SURFACE WAVES*

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Introduction. The theory of low gravity waves deals with the oscillations of a fluid under constant pressure when the inertia, viscous and capillary forces of the motion are small compared with gravitational forces. The study derives its interest particularly from problems of the propagation of waves, oscillations of ships in waves, and the design of harbors and breakwaters. The characteristics of gravity waves mentioned above are more or less satisfied in these problems, at any rate, if we consider regions sufficiently far from the breaker zone and the storm area where the waves are generated.

In the present paper we are concerned with two dimensional motion, that is, a motion where the crests of the waves are straight and parallel. Such motion may be expected to occur in regions whose distance from the storm area is large compared with the diameter of the storm area and with the diameter of the region considered.

In the text books, e.g. H. Lamb, *Hydrodynamics*, only wave motions in a canal of uniform depth are studied, and it is further supposed that the pressure is constant over the whole upper surface of the fluid. Now, in fact, not only do waves travel over uneven bottoms, but in some of the problems mentioned above we are interested just in what happens when there are obstacles in the bottom or on the surface, e.g. in the

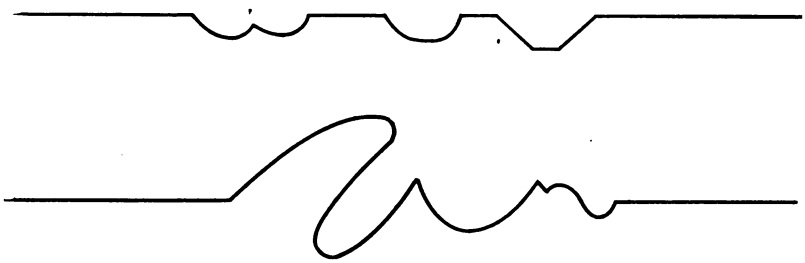

FIG. 1.

reflection[1] set up by reefs, moored ships, or breakwaters. Recently, reflection of waves has been studied by Dean, Lewy, Stoker, Ursell and others[2]. The work concentrated on motion in water very deep or very shallow compared with the wave length. These conditions are appropriate for studying ships in waves or the behavior of waves very near the coast. But while such investigations may give important information on how the reflection depends on the cross sectional shape of the obstacles (in deep or shallow water), naturally they say nothing about the effect of depth, and such information is

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[1]Another practically important problem is to determine the effect of the waves on the obstacle, the mean pressures set up by the waves. This does not seem to come out of the linear theory or simple modifications of it.

[2]Cf. e.g. J. J. Stoker, *Surface waves in water of variable depth*, Quart. Appl. Math. **5**, 1-54, 1947, F. Ursell, *The effect of a fixed vertical barrier on surface waves in deep water*, Proc. Camb. Phil. Soc. **43**, 3 1947; and references given there

necessary to decide, for example, where breakwaters are most effective (for waves of given period). We therefore try to marshal the reflection of obstacles in *arbitrary* depth, that is, we want to determine the reflected and transmitted waves system for a wave train coming up against (cylindrical) obstacles which lie in the bottom of a canal or are fixed in the surface.

Throughout the paper we consider simply harmonic oscillations; for, from such work the reflection of arbitrary incoming motion can be calculated, provided the reflection of incoming waves of short wave lengths is sufficiently small;[3] and if the reflection varies *slowly* with the wave length, the reflected wave at a point P at one time is determined by the incoming motion at P during a *short* interval. Among our results there will be only one where the reflection is sensitive to the wave length, that is, it is not well-defined by the incoming waves (if viscous and inertia forces are neglected).

Also we suppose that the depth of the canal is constant sufficiently far to either side of the obstacles.

Summary. In Sec. 1 we consider the *asymptotic behavior of wave motions and the definition of the reflection coefficient.* It was implicitly assumed above (where we spoke of incoming and reflected wave systems) that the wave motion is asymptotically a superposition of simple wave trains. More precisely we assume: if $\phi e^{i\sigma t}$ is the potential of a wave motion of period $2\pi/\sigma$ defined in the domain of the fluid, whose normal derivative vanishes on the bottom and on the cylinders, and whose pressure is nearly constant on the free surface ($\sigma^2\phi - g\ \partial\phi/\partial y = 0$ on the mean free surface, where y is measured along the vertical), then ϕ is asymptotically of the form $(ae^{ikx} + be^{-ikx}) \cosh k(y + h)$ (at the right hand infinity, say) where $\sigma^2 = gk \tanh kh$, and h is the right hand asymptotic depth of water. Further, in asking for *the* reflection we assume that a unique transmitted and reflected wave is consistent with a prescribed incoming wave. These assumptions are correct if and only if ϕ is restricted to be bounded.

We show in Th. I, using an expansion theorem of A. Weinstein, that at a few depths from the obstacle the potential is very nearly a superposition of simple wave trains, the error falling off exponentially with distance. Further, if for a given domain of fluid there is a potential which is not asymptotically zero, a right hand and a left hand reflection coefficient can be defined; they are unique, and equal to one another. (We calculate generally the left hand reflection coefficient.)

It can be shown that these results hold even if the depth is not constant, but only nearly so (differs exponentially little from a constant).

Section 2 provides a *general reduction of the problem.* To calculate the reflection coefficient for given obstacles a mixed boundary value problem for a potential in the domain of the fluid has to be solved and its asymptotic form determined. Since the solution of linear boundary value problems in a rectangular strip is relatively simple we transform the domain of the fluid into a strip whose width is equal to the asymptotic depth which from now on is assumed to be the same at either infinity. In Lemma I the problem is reduced to a linear integral equation for the potential on one boundary of the strip, and in Lemma II a solution by iteration is given (which converges for suitable obstacles).

