SUPERSONIC FLOW OVER BODIES OF REVOLUTION*

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1. Introduction. The purpose of this paper is to present a method of determination of the flow of a compressible fluid about the head of a pointed body of revolution. This method is based on the assumption that the actual flow may be approximated by a perturbation of the conical flow about the tip of the body, an assumption that will hold for contours of small curvature. The flow is furthermore supposed to be steady and axially symmetric; viscosity, heat conduction and external forces are considered to be negligible. Since the entropy perturbation is of the same order as the velocity perturbation, it must be taken into account.

2. Differential equations of the flow. Let us suppose that the velocity **q** of the flow and the local speed *a* of sound are measured in units of the speed *c* of efflux into a vacuum, that density ρ , pressure *p*, and absolute temperature *T* are measured in units of their respective stagnation values ρ_0 , p_0 , T_0 in the free stream, and that the entropy *S* is measured in units of c_v ; further we put $S_0 = 0$. Then we have, in view of our basic assumptions,¹

Bernoulli's equation:

$$1 - q^2 = \frac{2}{\gamma - 1} a^2 = T, \tag{1}$$

isentropy along streamlines:

$$p/\rho^{\gamma} = g(S) = e^{S}, \qquad (2)$$

the equation of continuity:

$$\nabla \cdot (\mathbf{q} T^{1/(\gamma-1)}) = 0, \tag{3}$$

and the vorticity equation:

$$(\nabla \times \mathbf{q}) \times \mathbf{q} = \frac{a^2}{\gamma(\gamma - 1)} \nabla S.$$
(4)

Introducing spherical coordinates r, θ , φ , so that the tip of the body lies at the origin r = 0, and its axis coincides with the line $\theta = 0$, and denoting the velocity components in the direction of increasing r, θ , φ by X, Y, Z, respectively, (see Fig. 1), we have for axially symmetric flow, since Z = 0, $\partial/\partial \varphi = 0$,

$$\nabla \cdot (\mathbf{q}T^{1/(\gamma-1)}) = \frac{\partial}{\partial r} \left(XT^{1/(\gamma-1)}\right) + \frac{2}{r} \left(XT^{1/(\gamma-1)}\right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left(YT^{1/(\gamma-1)}\right) + \frac{\cot \theta}{r} \left(YT^{1/(\gamma-1)}\right) = 0.$$

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¹See, cg., R. Sauer, Theoretische Einführung in die Gasdynamik, Springer, Berlin, 1943, p. 132.

Now



FIG. 1.

so that (3) becomes, after multiplication by $ra^2 T^{-1/(\gamma-1)}$,

$$rX_r(a^2 - X^2) - XY(X_{\theta} + rY_r) + Y_{\theta}(a^2 - Y^2) + a^2(2X + Y \cot \theta) = 0.$$
 (5)

The components of $\nabla \times \mathbf{q}$ in the r, θ , φ directions, respectively, are 0, 0, $(1/r)(X_{\theta} - rY_r - Y)$. Therefore,

$$(\nabla \times \mathbf{q}) \times \mathbf{q} = -\frac{1}{r} (X_{\theta} - rY_r - Y)(Y\mathbf{k}_1 - X\mathbf{k}_2),$$

where \mathbf{k}_1 and \mathbf{k}_2 are unit vectors in the r and θ directions. By Eq. (4) then,

$$X_{\theta} - rY_{r} - Y = -\frac{a^{2}}{\gamma(\gamma - 1)} \cdot \frac{rS_{r}}{Y} = \frac{a^{2}}{\gamma(\gamma - 1)} \cdot \frac{S_{\theta}}{X}.$$
 (6)

In case of a conical axially symmetric field, $X = U(\theta)$ and $Y = V(\theta)$ are functions of θ only, and the flow is necessarily irrotational, so that Eqs. (5) and (6) become²

$$-UV^{2} + (a_{0}^{2} - V^{2})V' + a_{0}^{2}(2U + V \cot \theta) = 0,$$
(7)

$$U' - V = 0, \qquad (8)$$

with $a_0^2 = (1/2)(\gamma - 1)(1 - U^2 - V^2), U' \equiv dU/d\theta$.

