—NOTES—

A NOTE ON THE VIBRATING STRING*

BY G. F. CARRIER (*Brown University*)

In a previous paper [1] certain characteristics of the behavior of an elastic string undergoing periodic vibrations of moderately large amplitude were established. In this note, the same basic result is obtained in a manner which is mathematically more satisfactory. Furthermore, a refinement of the solution associated with the periodic motion of lowest frequency is obtained.

It is convenient to postulate a material which obeys the stress-strain law

$$T - T_0 = EA\{[(1 + v_x)^2 + u_x^2]^{1/2} - 1\}. \tag{1}$$

Here, T_0 is the rest tension, A the rest cross-sectional area, E an elastic constant of the material, and T, u, v, θ, are defined[1] in Fig. 1. This law is probably as close to reality as any we could postulate for the general run of elastic materials. In any event, a modified Eq. (1) introduces into the results only higher order effects.

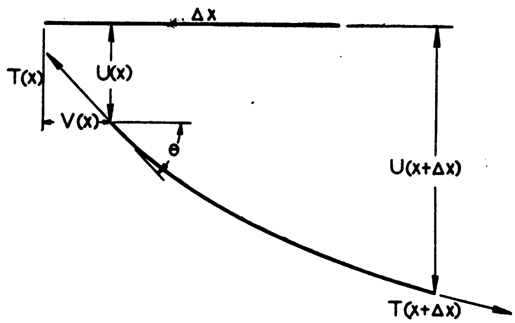

FIG. 1.

If we apply the conditions of dynamic equilibrium to an element of the string as shown in Fig. 1, we obtain (rigorously)

$$[T \sin \theta]_x = \rho A u_{tt} , \tag{2}$$

$$[T \cos \theta]_x = \rho A v_{tt} . \tag{3}$$

Now with $\tau = (T - T_0)/T_0$, $\xi = \pi x/l$, $\eta^2 = \pi^2\alpha^2 E t^2/\rho l^2$, $\alpha^2 = T_0/EA$, Eqs. (1), (2) and (3) can be combined to give

$$[(1 + \tau)e^{i\theta}]_{\xi\xi} = [(1 + \alpha^2\tau)e^{i\theta}]_{\eta\eta} . \tag{4}$$

The boundary conditions $u = 0$ and $v = 0$ at $\xi = 0$ and $\xi = \pi$ can be replaced by

$$\int_0^\pi (1 + \alpha^2\tau)e^{i\theta}\, d\xi = \pi. \tag{5}$$

These are essentially the basic equations used in [1].

*Received Feb. 25, 1948.

[1]We note that tan $\theta = u_x/(1 + v_x)$.

In order to obtain those solutions of Eqs. (4) and (5) in which we are interested, it is convenient to define $\theta = \alpha w$, carry out the differentiations indicated in Eq. (4), factor the exponential, and separate the real and imaginary parts. We obtain

$$(1 + \tau)w_{\xi\xi} - w_{\eta\eta} = -2w_\xi \tau_\xi + \alpha^2(2w_\eta \tau_\eta + \tau w_{\eta\eta}), \tag{6}$$

$$\tau_{\xi\xi} = \alpha^2[\tau_{\eta\eta} + (1 + \tau)w_\xi^2 - w_\eta^2] - \alpha^4 \tau w_\eta^2 . \tag{7}$$

Furthermore, Eq. (5) may be expanded to give

$$\int_0^\pi [(\tau - w^2/2) + \alpha^2(w^4/4! - w^2\tau/2) + \cdots]\, d\xi = 0. \tag{8}$$

Here, it is implied that we consider only functions w which are odd in ξ about the point $\pi/2$ and functions τ which are even about this point.

At this stage of the development, it is interesting to observe the analogy presented by this problem and the boundary layer problems in fluid dynamics [2]. In the boundary layer problem one replaces the coordinate normal to the boundary by a new coordinate which includes a natural parameter (i.e. the Reynold's number[2]) just as here α is incorporated in η. The velocity components of the fluid problem are of zero order and first order in α_1, whereas in our problem, τ and θ are respectively of these orders. Finally, in the fluid problem, once the foregoing items are incorporated into the equations, one solves the equations for the case $\alpha_1 = 0$ and anticipates that this asymptotic solution will represent the physical picture very well when $\alpha_1 \ll 1$. This, of course, will also be the procedure here.