In Sec. 3 we apply the results of Sec. 2 to determine the reflection from *obstacles in the bottom* where the surface remains free. We show in Th. I that if $z(\zeta)$ is a conformal

[3]For a discussion of the conditions see a forthcoming paper, *Some remarks on integral equations with kernels* $L(x_1 - \xi_1, \ldots, x_n - \xi_n)$, particularly Sec. 4.

transformation of the strip $0 > \eta > -h$ into the domain of the fluid where the infinities correspond and

$$\alpha = \frac{2k}{1 + 2kh/\sinh 2kh} \int_{-\infty}^{\infty} \left| \frac{dz}{d\zeta} - 1 \right|_{\eta=0} d\zeta + \max_{\eta=0} \left| \frac{dz}{d\zeta} - 1 \right| \frac{1 - 2kh/\sinh 2kh}{1 + 2kh/\sinh 2kh} < 1,$$

ϕ can be computed by iteration, and the reflection coefficient R is given by

$$\frac{R_0 - \alpha^2/2(1-\alpha)}{\{1 + [R_0 - \alpha^2/2(1-\alpha)]^2\}^{1/2}} < R < \frac{R_0 + \alpha^2/2(1-\alpha)}{\{1 + [R_0 + \alpha^2/2(1-\alpha)]^2\}^{1/2}}$$

where

$$R_0 = \frac{k}{1 + 2kh/\sinh 2kh} \left| \int_{-\infty}^{\infty} \left(\frac{dz}{d\zeta} - 1\right)_{\eta=0} \exp(2ik\zeta)\, d\zeta \right|.$$

In general, the α of the obstacle in which we are interested will not be easily calculated. It is therefore desirable to show how α varies with the shape of the obstacle. This is done in two (simple) general theorems.

Theorem II: If the domains D_1, D_2 both lie in the strip $0 > y > -h$, and D_1 is included in D_2, we have

$$\alpha_1 \geq \alpha_2 .$$

FIG. 2.

Theorem III: If D_1, D_2 are the intersection and join of a striplike domain D with the strip $0 > y > -h$, we have

$$\alpha \leq \alpha_1 + \alpha_2 .$$

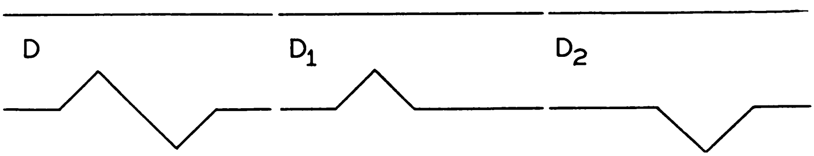

FIG. 3.

In Sec. 4 *obstacles in the surface* are considered. If the draught of an obstacle in the surface is small compared with the depth and the wave length, and the beam is moderately short compared with the wave length, the reflection from the obstacle tends to the reflection by a "plate" of the same beam in the surface, and

$$\frac{2k[(\sin 2k)/2k - \alpha^2/(1-\alpha)]/(1 + 2kh/\sinh 2kh)}{\{1 + 4k^2[(\sin 2k)/2k - \alpha^2/(1-\alpha)]^2/(1 + 2kh/\sinh 2kh)^2\}^{1/2}}$$

$$< R < \frac{2k[(\sin 2k)/2k + \alpha^2/(1-\alpha)]/(1 + 2kh/\sinh 2kh)}{\{1 + 4k^2[(\sin 2k)/2k + \alpha^2/(1-\alpha)]^2/(1 + 2kh/\sinh 2kh)^2\}^{1/2}}$$

where

$$\alpha < 4 \sin^2 k/(1 + 2kh/\sinh 2kh) + (1 - 2kh/\sinh 2kh)/(1 + 2kh/\sinh 2kh) < 1$$

or

$$\alpha < 4 \sin^2 k/(1 + 2kh/\sinh 2kh) + 2[2^{-1/2} \exp(-2^{1/2}kh) + 1/kh + 2\pi^{-1/2}k^{1/2}] < 1$$

and $k = \pi$ *beam/wave length.* The former bound for α is useful in shallow water, the latter in deep water.

Discussion. Throughout, explicit bounds for the error in our calculations are given, and the iterative method is expected to be of use in numerical work, particularly in the problem of Sec. 4 where we get an integral equation over a finite range. The bounds are good in shallow water since then the term $(1 - 2kh/\sinh 2kh)/(1 + 2kh/\sinh 2kh)$ in α is small. But it seems worthwhile to collect some results which are easily obtained from the algebra without further calculations.

The reflection coefficient for obstacles in the bottom takes a particularly simple form if the wave length is large compared with the depth and with the dimensions of the obstacle. Then

$$R_0 = \frac{k}{1 + 2kh/\sinh 2kh} \left| \int_{-\infty}^{\infty} \left(\frac{dz}{d\zeta} - 1\right)_{\eta=0} \exp(2ik\zeta)\, d\zeta \right|,$$

and if the waves are long, $2k\zeta$ is nearly constant over the range where $(dz/d\zeta - 1)_{\eta=0}$ is noticeable, so that

$$R_0 \sim \tfrac{1}{2}k \left| \int_{-\infty}^{\infty} \left(\frac{dz}{d\zeta} - 1\right) d\zeta \right|,$$

i.e., the reflection coefficient is $k/2$ *times the contraction constant of the transformation which maps the strip* $0 > \eta > -h$ *into the domain of the fluid.*

We compare the reflection by a vertical barrier and a long low reef.