It will in the following be assumed that U, V are the velocity components in the ²These equations are identical with those first derived by Taylor and Maccoll, Proc. Roy. Soc. (A) 139, 278 (1939).

conical field of the nose of the body of revolution, and that they have already been determined as functions of θ . We now assume that

$$X = U + u, \qquad Y = V + v, \tag{9}$$

and that the velocity perturbations u, v are small compared with U, V, making second order terms such as u^2 , uu_r , etc., negligible. Under these conditions Eqs. (5) and (6) become, after some simplifications,

$$ArU_r + B(u_\theta + rv_r) + Cv_\theta + Du + Ev = 0, \qquad (10)$$

 $u_{\theta} - rv_{\tau} + Fu + Gv + H = 0,$

 \mathbf{with}

$$\begin{split} A &= a_0^2 - U^2, \quad B = -UV, \quad C = a_0^2 - V^2, \\ D &= 2a_0^2 - V^2 - (\gamma - 1)U(2U + V \cot \theta) - (\gamma - 1)UV', \\ E &= a_0^2 \cot \theta - (\gamma - 1)V(2U + V \cot \theta) - (\gamma + 1)VV', \\ F &= \frac{S_{\theta}}{\gamma} \left(\frac{a_0^2}{(\gamma - 1)U^2} + 1 \right), \quad G = \frac{S_{\theta}}{\gamma} \frac{V}{U} - 1, \quad H = -\frac{S_{\theta}}{\gamma} \frac{a_0^2}{(\gamma - 1)U} \,. \end{split}$$

3. System of characteristic differential equations. Since $u = u(r, \theta)$, $v = v(r, \theta)$, whence $du = u_r dr + u_{\theta} d\theta$, $dv = v_r dr + v_{\theta} d\theta$, the characteristic equations³ of the system (10) and (11) may be obtained from the condition that the matrix

Ar	В	Br	C	Du + Ev
0	1	-r	0	Fu + Gv + H
dr	$d\theta$	0	0	-du
0	0	dr	$d\theta$	-dv

has the rank 3.

The vanishing of the determinant of the first four columns leads to

$$Ar^2 d\theta^2 - 2Br dr d\theta + Cdr^2 = 0,$$

whence⁴

$$dr - \frac{B \pm R}{C} r \, d\theta = 0, \tag{12}$$

with

$$R = (B^{2} - AC)^{1/2} = [a_{0}^{2}(q_{0}^{2} - a_{0}^{2})]^{1/2}, \qquad q_{0}^{2} = U^{2} + V^{2}.$$

Putting

$$m(\theta) = (B + R)/C,$$
 $n(\theta) = (B - R)/C,$

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(11)

³For the theory of characteristics see R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, J. Springer, Berlin, 1937, vol. 2, ch. V.

⁴Note that for supersonic flow always C > 0.

we obtain for the equation of the first⁵ family $\eta = \text{const.}$ of characteristics

$$dr - rm \, d\theta = 0, \tag{13}$$

and for the second⁵ family $\xi = \text{const.}$

$$dr - rn \, d\theta = 0. \tag{14}$$

It is easy to show that Eqs. (13) and (14) are also the equations for the characteristics of the conical field, as follows. Since (see Fig. 2)



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 $\omega_1 = \alpha - \beta, \qquad \omega_2 = -\alpha - \beta,$

and $\tan \alpha = a_0(q_0^2 - a_0^2)^{-1/2}$, $\tan \omega_1 = rd\theta/dr$, $\tan \beta = -V/U$, we get

$$\pm \frac{r \, d\theta}{dr} = \frac{a_0^2 - V^2}{\mp UV + a_0(q_0^2 - a_0^2)^{1/2}} = \frac{C}{\pm B + R}$$

along a characteristic of the first and second family, respectively. These relationships, however, are identical with Eqs. (13) and (14).

