A CRITERION OF OSCILLATORY STABILITY*

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1. In the linear differential equation

$$x'' + f(t)x = 0, \tag{1}$$

let $f(t)$ be a real-valued, continuous function for large positive t, say for $t_0 \leqq t < \infty$. Consider only those solutions $x(t)$ of (1) which are real-valued and distinct from the trivial solution (identically zero). Then, if $N(t)$ denotes the number of zeros of $x(s)$ on the interval $t_0 \leqq s \leqq t$, it is clear from Sturm's separation theorem that

$$N(t) \to \infty \qquad \text{as} \qquad t \to \infty \tag{2}$$

cannot hold for a particular solution $x(t)$ unless it holds for every solution $x(t)$. In this case, (1) is called oscillatory. Thus (1) is called non-oscillatory if one (hence every) non-trivial solution fails to acquire an infinity of zeros, as $t \to \infty$.

In the applications, the importance of the classification of the differential equations (1) into the oscillatory and non-oscillatory categories is due to the following well-known fact: A non-trivial solution of (1) must change its sign whenever it vanishes, since $x(t)$ and $x'(t) = dx\,(t)/dt$ cannot vanish simultaneously.

A general criterion has been developed† for the type of stability defined by (2). In what follows, a sufficient criterion that $x(t)$ be oscillatory will be given which is not contained in known explicit tests.

2. It is easy to see that, if

$$f(t) \geqq 0, \tag{3}$$

(1) must be oscillatory whenever the indefinite integral

$$F(t) = \int^t f(s)\, ds \tag{4}$$

satisfies the condition

$$F(t) \to \infty \qquad \text{as} \qquad t \to \infty. \tag{5}$$

In fact, (3) and (1) imply that the graph of every solution $x = x(t)$ must always turn its concavities toward the t-axis of the (t, x)-plane. Hence, if (1) is non-oscillatory, and if the solution is so chosen that $x(t) > 0$ as $t \to \infty$, it is clear that $x(t) \geqq c$ holds for every sufficiently large t and for a positive constant c. Consequently, from (1) and (3),

$$x'(t) - \text{const.} = -\int^t x(s)f(s)\, ds \leqq -c \int^t f(s)\, ds.$$

Since $-c < 0$, it now follows from (4) and (5) that, if t is large enough, $x'(t)$ is negative, hence $x(t)$ is decreasing. But this contradicts the assumption that $x = x(t)$ is ultimately positive, and concave toward the t-axis, as $t \to \infty$.

By a substantial refinement of this argument, it will be shown below that, *if* (4) *satisfies* (5), *then* (1) *must be oscillatory, whether* (3) *is assumed or not.*

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†A. Wintner, Quart. Appl. Math. **5**, 232-236 (1947).

The usefulness of this criterion becomes clear if it is observed that the following fact results as a corollary: If $f(t)$ is periodic, or just almost-periodic, and if the mean-value of $f(t)$, that is, the constant term, a_0, in the (harmonic or anharmonic) Fourier expansion,

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(\lambda_n t - \alpha_n), \qquad (\lambda_n \neq 0),$$

of $f(t)$ is positive, then (1) must be oscillatory. In fact, the function (4) is now asymptotically equal to $a_0 t$, and so (5) is satisfied if $a_0 > 0$.

3. Since (5) implies that

$$\left\{ \int^t F(s)\, ds \right\} / t \to \infty \qquad \text{as} \qquad t \to \infty, \tag{6}$$

more than the last italicized statement is contained in the following theorem:

Equation (1) *must be oscillatory if* (4) *satisfies* (6).

In order to prove this theorem, suppose that (1) is non-oscillatory. Then, if $x(t)$ is any real-valued, non-trivial solution of (1), and if t is large enough, it can be assumed that $x(t) > 0$ (in fact, if $x(t)$ is a solution, then $-x(t)$ is). Since $x(t)$ does not vanish, it has a logarithmic derivative which, in view of (1), satisfies Riccati's differential equation $q' = -f - q^2$, where $q = x'/x$. This differential equation implies the inequality $f \leqq -q'$. Hence, two quadratures and the definition (4) show that

$$\int^t F(s)\, ds \leqq - \int^t q(s)\, ds + at + b,$$

where a, b are integration constants. Since $q(t) = x'(t)/x(t)$ and $x(t) > 0$, it follows that, if b is suitably chosen,

$$\int^t F(s)\, ds - at - b \leqq - \log \mid x(t) \mid. \tag{7}$$

If (6) is assumed, then (7) implies that $\log \mid x(t) \mid \to -\infty$, i.e., that

$$x(t) \to 0 \qquad \text{as} \qquad t \to \infty. \tag{8}$$

Hence, if (1) is non-oscillatory, then (6) cannot hold unless every solution of (1) satisfies (8). Consequently, in order to complete the proof of the last italicized theorem, it will be sufficient to verify the following assertion: (1) must be oscillatory if every solution of (1) satisfies (8).