In Sec. 3, we determine the transformation function $z(\zeta)$ for a strip of width h with a cut of length r in the bottom (a barrier of height r). Then α becomes

$$\frac{8kh}{\pi(1 + 2kh/\sinh 2kh)} \log \sec \frac{r\pi}{2h} + 2 \sin^2 \frac{r\pi}{4h} \frac{1 - 2kh/\sinh 2kh}{1 + 2kh/\sinh 2kh},$$

and if $kh \to 0$ we have

$$R_0 \to \frac{2}{\pi} kh \log \sec \frac{r\pi}{2h}$$

To get the reflection by a low gently sloping reef of the form $y = -h + \epsilon f(x)$ we use an approximate expression for $z(\zeta)$ in terms of $f(x)$ to show that

$$R_0 \sim \frac{\epsilon}{h} \frac{2kh}{\sinh 2kh(1 + 2kh/\sinh 2kh)} k \left| \int_{-\infty}^{\infty} f(x) \exp(2ikx)\, dx \right|.$$

For a horizontal reef of width a and height ϵ

$$R_0 \sim \frac{\epsilon}{h} \frac{2kh \mid \sin 2ka \mid}{\sinh 2kh(1 + 2kh/\sinh 2kh)}.$$

(No details are given in the paper.)

If the width of the reef is large compared with the wave length, R_0 varies rapidly with the wave length. To calculate the reflection at a point P at one time we have to know the incoming motion at P over an interval of order $a(gh)^{-1/2}$, for fixed ϵ/h.

In practical problems it is often important to understand how the reflection of a *given* obstacle in the incoming waves varies when it is placed in different depths. The period of the waves remains constant so that it is necessary to plot the reflection against the depth. Since however kh is a monotone function of h for any σ^2, we get the general shape of the curve by plotting the reflection against kh.

We observe that the reflection by a horizontal reef of width a decreases rapidly as the depth increases, provided $ka < \pi/2$.

$$\left(k = \frac{\sigma^2}{g \tanh kh} \quad \text{and} \quad R_0 \sim \frac{\epsilon\sigma^2}{g} \frac{\sin(2\sigma^2 a/g \tanh kh)}{\sinh^2 kh(1 + 2kh/\sinh 2kh)}\right).$$

Lastly consider the variation of the reflection of waves of given period by a plate of given beam in the surface of water,

$$R \sim \frac{k \times \text{beam}}{1 + 2kh/\sinh 2kh}, \qquad \text{i.e.,} \qquad R \sim \frac{\sigma^2 \times \text{beam}}{g(\tanh kh + kh \operatorname{sech}^2 kh)}.$$

We find that the reflection first decreases and then increases as we go from shallow into deep water (or vice versa):

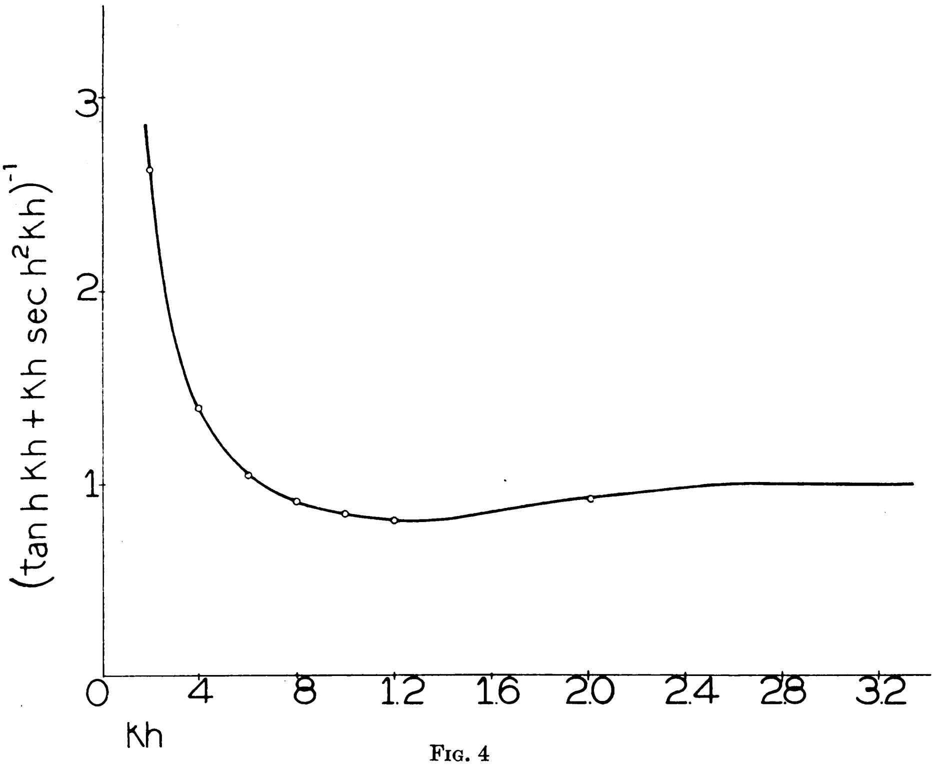

FIG. 4

But more tedious algebra would be needed to give information on the effect of draught in different depths of water (Th. II of Sec. 4). Also, what happens when the dimensions of the obstacle are nearly equal to the wave length is undecided by the present work.

1. Asymptotic behavior and definition of the reflection coefficient. To discuss two dimensional wave motion in a domain sketched in Fig. 1[4] we choose a system of coordinates so that the z-axis is parallel to the axes of the cylinders, the mean free surface lies in $y = 0$, and y is measured vertically upwards.

By the theory of small oscillations the potential $\phi(x, y)e^{i\sigma t}$ must satisfy the following conditions:

(A) ϕ is bounded and harmonic in the domain of the fluid,
$\partial\phi/\partial n = 0$ on the lower boundary and on the fixed cylinders, where $\partial/\partial n$ is the derivative along the normal to the boundary,
$\partial\phi/\partial y$ is bounded in the neighborhood of the mean free surface,
$\sigma^2\phi - g\,\partial\phi/\partial y = 0$ on the mean free surface, where g denotes the acceleration due to gravity.