Integration of these equations leads to

$$r/r_1 = \exp \int_{\theta_1}^{\theta} m(\theta) \ d\theta,$$

 $r/r_2 = \exp \int_{\theta_2}^{\theta} n(\theta) \ d\theta,$

find

⁵Looking in the direction of \mathbf{q} , the "first" family is the family running forward to the left of \mathbf{q} , the "second" family is the one running to the right of \mathbf{q} .

fi 1, 2, P are three points located as shown in Fig. 3. These relationships may also be expressed in the form



FIG. 3. $r/r_1 = M(\theta)/M(\theta_1),$ (15)

$$r/r_2 = N(\theta)/N(\theta_2), \tag{16}$$

For later use we note here also that

$$r_2/r_1 = (r_2/r)(r/r_1) = \exp\left(\int_{\theta}^{\theta_2} nd\theta + \int_{\theta_2}^{\theta} md\theta\right),$$

 $M(\theta) = \exp \int_{\theta_{\star}}^{\theta} m(\theta) \ d\theta, \qquad N(\theta) = \exp \int_{\theta_{\star}}^{\theta} n(\theta) \ d\theta.$

whence

$$M(\theta)/N(\theta) = (r_2/r_1)M(\theta_1)/N(\theta_2).$$
(17)

To determine the velocity perturbations u, v at P we make use of the fact that also necessarily

$$\begin{vmatrix} Ar & B & C & Du + Ev \\ 0 & 1 & 0 & Fu + Gv + H \\ dr & d\theta & 0 & -du \\ 0 & 0 & d\theta & -dv \end{vmatrix} = 0.$$

In view of mn = A/C, m + n = 2B/C this results in

$$dv + ndu + Kd\theta = 0 \text{ along } \eta = \text{const.}, \tag{18}$$

$$dv + mdu + Ld\theta = 0 \text{ along } \xi = \text{ const.}, \tag{19}$$

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with

$$K = C^{-1}[(D - RF)u + (E - RG)v - RH],$$
$$L = C^{-1}[(D + RF)u + (E + RG)v + RH].$$

Finally, since $dS = S_r dr + S_{\theta} d\theta$, and, by Eq. (6), $-rXS_r = YS_{\theta}$, we have

$$dS = S_{\theta} d\theta (1 - mY/X) \qquad \text{along } \eta = \text{const.}, \tag{20}$$

and

$$dS = S_{\theta} d\theta (1 - nY/X) \qquad \text{along } \xi = \text{const.}$$
(21)

It is thus seen that in order to compute K, L, and dS it is necessary to know S_{θ} . This quantity may be found as follows. Since

$$S = S[\xi(r, \theta), \eta(r, \theta)],$$
 we have $S_{\theta} = S_{\xi}\xi_{\theta} + S_{\eta}\eta_{\theta}$.

But

$$egin{pmatrix} \xi_r & \xi_ heta\ \eta_ au & \eta_ heta \end{pmatrix} = rac{1}{J} egin{pmatrix} heta_\eta & -r_\eta\ - heta_\xi & r_\xi \end{pmatrix}$$

with $J = \partial(r, \theta) / \partial(\xi, \eta)$. By Eqs. (13), (14) $r_{\xi} = rm\theta_{\xi}$, $r_{\eta} = rn\theta_{\eta}$, whence

$$J = (m - n)r\theta_{\xi}\theta_{\eta} \quad \text{and} \quad S_{\theta} = \frac{1}{m - n} \left(m \frac{S_{\eta}}{\theta_{\eta}} - n \frac{S_{\xi}}{\theta_{\xi}} \right).$$
(22)

If, then, the quantities θ and S are known at the points 1, 2, and P of a characteristic net (see Fig. 3), then approximately

$$S_{\theta} = \frac{1}{m-n} \left(m \frac{S-S_2}{\theta-\theta_2} - n \frac{S-S_1}{\theta-\theta_1} \right).$$
(23)

4. Boundary conditions. A. At the body. Let the equation of the contour of the body of revolution in a meridian plane be

$$r = f(\theta). \tag{24}$$

Since the contour must be a streamline, we have

$$f'(\theta) = rX/Y$$

Introducing the perturbations u, v defined in Eqs. (9), we obtain

$$ru - f'(\theta)v + rU - f'(\theta)V = 0.$$
⁽²⁵⁾

B. At the shock wave. Here the laws of conservation of mass, of momentum, and of energy must be satisfied, and, moreover, the change in entropy must be calculated.

From the three conservation laws it is easy to deduce first the continuity of the tangential velocity component: $q_t = \dot{q_{1t}}$. (Fig. 4) Along the shock, then,

$$\frac{dr}{d\theta} = r \frac{q_1 \sin \theta + Y}{q_1 \cos \theta - X},\tag{26}$$

where q_1 is the free stream velocity. This relationship may also be expressed in the form

$$\tan \theta_w = \frac{q_1 - X \cos \theta + Y \sin \theta}{X \sin \theta + Y \cos \theta}.$$
 (27)



Fig. 4.