Returning to the problem then, we can deduce consecutively from Eqs. (7), (8), (6), that, for $\alpha = 0$,

$$\tau = \tau(\eta), \tag{9}$$

$$\tau = \int_0^\pi (w^2/2\pi)\, d\xi, \tag{10}$$

$$\left[1 + \int_0^\pi (w^2/2\pi)\, d\xi\right] w_{\xi\xi} - w_{\eta\eta} = 0. \tag{11}$$

Equation (11) corresponds to Eq. (17) of [1]. It leads directly to the solution for the low frequency periodic oscillation (as given in [1])

$$w = 2\epsilon u_0(\eta) \cos \xi \tag{12}$$

and

$$\tau = \epsilon^2 u_0^2(\eta), \tag{13}$$

where

$$u_0(\eta) = \mathrm{cn}[(1 + \epsilon^2)^{1/2}\eta, k_0] \tag{14}$$

and

$$k_0 = \epsilon/[2(1 + \epsilon^2)]^{1/2}. \tag{15}$$

[2]We shall call the Reynold's number α_1^{-2} here.

This solution can be expected to be a good approximation to the rigorous result for small α. It is interesting nevertheless to investigate the periodic solution which is valid to terms of order α^4. The procedure is necessarily somewhat unorthodox. It is convenient to write

$$w = 2\epsilon\varphi(\eta)\cos\xi + \alpha^2 \sum_{j=3,5,7}^{\infty} u_j(\eta)\cos j\xi, \tag{16}$$

$$\tau = \epsilon^2\varphi^2(\eta) + \alpha^2\tau_1(\xi, \eta) = \tau_0 + \alpha^2\tau_1 . \tag{17}$$

If these forms of w and τ are substituted into Eqs. (6), (7), (8), we obtain from Eq. (7) (discarding terms of order $\alpha^4, \cdots$)

$$(\tau_1)_{\xi\xi} = (\tau_0)_{\eta\eta} + 4\epsilon^2(1 + \tau_0)\varphi^2 \sin^2\xi - 4\epsilon^2\cos^2\xi(\varphi_\eta)^2. \tag{18}$$

Using Eq. (8) to evaluate the arbitrary function of η which arises when we integrate Eq. (18), we obtain for τ_1

$$\begin{aligned}\tau_1 = (\tau_0)_{\eta\eta}[(\xi - \pi/2)^2/2 - \pi^2/24] + \epsilon^2(1 + \tau_0)\varphi^2[(\xi - \pi/2)^2 \\ - \pi^2/12 + (\cos 2\xi)/2] - \epsilon^2\varphi_\eta^2[(\xi - \pi/2)^2 - \pi^2/12 \\ - (\cos 2\xi)/2] + \frac{3}{4}\epsilon^4\varphi^4.\end{aligned} \tag{19}$$

We may now expand the functions of ξ which occur in Eq. (19) in Fourier series containing only terms of the form $\cos 2n\xi$. Using these, the terms of order α^2 in Eq. (6) may be evaluated. In fact, we can write them so that they consist of series of the form $F_n \cos(2n + 1)\xi$, where the F_n are combinations of φ, τ_0, and their derivatives. When this is done we may combine all terms and equate the coefficient of $\cos(2n + 1)\xi$ to zero for each n. The equation associated with $n = 0$ is that which defines φ and (when terms in $\alpha^4, \cdots$, are discarded) is

$$\varphi_{\eta\eta} + \varphi + \epsilon^2\varphi^3 + \alpha^2[13\epsilon^2\varphi\varphi_\eta^2 - 2\epsilon^2\varphi_{\eta\eta}^2 - 9\epsilon^2\varphi^3 - 6\epsilon^4\varphi^5]/4 = 0. \tag{20}$$

This equation is not easy to integrate but a simple substitution renders it tractable. For moderately small α one expects that φ will closely resemble u_0. Let us observe then, that

$$(u_0)_\eta^2 = (1 + \epsilon^2/2) - u_0^2 - \epsilon^2 u_0^4/2 \tag{21}$$

and

$$(u_0)_{\eta\eta} = -u_0 - \epsilon^2 u_0^3 , \tag{22}$$

and let us assume (or rather hope) that

$$\varphi_\eta^2 = 2(1 + \epsilon^2/2) - \varphi^2 - \epsilon^2\varphi^4/2 + O(\alpha^2), \tag{23}$$

$$\varphi_{\eta\eta} = -\varphi - \epsilon^2\varphi^3 + O(\alpha^2). \tag{24}$$

If these relations were true, then we could replace φ_η^2 and $\varphi_{\eta\eta}$ *in the coefficient of* α^2 in Eq. (20) and obtain an equation which is still accurate to order α^4. Since Eq. (20) is already in error to terms in α^4, this is a consistent procedure. Let us make this substitution then, with the reservation that we must substitute the solution obtained under this assumption into Eq. (20) and verify that the terms "left over" are of order α^4. One can readily verify that the forthcoming answer satisfies this condition. When Eq. (20) is modified in this manner, it becomes

$$\varphi_{\eta\eta} + a_1\varphi + a_3\varphi^3 + a_5\varphi^5 = 0, \tag{25}$$

where

$$a_1 = 1 + 13\alpha^2\epsilon^2(2 + \epsilon^2)/8,$$

$$a_3 = \epsilon^2(1 - 5\alpha^2),$$

$$a_5 = -21\epsilon^4\alpha^2/8,$$

and the solution such $\varphi(0) = 1$, $\varphi'(0) = 0$, is[3]