In analyzing possible motions satisfying (A) it is convenient to use the following lemmas:

Lemma I (A. Weinstein, C. R., 1927). The functions

$$\{\cosh k(y + h), \cos k_n(y + h)\}$$

are a complete set of orthogonal functions which satisfy

$$\sigma^2 f_n - g\, df_n/dy = 0 \quad \text{for } y = 0,$$
$$df_n/dy = 0 \quad \text{for } y = -h,$$
$$f_n \in L(-h, 0)$$

if $\sigma^2 = gk \tanh kh$, $\sigma^2 = -gk_n \tan k_n h$, $(n - 1/2)\pi < k_n h < n\pi$.

Lemma II (ibid.). A harmonic function $\phi(x, y)$, bounded in the strip $0 > y > -h$, which satisfies

$$\sigma^2\phi - g\,\partial\phi/\partial y = 0 \text{ for } y = 0$$

$\partial\phi/\partial y$ bounded in the strip and zero on $y = -h$,

is of the form

$$(ae^{ikx} + be^{-ikx}) \cosh k(y + h).$$

Lemma III. A harmonic function $\phi(x, y)$, bounded in the half strip $0 > y > -h$, $x < X_0$, which satisfies

$\partial\phi/\partial y$ bounded in the half strip and zero on $y = -h$, $r < X_0$,
$\sigma^2\phi - g\,\partial\phi/\partial y$ prescribed on $y = 0$, $x < X_0$,
ϕ prescribed on $x = X_0$, $0 > y > -h$,

is defined uniquely except for a term

$$a \sin k(x - X_0) \cosh k(y + h).$$

[4]The (striplike) domains considered below are (1) bounded by $y = 0$ for $|x| > X_0$; $y = -h$ for $x < -X_0$; $y = -h'$ for $x > X_0$; a Jordan arc joining $(\pm X_0, 0)$; a Jordan arc joining $(-X_0, -h)$ to $(X_0, -h')$; (2) the boundary is a simple curve.

For, the difference Φ between two expressions satisfying these conditions is bounded in the half strip; also

$$\begin{array}{ll} \partial\Phi/\partial y & \text{is bounded and zero on } y = -h,\ x < X_0\,, \\ \sigma^2\Phi - g\,\partial\Phi/\partial y = 0 & \text{on } y = 0,\ x < X_0\,, \\ \Phi = 0 \text{ on} & x = X_0\,,\ 0 > y > -h. \end{array}$$

Since Φ is bounded[5] near the line $x = X_0$, $0 > y > -h$, and zero on it, Φ can be continued across it, and $\Phi(X_0 + x, y) = -\Phi(X_0 - x, y)$. Thus Φ satisfies the conditions of Lemma II in the whole strip $0 > y > -h$, and is zero on $x = X_0$. Hence the lemma. The result also holds if $X_0 = \infty$.

Theorem I. If a potential satisfies (A), then over the flat portion at either infinity it is of the form

$$(ae^{ikx} + be^{-ikx}) \cosh k(y + h) + \sum_1^\infty a_n \exp(-k_n \mid x \mid) \cos k_n(y + h), \tag{1.1}$$

where h is the asymptotic depth at the infinity considered. Also

$$\sum_1^\infty \mid a_n \mid \exp(-k_n \mid x \mid) = O(\exp\{-\pi \mid x \mid/2h\}).$$

Consider the flat portion at the left hand infinity.

Along any vertical $x = -X_0$, $0 > y > -h$, by Lemma I, ϕ can be expanded in the form

$$\phi(-X_0\,, y) = \alpha \cosh k(y + h) + \sum_1^\infty \alpha_n \cos k_n(y + h), \tag{1.2}$$

$$h(1 + \sinh k_n h/2k_n h)\alpha_n = \int_{-h}^0 \phi(-X_0\,, y) \cos k_n(y + h)\, dy. \tag{1.3}$$

By analytic continuation across $y = 0$ and $y = -h$, ϕ is analytic on $x = -X_0$, $0 \geq y \geq -h$, so that repeated integration by parts of

$$\int_{-h}^0 \phi(-X_0\,, y) \cos k_n(y + h)\, dy$$

is allowed, and $\alpha_n = O(n^{-4})$. Thus (1.2) is uniformly convergent, and the series

$$(a_0e^{ikx} + b_0e^{-ikx}) \cosh k(y + h) + \sum_1^\infty \alpha_n e^{k_n x} e^{k_n X_0} \cos k_n(y + h),$$

with

$$a_0e^{ikX_0} + b_0e^{-ikX_0} = \alpha \tag{1.4}$$

is bounded and harmonic in $x < -X_0$, $0 > y > -h$, and equal to (1.2) on $x = -X_0$. By Lemma III, (1.4) is the potential in the half strip $0 \geq y \geq -h$, $x < -X_0$. Similarly for the right hand infinity.

Since $(n - 1/2)\pi < k_n h < n\pi$, the infinite series is less than

$$\exp(\pi \mid x \mid/2h) \sum_1^\infty \mid \alpha_n \mid \exp(-\pi \mid x + X_0 \mid/h).$$

[5]The symmetry principle for harmonic functions does not require the function to be continuous near the arc across which it is continued, but it is sufficient that it is bounded.

If $|\phi(-X_0, y)|_{0>y>-h} < M$, $|\alpha_n| < 2M/(1 - 1/\pi)$ by (1.3), and the series is less than $2M(1 - 1/\pi)^{-1} \exp[-1/2\, \pi(|x| - X_0)/h]\{1 - \exp[-\pi(|x| - X_0)/h]\}^{-1}$.

Theorem II. Suppose at the two infinities the asymptotic forms of a potential satisfying (A) are

$$(ae^{ikx} + be^{-ikx}) \cosh k(y + h), \qquad (a'e^{ik'x} + b'e^{-ik'x}) \cosh k'(y + h').$$

Then

$$(|a|^2 - |b|^2)kh(1 + \sinh 2kh/2kh) = (|a'|^2 - |b'|^2)k'h'(1 + \sinh 2k'h'/2k'h').$$

This is the principle of the constancy of transmission of energy.