The velocity perturbations u, v along the shock wave may be found as follows. Let q_h and q_v be the horizontal and vertical components of \mathbf{q} ; then (see Fig. 4)

$$q_h = X \cos \theta - Y \sin \theta, \qquad q_v = X \sin \theta + Y \cos \theta.$$
 (28)

Along the shock wave q_h and q_v satisfy the equation of the shock polar

$$(q_1 - q_h)^2 (q_h - b) - q_v^2 (e - q_h) = 0$$
⁽²⁹⁾

with

$$b = \mu/q_1$$
, $e = \mu/q_1 + (1 - \mu)q_1$, $\mu = (\gamma - 1)/(\gamma + 1)$

Making use of Eqs. (28) and introducing the perturbations u, v we find that Eq. (29) leads to

$$\Gamma u + \Delta v + \Lambda = 0, \tag{30}$$

where

$$\begin{split} \Gamma &= (3U^2 + V^2) \cos \theta - 2UV \sin \theta - 2U[(2q_1 + b) \cos^2 \theta + e \sin^2 \theta] \\ &\quad + q_1(1 + \mu)V \sin 2\theta + (2\mu + q_1^2) \cos \theta, \\ \Delta &= -(U^2 + 3V^2) \sin \theta + 2UV \cos \theta - 2V[(2q_1 + b) \sin^2 \theta + e \cos^2 \theta] \\ &\quad + q_1(1 + \mu)U \sin 2\theta - (2\mu + q_1^2) \sin \theta, \\ \Lambda &= U^3 \cos \theta - V^3 \sin \theta - UV(U \sin \theta - V \cos \theta) - U^2[(2q_1 + b) \cos^2 \theta + e \sin^2 \theta] \\ &\quad - V^2[(2q_1 + b) \sin^2 \theta + e \cos^2 \theta] + q_1(1 + \mu)UV \sin 2\theta \\ &\quad + (2\mu + q_1^2)(U \cos \theta - V \sin \theta) - \mu q_1 \,. \end{split}$$

To determine the entropy variation along the shock we notice first that

$$\frac{dS}{d\theta} = S_r \frac{dr}{d\theta} + S_\theta = S_\theta \left(1 - \frac{Y}{X} \frac{1}{r} \frac{dr}{d\theta} \right). \tag{31}$$

Now from

$$S = \log (p/\rho^{\gamma}) = \log (p/p_1) + \gamma \log (\rho_1/\rho) + S_1, \qquad (32)$$

and the Rankine-Hugoniot equations

$$\frac{p}{p_1} = \frac{2\gamma M_1^2 \sin^2 \theta_w - (\gamma - 1)}{\gamma + 1},$$
$$\frac{\rho_1}{\rho} = \frac{(\gamma - 1) \sin^2 \theta_w + 2/M_1^2}{(\gamma + 1) \sin^2 \theta_w}$$

it may be shown that

$$\frac{dS}{d\theta} = \frac{2\gamma ks}{t^2} \cdot \frac{(\tan^2 \theta_w - t)^2}{\tan \theta_w (\tan^2 \theta_w - k)(\tan^2 \theta_w + s)} \cdot \frac{d\theta_w}{d\theta}$$
(33)

with $t^{-1} = M_1^2 - 1$, $k^{-1} = [2\gamma/(\gamma - 1)]M_1^2 - 1$, $s^{-1} = [(\gamma - 1)/2]M_1^2 + 1$.

Once θ_w is known, Eq. (33) permits the calculation of $dS/d\theta$, and then Eq. (31) that of S_{θ} . This will be described in more detail in the following section.

5. Computational procedure. Let the value of $M_1 > 1$ and the contour of the body of revolution be given (see Fig. 5). We replace the nose 0'10" of the body by a small tangential cone 010", and compute the conical flow U, V determined by M_1 , θ_* , thus obtaining a number of starting points 2, 3, 4, \cdots on the initial η -characteristic 11'. We use these points to get a characteristic net as shown in Fig. 5. Each point of this net belongs to one of these four types:

- a. It lies on the initial η characteristic 11', e.g. 2;
- b. It lies on the contour, e.g. 10;
- c. It lies in the "interior", e.g. 5';
- d. It lies on the shock wave, e.g. 6'.



