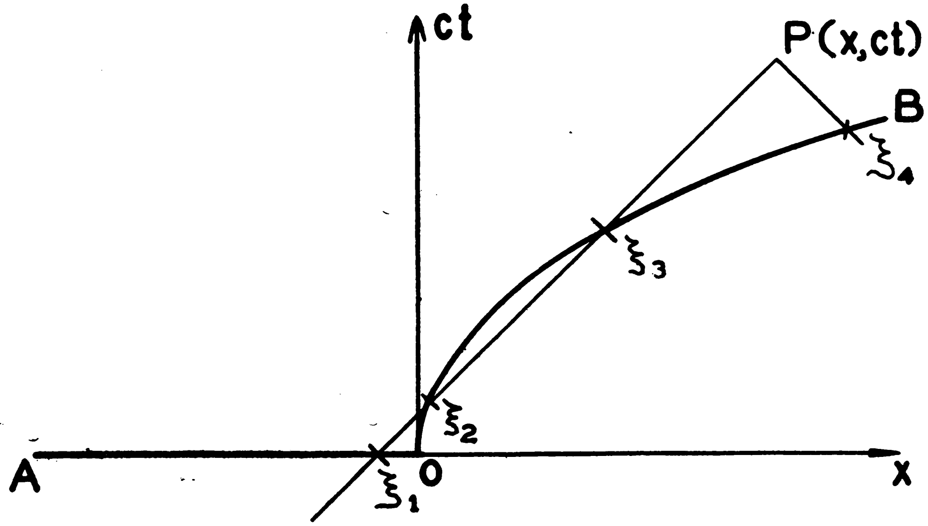

FIG. 2.

We shall not write explicitly the integrals in the general case but will focus our attention on the pressure distribution when the wedge reaches the speed of sound. In that case points ξ_2 and ξ_3 disappear. Expression (8) becomes

$$p(x) = p_1 + p_2 + p_3 \tag{11}$$

$$p_1 = \frac{\alpha\rho c^2}{2\pi} \int_{a-2}^{0} d\lambda \log \frac{2 + [4 - (a - \lambda)^2]^{1/2}}{|a - \lambda|},$$

$$p_2 = \frac{\alpha\rho c^2}{\pi} \int_0^{[-1+(3+a)^{1/2}]^2} \frac{\lambda^{1/2}\, d\lambda}{\{(2 - 2\lambda^{1/2})^2 - (a - \lambda)^2\}^{1/2}},$$

$$p_3 = \frac{\alpha\rho c^2}{2\pi} \int_0^{[-1+(3+a)^{1/2}]^2} d\lambda \log \frac{2 - 2\lambda^{1/2} + \{(2 - 2\lambda^{1/2})^2 - (a - \lambda)^2\}^{1/2}}{|a - \lambda|},$$

where $x/x_1 = a$ and $\xi/x_1 = \lambda$ are non-dimensional coordinates and $x_1 = c^2/2\gamma$ is the distance traveled by the wedge when it reaches the speed of sound. It is also assumed here that the coordinate $x > -ct$.

The value $a = 1$ corresponds to the nose. Note that for this value of a, p_1 and p_3 are finite, while p_2 becomes infinite. Hence p_2 is the preponderant contribution to the pressure for points near the nose. Because of the scale factor x_1, this is also equivalent to saying that p_2 is preponderant when the acceleration is small. We shall, therefore, assume that p_2 represents the pressure and evaluate the corresponding integrals in the vicinity of $a = 1$. By the change of variable

$$1 - \lambda^{1/2} = z(1 - a)^{1/2}$$

and for small values of $(1 - a)$ it is possible to show that the integral for p_2 tends toward the value

$$p_2 = \frac{4\alpha\rho c^2}{\pi(1 - a)^{1/4}} \int_0^{\infty} \frac{dz}{[(z^2 + 1)z]^{1/2}} = \frac{4.78}{(1 - a)^{1/4}}\, \alpha\rho c^2. \tag{12}$$

If we denote by x_2 the distance from the nose,

$$p_2 = 9.56 \frac{\alpha \rho c^2}{2} (x_1/x_2)^{1/4}. \tag{13}$$

The drag for a length l is

$$D = 2\alpha \int_0^l p_2 \, dx_2 = 25.5\alpha^2 (x_1/l)^{1/4} \frac{\rho c^2}{2} l. \tag{14}$$

The drag coefficient is

$$c_D = 25.5\alpha^2 (x_1/l)^{1/4} = 25.5\alpha^2 (c^2/2\gamma l)^{1/4}. \tag{15}$$

The drag depends on the ratio of the length of the body to the distance traveled from rest to reach the speed of sound. The presence of an acceleration causes a finite drag at the speed of sound in contrast to the infinite value in the steady case. As the value of the acceleration decreases the drag tends to infinity as the inverse fourth root of the acceleration. Another difference with the steady case is the concentration of infinite pressure at the nose. In this connection it may be concluded that the lift distribution on an accelerated wing will introduce a stalling moment in going through the speed of sound. It may be seen from formula (15) that extremely high values of the acceleration are needed for usual body sizes before the effect becomes appreciable.

It must be added that the methods presented in this paper are not restricted to the acceleration of a wedge. By superposition of positive and negative wedges the method solves the problem for a symmetric body of arbitrary shapes with constant acceleration. Furthermore, it will be noted that expressions (6), (7) and (8) may easily be generalized to cover not only the symmetric body of arbitrary shape but also the case of completely arbitrary motion. The present paper indicates how the pressure distribution in such cases may be completely expressed by quadratures.

A GENERALIZATION OF THE WIENER-HOPF TECHNIQUE*

By G. F. CARRIER (*Brown University*)

1. Introduction. Many of the problems of mathematical physics require the solution of an integral equation of the type

$$u(x) = \varphi(x) + \int_a^\infty K(x, x_0) f(x_0) \, dx_0 ,$$

where $u(x) = 0$ when $x > a$, and $f(x) = 0$ when $x < a$. When $K(x, x_0) = K(x - x_0)$ the equation is a Wiener-Hopf integral equation and the technique by which $f(x)$ may be found is well-known (cf. [1]). However, in many of the problems which arise K is not a function of $(x - x_0)$ and thus it seems desirable to generalize the Wiener-Hopf technique to include a more general family of kernels. In this paper, we shall concern ourselves with kernels which arise as the Green's functions of a certain family of partial differential equations. Although we shall choose as a basic problem a certain boundary

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