Let X_0 be so large that the bottom is flat for $|x| \geq X_0$. If $\bar{\phi}$ is the complex conjugate of ϕ, $\bar{\phi}$ also satisfies (A). Consider

$$\int_C (\phi \nabla \bar{\phi} - \bar{\phi} \nabla \phi) \cdot d\mathbf{S},$$

where C is the contour of the domain of fluid between $x = \pm X_0$, and $d\mathbf{S}$ is a surface element of the boundary. Since ϕ and $\bar{\phi}$ are harmonic and bounded in the interior of C, the integral is zero by Green's theorem. The integrand is zero on the lower boundary and on the fixed cylinders since $\nabla\phi \cdot d\mathbf{S} = 0$, and on the mean free surface since

$$\nabla\phi \cdot d\mathbf{S} = \frac{\partial \phi}{\partial y}\, dx = \frac{\sigma^2}{g}\, \phi\, dx.$$

Therefore the integrals along the vertical portions $x = X_0$, $0 > y > -h$, and $x = -X_0$, $0 > y > -h'$ are equal and opposite. By the orthogonality of the expansion of Lemma I only the asymptotic waves contribute to the integral, and

$$\int_{X_0 - ih}^{X_0} \left(\phi \frac{\partial \bar{\phi}}{\partial x} - \bar{\phi} \frac{\partial \phi}{\partial x}\right) dy = 2ik(|b|^2 - |a|^2) \int_{-h}^{0} [\cosh k(y + h)]^2\, dy.$$

Hence the theorem.

Notation: $\{a, b; a', b'\}$ denotes a potential whose asymptotic form on $y = 0$ is $(ae^{ikx} + bc^{-ikx})$ as $x \to -\infty$, and $(a'e^{ik'x} + b'e^{-ik'x})$ as $x \to +\infty$.

Theorem III. (1) If $b = 0$ and $a \neq 0$, then a'/a, b'/a are unique.

(2) If $b = 0$ and $a \neq 0$, there is also a solution with $b' = 0$, $a' \neq 0$.

To prove (1):

Since the problem is linear we may take $a = 1$ without loss of generality. Suppose $\{1, 0; a', b'\}(=\phi)$ and $\{1, 0; a_1', b_1'\}(=\phi_1)$ are two solutions. By linear superposition $a'\phi_1 - a_1'\phi$, i.e. $\{a' - a_1, 0; 0; b_1'a' - b'a_1'\}$ is also a solution. By Th. II

$$|a' - a_1'|^2 kh(1 + \sinh 2kh/2kh)(\cosh k'h')^2$$
$$= -|b_1'a' - b'a_1'|^2 k'h'(1 + \sinh 2k'h'/2k'h')(\cosh kh)^2$$

so that $a' = a_1'$, and, since $a' \neq 0$, $b_1' = b'$.

To prove (2):

Recall that if $\phi(=\{1, 0; a', b'\})$ is a solution of (A), so is $\bar{\phi}(=\{0, 1; \bar{b}', \bar{a}'\})$ By Th. II the determinant

$$\begin{vmatrix} a' & b' \\ \bar{b}' & \bar{a}' \end{vmatrix} = |a'|^2 - |b'|^2$$

is not zero, so that $\bar{a}'\phi - b'\bar{\phi}(=\{\bar{a}', -b'; c, 0\})$ has a non-zero c. (If there is a progressive potential at $-\infty$, there is also one at $+\infty$).

Definition. We define the *right hand reflection coefficient* (R_R) to be the ratio of the amplitudes $|b'|/|a'|$, when in the potential at $-\infty$ $b = 0$, $a \neq 0$ (the potential at $-\infty$ is progressive). By Th. III(1) this ratio is unique.

We define a *left hand reflection coefficient* (R_L) in an analogous manner. By Th. III(2) R_L exists if, and only if, R_R exists. Now R_L and R_R are equal since $R_L = |-b'|/|\bar{a}'|$. Note that this is true generally, i.e. also if the asymptotic depths are different.

2. Two lemmas in the theory of surface waves. In the general reduction of wave problems to an (one dimensional) integral equation we need

Lemma I. (a) If ϕ is bounded and harmonic in $0 > \eta > -h$, $-\infty < \xi < \infty$,

$$\partial\phi/\partial\eta = 0 \quad \text{on} \quad \eta = -h,$$
$$\phi \to 0 \quad \text{as} \quad \xi \to +\infty,$$

$\sigma^2\phi - g\,\partial\phi/\partial\eta$ is L-integrable on $\eta = 0$, and $O(e^{-c|\xi|})$, for some $c > 0$,

$$f(x, y) = \frac{1}{2\pi}\int_{-\infty+i\rho}^{\infty+i\rho} \frac{\cosh k(y+h)}{\cosh kh}\,\frac{e^{ikx}\,dk}{\sigma^2 - gk\tanh kh},$$
$$0 < \rho < \min(\pi/2h, c),$$

then

$$\phi(\xi, \eta) = \int_{-\infty}^{\infty}\left[\sigma^2\phi(\xi', 0) - g\frac{\partial\phi}{\partial\eta'}(\xi', 0)\right] f(\xi - \xi', \eta)\,d\xi'.$$

(If $\phi \to 0$ as $\xi \to -\infty$, ρ must lie between 0 and $-\pi/2h$ in the definition of f).