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Type a: point 2. Then $r = r_1 M(\theta_2)$, u = 0, v = 0; S may be found by Eq. (32) and the Rankine-Hugoniot equations. This value of the entropy prevails in the whole region bounded by the straight shock 01', the contour, and the streamline through 1'.

Type b: point 10. By Eqs. (16) and (24) $r = f(\theta) = r_2 N(\theta) / N(\theta_7)$, which permits the determination of θ , and then r. Further, by (19) and (25),

$$(v - v_7) + m(u - u_7) + L_m(\theta - \theta_7) = 0,$$

 $ru - f'(\theta)v + rU - f'(\theta)V = 0.$

Here r and θ are to be taken at 10, and the coefficient L_m is first computed at $u_m = u_7$, $v_m = v_7$. Since $S_{\theta} = 0$, $L_m = C^{-1}(Du_m + Ev_m)$. The solutions $u^{(1)}$, $v^{(1)}$ of the last two equations may then be improved by computing L_m at $u_m = (u^{(1)} + u_7)/2$, $v_m = (v^{(1)} + v_7)/2$, getting solutions $u^{(2)}$, $v^{(2)}$, and then forming L_m at $u_m = (u^{(2)} + u_7)/2$, $v_m = (v^{(2)} + v_7)/2$, etc., until the desired accuracy has been achieved. Type c: point 5'. By Eq. (17) $M(\theta)/N(\theta) = (r_{4'}/r_{3'})M(\theta_{3'})/N(\theta_{4'})$.

Further, $r = r_{3'}M(\theta)/M(\theta_{3'})$. By Eqs. (18) and (19)

$$(v - v_{3'}) + n(u - u_{3'}) + K_m(\theta - \theta_{3'}) = 0,$$

$$(v - v_{4'}) + m(u - u_{4'}) + L_m(\theta - \theta_{4'}) = 0,$$

with K_m and L_m evaluated at $u_m = (u_{4'} + u_{3'})/2$, $v_m = (v_{4'} + v_{3'})/2$, and $S_{\theta,m} = (S_{\theta,4'} + S_{\theta,3'})/2$. Finally, by Eq. (20) $S - S_{3'} = S_{\theta,m}(\theta - \theta_{3'})(1 - mY/X)$. The value of S_{θ} at 5' is obtained by Eq. (23).

Type d: point 6'. By Eqs. (26) and (13), approximately,

$$\frac{r-r_{4'}}{\theta-\theta_{4'}} = \left(r\frac{q_1}{q_1}\frac{\sin \theta+Y}{\cos \theta-X}\right)_{4'}, \qquad \frac{r-r_{5'}}{\theta-\theta_{5'}} = (rm)_{5'}.$$

If the solutions of these equations are called $r^{(1)}$ and $\theta^{(1)}$, a better approximation may be obtained by forming the right sides of the equations above at $(r^{(1)} + r_{4'})/2$, $(\theta^{(1)} + \theta_{4'})/2$ and $(r^{(1)} + r_{5'})/2$, $(\theta^{(1)} + \theta_{5'})/2$, respectively.

For the determination of u and v at 6' we have by Eqs. (18) and (30)

$$(v - v_{5'}) + n(u - u_{5'}) + K_m(\theta - \theta_{5'}) = 0$$

$$\Gamma u + \Delta v + \Lambda = 0;$$

here we may again use successive approximations, as follows. Using at first in K_m the values of u, v, and S_{θ} at 5' we obtain as solutions of the last two equations $u = u^{(1)}$, $v = v^{(1)}$. Equation (31) will now give $S^{(1)}$, if the value of $dS/d\theta$ is computed by Eq. (33) by means of (27). For the second approximation we use in $K_m u_m = (u^{(1)} + u_{5'})/2$, $v_m = (v^{(1)} + v_{5'})/2$, $S_{\theta,m} = (S_{\theta}^{(1)} + S_{\theta,5'})/2$, obtain $u^{(2)}$, $v^{(2)}$, $S_{\theta}^{(2)}$, etc., and continue until the desired accuracy has been achieved.