4. Since what is claimed by this assertion is just the truth of (2), more than what is needed is contained in the following fact:

Equation (8) *cannot hold for every solution of* (1) *unless*

$$N(t)/t \to \infty \qquad \text{as} \qquad t \to \infty. \tag{9}$$

The deduction of the refinement (9) of (2) proceeds as follows:

Let $x = x(t)$ and $x = y(t)$ be two linearly independent solutions of (1). Their Wronskian, being a non-vanishing constant, can be assumed to be 1. Then $r^2\phi' = 1$ is an identity in t, if $r = r(t) > 0$ and a (continuous) $\phi = \phi(t)$ are defined by placing $x = r \cos \phi$ and $y = r \sin \phi$.

Suppose that (8) holds for every solution of (1). Then $x(t) \to 0$ and $y(t) \to 0$ as $t \to \infty$. It follows therefore from $r^2(t)\phi'(t) = 1$, where $r^2 = x^2 + y^2$, that $\phi'(t) \to \infty$.

Hence, a quadrature shows that $\phi(t)/t \to \infty$, as $t \to \infty$. In view of $x(t) = r(t) \cos \phi(t)$, this implies the truth of (9) for the number of the zeros of $x(t)$.

It follows in the same way that

$$N(t)/t > \text{const.} > 0 \quad \text{as} \quad t \to \infty, \tag{10}$$

if every solution of (1), instead of satisfying (8), remains just bounded as $t \to \infty$. Similarly, *if* $x(t)/t^{1/2}$ *remains bounded for every solution of* (1), *then* (1) *must be oscillatory,* since (2) then follows in the same way as (9) and (10) did.

THE DISTRIBUTION OF PLANE ANGLES OF CONTACT*

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Let $R = \omega(\theta)$ be a closed, simple, convex, plane curve, with a continuous tangent and the origin in its interior, and with R, θ its polar coordinates. A particle A moves in a straight line at constant speed k from time $t = 0$ to $t = T$. Another particle B moves with unit speed in a straight line coplanar with the path of A. The probability that at time $t = 0$ the particle B will be in any sub-region g of a sufficiently large region G containing the starting point of A (at time $t = 0$) is proportional to the area of g. The probability that the azimuth of B's motion will lie between the angles α, β, is $(\beta - \alpha)/2\pi$ and is independent of the starting position of B. Let θ be the angle from the direction of A's motion to the particle B, and R the distance from A to B. When $R = \omega(\theta)$ a "contact" is said to occur. Let ϕ be the angle from the radius vector from A to B to the direction of B's motion. The couple (θ, ϕ) characterize a contact. Naturally no contact need ever occur between $t = 0$ and $t = T$, or at any other time, for that matter. In this note we shall be concerned with the probability distribution of (θ, ϕ) in the totality of contacts which *do* occur between $t = 0$ and $t = T$, and excluding the possibility that, at $t = 0$, $R < \omega(\theta)$. We will show that the probability density $f(\theta, \phi)$ of (θ, ϕ) is $c(D^* + |D^*|)$ where

$$D^* = -\omega(\theta) \cos \phi + \frac{d\omega}{d\theta} \sin \phi + k\left[\omega(\theta) \cos \theta + \frac{d\omega}{d\theta} \sin \theta\right]$$

and c is a constant defined by

$$\int_0^{2\pi} \int_0^{2\pi} f(\theta, \phi)\, d\theta\, d\phi = 1.$$

The curve $R = \omega(\theta)$ may be thought of as a field of force surrounding the particle A. If A and B are circular disks and $\omega(\theta)$ identically equal to the sum of their radii, a "contact" would be literally such. The following should also be noted:

a) The result is independent of T. The requirement that G should be sufficiently large means that G should contain a circle, centered at the position of A at $t = 0$, and of radius sufficiently large (the minimum radius depends upon T). Once G is sufficiently large, $f(\theta, \phi)$ does not depend upon G.

b) The result depends only upon the ratio k of the speeds. The constant c depends

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