(b) Define $f(x, 0) = f(x)$. If $x > 0$, $-f(x) \geq 0$ and

$$\int_0^{\infty} |f(x)|\,dx = \frac{1}{2\sigma^2}\,\frac{1 - 2kh/\sinh 2kh}{1 + 2kh/\sinh 2kh},$$

$$f(x)\,| < \frac{4\exp(\pi x/2h)}{\pi g}\log(1 - \exp\{-\pi x/h\}).$$

$$\text{If } x < 0, \qquad f(x) = f(-x) + i\,\frac{e^{ikx} - e^{-ikx}}{g\tanh kh(1 + 2kh/\sinh 2kh)}.$$

(c) The asymptotic form of $\phi(\xi, 0)$ as $\xi \to -\infty$ is

$$\frac{i}{g\tanh kh(1 + 2kh/\sinh 2kh)}\left\{e^{ik\xi}\int_{-\infty}^{\infty}\left[\sigma^2\phi(\xi', 0)\right.\right.$$

$$\left. - g\frac{\partial\phi}{\partial\eta'}(\xi', 0)\right]e^{-ik\xi'}\,d\xi' - e^{ik\xi}\int_{-\infty}^{\infty}\left[\sigma^2\phi(\xi', 0) - g\frac{\partial\phi}{\partial\eta'}(\xi', 0)\right]$$

$$\left.\cdot e^{ik\xi'}\,d\xi'\right\}.$$

(a) Observe that by Lemma III, Sec. 1, there is at most one potential satisfying conditions (a). We construct its Fourier integral expression.

Since $\sigma^2\phi - g\ \partial\phi/\partial\eta$ is L_1 and $O(e^{-c|\xi|})$ for large $|\xi|$, its Fourier transform $G(k)$ exists in $c - \epsilon \geq \Im(k) \geq -c + \epsilon$, is bounded there, and

$$\phi(\xi, \eta) = (2\pi)^{-1/2} \int_{-\infty+i\rho}^{\infty+i\rho} \frac{G(k)}{\sigma^2 - gk \tanh kh} \frac{\cosh k(\eta + h)}{\cosh kh} e^{ik\xi}\, dk$$

$$c > \rho > 0, \qquad \pi/2h > \rho,$$

satisfies (a).

We convert the Fourier integral into a linear transform of

$$\sigma^2\phi(\xi', 0) - g \frac{\partial\phi}{\partial\eta'}(\xi', 0).$$

We apply the Faltung theorem to

$$G(k) \cosh k(\eta + h)[\cosh kh(\sigma^2 - gk \tanh kh]^{-1}$$

which is permissible since $G(k)$ is bounded on $\Im(k) = \rho$ and the transform is L_2 in $0 \geq \eta \geq -h$.

(b) Consider $x > 0$.

(i) By the theory of residues

$$f(x) = -\sum_1^\infty \frac{e^{-k_r x}}{g \tan k_r h(1 + 2k_r h/\sin 2k_r h)}$$

where ik_r are the imaginary roots of $\sigma^2 = gk \tanh kh$, and $(r - 1/2)\pi < k_r h < r\pi$ (cf. Lemma I of Sec. 1). Now

$$g \tan k_r h(1 + 2k_r h/\sin 2k_r h) = (g/k_r h)[k_r^2 h^2 + (\sigma^2 h/g)(\sigma^2 h/g - 1)]$$
$$> (g/k_r h)[(r - 1/2)^2\pi^2 - 1/4]$$

since $(\sigma^2 h/g)(\sigma^2 h/g - 1)$ attain its minimum at $\sigma^2 h/g = 1/2$.

Thus all the terms of the series are positive and $f(x) < 0$.

Since, further, $k_r h > 1$,

$$g[k_r^2 h + (\sigma^2/g)(\sigma^2 h/g - 1)]/k_r > \frac{1}{4} g\pi(2r - 1) > \frac{1}{4} g\pi r, \qquad \text{and}$$

$$|f(x)| < \frac{4}{\pi g} \sum_{r=1}^\infty \exp\{-(2r - 1)\pi x/2h\}/r = \frac{4}{\pi g} \exp\{\pi x/2h\} \log(1 - \exp\{-\pi x/h\}).$$

(ii) Observe that

$$\int_0^\infty |f(x)|\, dx = -\int_0^\infty f(x)\, dx = \int_0^\infty dx\left(\frac{1}{2\pi} \int_{-\infty+i\rho}^{\infty+i\rho} \frac{e^{ikx}}{\sigma^2 - gk \tanh kh}\, dk\right)$$

$$= -\frac{1}{2\pi i} \int_{-\infty+i\rho}^{\infty+i\rho} \frac{dk}{k(\sigma^2 - gk \tanh kh)}$$

where we may invert the order of integration since $|e^{ikx}| = e^{-\rho x}$.

We evaluate the integral by contour integration round a large upper and lower semi-circle with centre $i\rho$. The only singularities in the upper semi-circle are at ik_n and the integral is equal to

$$-\, g^{-1} \sum_{1}^{\infty} [k_r \tan k_r h(1 + 2k_r h/\sin 2k_r h)]^{-1}.$$

The singularities in the lower semi-circle are at 0, $\pm k$ and $-ik_n$, and the integration is clockwise. We get

$$-\left\{-\sigma^{-2} + 2[gk \tanh kh(1 + 2kh/\sinh 2kh)]^{-1}\right.$$

$$\left. - \sum_{1}^{\infty} [gk_r \tan k_r h(1 + 2k_r h/\sin 2k_r h)]^{-1}\right\}.$$

Equating the two we find

$$-\, g^{-1} \sum_{1}^{\infty} [k_r \tan k_r h(1 + 2k_r h/\sin 2k_r h)]^{-1} = -\frac{1}{2\sigma^2}\frac{1 - 2kh/\sinh 2kh}{1 + 2kh/\sinh 2kh}.$$