$$\varphi = \frac{\mathrm{cn}\ (\beta\eta, k)}{1 - A\ \mathrm{sn}^2\ (\beta\eta, k)}, \tag{26}$$

where

$$\beta^2 = 1 + \epsilon^2 + \frac{3}{4}\alpha^2\epsilon^4,$$

$$k^2 = [\epsilon^2/2(1 + \epsilon^2)][1 - 5\alpha^2 - 7\alpha^2\epsilon^2/4 - 3\alpha^2\epsilon^4/4(1 + \epsilon^2)],$$

$$A = 7\alpha^2\epsilon^2/8,$$

when terms of order α^4 are discarded.

Substitution of this result into Eq. (20) will reveal that the equality fails only in terms of order α^4, and thus the desired result has been obtained. It is evident that for small $\alpha^2(\alpha^2 = .002$ is large for the usual elastic medium) the frequency and wave form differ very little from those of the first order solution.

There still remains the problem of finding u_3, u_5, $\cdots$. Again, substitution of Eq. (16) and (20) into Eq. (6) leads to the equation for u_3:

$$(u_3)_{\eta\eta} + 9(1 + \epsilon^2\varphi^2)u_3 = G(\eta). \tag{27}$$

Here, $G(\eta)$ is a group of terms of the type $(\tau_0)_{\eta\eta}$, φ, φ^3, $\cdots$. That is, $G(\eta)$ has the same period as φ.

It is known that the homogeneous solutions associated with Eq. (27) are of the type

$$u_3 = e^{\pm\lambda\eta}f(\eta),$$

where $f(\eta)$ has the period of φ^2. *If* λ *is purely imaginary, such solutions are stable and the natural frequencies are different from that of* φ^2. In such cases, the non-homogeneous solution of Eq. (27) can rarely exhibit a resonant effect. Since φ^2 does not differ greatly from a trigonometric function, the results obtained by Lubkin & Stoker [3] can be used to estimate whether stability of the homogeneous solution of Eq. (27) is implied. It is evident by inspection of their results that the solution is stable when ϵ is not large compared to unity. Therefore, we can conclude that the functions u_3, u_5, $\cdots$, are bounded and that we are dealing with a periodic solution of Eq. (6). The difficulties

[3]Elementary integration and the use of Pierce's integral tables are the only processes required here.

in actually obtaining u_3, u_5, $\cdots$, etc. are so great that we do not feel justified in pursuing this point here.

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TRANSONIC DRAG OF AN ACCELERATED BODY*

By M. A. BIOT (*Brown University*)

It is known from the linear theory that the steady state drag of a body at the speed of sound is infinite. The occurrence of this infinite value may be interpreted as due to a resonance phenomenon and the accumulation of disturbances over an infinite interval of time. In non-steady motion, however, this resonance does not occur, and a finite value must be expected for the drag, which becomes smaller as the acceleration increases. The investigation of this phenomenon is the object of the present paper. An investigation of the drag of an accelerated body was made by F. J. Frankl.[1] His method however is approximate and does not apply at the speed of sound.

We consider a two-dimensional symmetric wedge of vertex angle 2α moving along the x-axis. The wedge is uniformly accelerated with an acceleration γ. The coordinate of the vertex O as a function of time t is (Fig. 1).

$$x = 1/2\,\gamma t^2. \tag{1}$$

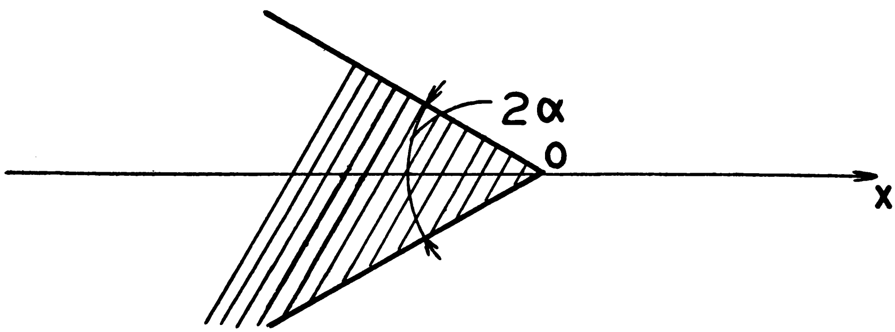

Fig. 1.

We shall simulate the motion of the solid wedge by distributing variable sources along the x-axis in such a way that the velocity component normal to x is the same as that

*Received Dec. 13, 1948. Presented at the 7th International Congress of Applied Mechanics, London, 1948.

[1]F. J. Frankl, *Influence of the acceleration of oblong bodies of revolution upon the resistance of the gas*, Inst. of Mech., Acad. Sci. USSR Appl. Math. Mech., Vol. X, 1946.