(iii) Consider $x < 0$. Integrating clockwise round the lower semi-circle we find

$$f(x) = -\sum_{1}^{\infty} \frac{e^{-k_r|x|}}{g \tan k_r h(1 + 2k_r h/\sin 2k_r h)} + i\,\frac{e^{ikx} - e^{-ikx}}{g \tanh kh(1 + 2kh/\sinh 2kh)}.$$

(c) follows immediately from the fact that in the integral

$$\int_{-\infty}^{\infty} \left(\sigma^2\phi - g\,\frac{\partial\phi}{\partial\eta'}\right) f(\xi - \xi', 0)\, d\xi'$$

the contribution from the infinite series part of $f(\xi - \xi')$ is small when $|\xi|$ is large: if $|\xi - \xi'|$ is small, $\sigma^2\phi - g\,\partial\phi/\partial\eta'$ is small since $|\xi'|$ is large, and $\sigma^2\phi(\xi', 0) - g\,\partial/\partial\eta'[\phi(\xi', 0)]$ is $O(e^{-c|\xi'|})$; if $|\xi - \xi'|$ is large, the infinite series in f is exponentially small by (b). This proves the lemma.

If the asymptotic depths of the domain of fluid in Fig. 1 are equal, and we transform it conformally into the strip $0 > \eta > -h$, the resulting boundary value problem for the potential $\phi(\xi, \eta)c^{i\sigma t}$ is essentially of the following form:

(B) ϕ bounded and harmonic in $0 > \eta > -h$,

$\partial\phi/\partial\eta = 0$ on $\eta = -h$

$\sigma^2\phi - g\,\partial\phi/\partial\eta = g(\xi)\phi + h(\xi)$ on $\eta = 0$

$\phi \to 0$ as $\xi \to +\infty$, $0 > \eta > -h$.

By the function $f(x, y)$ of Lemma I the two dimensional problem (B) is reduced to the integral equation

$$\phi(\xi, 0) = \int_{-\infty}^{\infty} g(\xi')\phi(\xi', 0) f(\xi - \xi')\, d\xi' + \int_{-\infty}^{\infty} h(\xi') f(\xi - \xi')\, d\xi'.$$

This equation is solved for suitable $g(\xi)$ (suitable obstacles) and integrable $h(\xi)$ by *Lemma II*. If $g(\xi) = O(e^{-c|\xi|})$ and

$$\alpha = [g \tanh kh(1 + 2kh/\sinh 2kh)]^{-1} \max_{\xi} \int_{-\infty}^{\xi} | g(\xi') | | e^{ik(\xi-\xi')} - e^{-ik(\xi-\xi')} | d\xi'$$

$$+ \frac{\max | g(\xi') |}{\sigma^2} \left(\frac{1 - 2kh/\sinh 2kh}{1 + 2kh/\sinh 2kh}\right) < 1$$

the integral equation above can be solved for all ξ by iteration and

$$| \phi | < \frac{\max_{\xi} \left| \int_{-\infty}^{\infty} h(\xi')f(\xi - \xi') \, d\xi' \right|}{1 - \alpha}.$$

Also its solution is unique.
Proof. Write

$$\phi_0 = \int_{-\infty}^{\infty} h(\xi')f(\xi - \xi') \, d\xi', \qquad \phi_{n+1} = \int_{-\infty}^{\infty} \phi_n(\xi', 0)g(\xi')f(\xi - \xi') \, d\xi'.$$

It is readily seen that

$$| \phi_n(\xi, 0) | < \alpha^n \max_{\xi} \left| \int_{-\infty}^{\infty} h(\xi')f(\xi - \xi') \, d\xi' \right|.$$

By dominated convergence it follows that $\sum_0^{\infty} \phi_n(\xi, 0)$ converges to a solution of the integral equation. Further, the solution is unique if ϕ is to be bounded, for the difference Φ between any two solutions satisfies

$$\Phi = \int_{-\infty}^{\infty} \Phi(\xi', 0)g(\xi')f(\xi - \xi') \, d\xi',$$

and therefore

$$| \Phi | \leq \max | \Phi | \int_{-\infty}^{\infty} | g(\xi')f(\xi - \xi') | \, d\xi' \leq \alpha \max | \Phi |.$$

If $\alpha < 1$ this means that $\Phi \equiv 0$.

Thus, if $\alpha < 1$, not only the asymptotic potentials but also the potential in the whole domain of the fluid is defined by the asymptotic potential at one infinity.

3. Reflection of waves by submerged obstacles. In the present section we apply the results of Sec. 2 to calculate how the reflection coefficient depends on the cross sectional shape of the submerged obstacle.

By the usual theory, a potential $\phi e^{-i\sigma t}$ which is progressive at $+\infty$ must satisfy the following conditions:

(C) ϕ bounded and harmonic in the domain of the fluid,

$\sigma^2\phi - g\,\partial\phi/\partial y = 0$ on $y = 0$,

$\partial\phi/\partial n = 0$ on the lower boundary,

$\phi \to ae^{ikx} \cosh k(y + h)/\cosh kh$ as $x \to +\infty$.

We map the strip $0 > \eta > -h$ on the domain of the fluid (so that the infinities correspond) by the conformal transformation $z(\zeta)(\zeta = \xi + i\eta)$ which is unique except for a linear shift in ξ. The boundary conditions then become

(C′) ϕ bounded and harmonic in $0 > \eta > -h$,

$\sigma^2\phi - g\,\partial\phi/\partial\eta + \sigma^2(dz/d\zeta - 1)\phi = 0$ on $\eta = 0$,

$\partial\phi/\partial\eta = 0$ on $\eta = -h$,

$\phi \to ae^{ik\xi}\cosh k(\eta + h)/\cosh kh$ as $\xi \to +\infty$.

Theorem I. There is a unique ϕ satisfying C' provided (e.g.)

$$\alpha = \frac{2k}{1 + 2kh/\sinh 2kh}\int_{-\infty}^{\infty}\left|\frac{dz}{d\zeta} - 1\right|_{\eta=0} d\zeta + \max_{\eta=0}\left|\frac{dz}{d\zeta} - 1\right|\frac{1 - 2kh/\sinh 2kh}{1 + 2kh/\sinh 2kh} < 1. \tag{3.1}$$

ϕ can then be computed by iteration, and the reflection coefficient R is given by

$$\frac{R_0 - \alpha^2/2(1-\alpha)}{\{1 + [R_0 - \alpha^2/2(1-\alpha)]^2\}^{1/2}} < R < \frac{R_0 + \alpha^2/2(1-\alpha)}{\{1 + [R_0 + \alpha^2/2(1-\alpha)]^2\}^{1/2}},$$

where

$$R_0 = \frac{k}{1 + 2kh/\sinh 2kh}\left|\int_{-\infty}^{\infty}\left(\frac{dz}{d\zeta} - 1\right)e^{2ik\zeta}\,d\zeta\right|.$$

Proof. Write

$$\phi = \phi_1 + ae^{ik\xi}\cosh k(\eta + h)/\cosh kh.$$

Then the conditions for ϕ_1 are of the form (B) of Sec. 2, if $-\sigma^2(dz/d\zeta - 1)$ is substituted for $g(\xi)$, and $-\sigma^2(dz/d\zeta - 1)ae^{ik\xi}$ for $h(\xi)$. By Lemma II of Sec. 2 ϕ_1 can be calculated by iteration if
$|dz/d\zeta - 1| = O(e^{-c|\xi|})$, and the α of Lemma II, Sec. 2 is less than 1.
These conditions are checked in (i) and (ii).
To get bounds for the reflection coefficient we get in (iii) bounds in terms of α for the reflected wave.

(i) $$\left|\frac{dz}{d\zeta} - 1\right| = O(\exp\{-\pi|\xi|/h\}).$$

Consider the inverse $\zeta(z)$ of $z(\zeta)$, for definiteness in $x < -X_0$. The imaginary part of $\zeta(z)$ is harmonic and bounded in $0 > y > -h$, $x < -X_0$. Also $\Im[\zeta(z) - z] = 0$ on $y = 0$ and $y = -h$ for $x < -X_0$. Thus by the reflection principle ζ can be continued across $y = 0$ and $y = -h$, and is therefore analytic on $0 \geq y \geq -h$, $x = x_1$ if $x_1 < -X_0$. We can therefore expand

$$\Im[\zeta(z) - z] = \sum_1^{\infty} f_n(x)\sin(n\pi y/h)$$

for $x < -X_0$ where

$$f_n(x) = 2/(\pi h)\int_{-h}^{0}[\eta(x, y) - y]\sin(n\pi y/h)\,dy.$$

By repeated integration by parts

$d^2f_n/dx^2 = (n\pi/h)^2 f_n$

and hence

$f_n = a_n e^{n\pi x/h} + b_n e^{-n\pi x/h}.$

Since $\eta(x, y) - y$ is bounded, $b_n = 0$ and $a_n e^{-n\pi X_0/h}$ is bounded. Thus

$$\left| \frac{d\zeta}{dz} - 1 \right| \leq \sum_1^\infty (n\pi/h) a_n \exp \{n\pi x/h\} = O(\exp\{-\pi(|x| - X_0)/h\}), \text{ and } \quad \zeta(z) \sim z.$$

Similarly for $x > +X_0$.

(ii) It is readily verified that (3.1) is a bound for α of Sec. 2.

(iii) By Lemma I of Sec. 2 the reflected wave is asymptotically

$$e^{-ik\xi}[g \tanh kh(1 + 2kh/\sinh 2kh)]^{-1} \int_{-\infty}^{\infty} \left(\sigma^2\phi - g\frac{\partial \phi}{\partial \eta'}\right) e^{ik\xi'}\, d\xi'$$

$$= e^{-ik\xi}\sigma^2[g \tanh kh(1 + 2kh/\sinh 2kh)]^{-1} \int_{-\infty}^{\infty} \phi\left(\frac{dz}{d\zeta'} - 1\right) e^{ik\xi'}\, d\xi'.$$

If

$$\phi_1 = \sum_1^\infty \phi_n^{(1)},$$

$| \phi_n^{(1)} | < a\alpha^n$ and $| \phi_1 | < a\alpha/(1 - \alpha)$,

and the amplitude of the reflected wave differs from

$$ak(1 + 2kh/\sinh 2kh)^{-1} \int_{-\infty}^{\infty} \left(\frac{dz}{d\zeta'} - 1\right) e^{2ik\xi'}\, d\xi'$$

by less than

$$ak\alpha(1 - \alpha)^{-1}(1 + 2kh/\sinh 2kh)^{-1} \int_{-\infty}^{\infty} \left| \frac{dz}{d\zeta'} - 1 \right| d\xi'.$$

Since

$$k(1 + 2kh/\sinh 2kh)^{-1} \int_{-\infty}^{\infty} \left| \frac{dz}{d\zeta'} - 1 \right| d\xi' < \frac{1}{2}\alpha,$$

by Th. II of Sec. 1 the incoming wave lies between $a\{1 + [R_0 \pm \alpha^2/2(1 - \alpha)]^2\}^{1/2}$ and

$$\frac{R_0 - \alpha^2/2(1 - \alpha)}{\{1 + [R_0 - \alpha^2/2(1 - \alpha)]^2\}^{1/2}} < R < \frac{R_0 + \alpha^2/2(1 - \alpha)}{\{1 + [R_0 + \alpha^2/2(1 - \alpha)]^2\}^{1/2}}.$$

Example. Consider a vertical barrier of height r in the bottom of a canal of depth h. Let $r\pi/h = \epsilon$. Then

$$\alpha = 8kh[\pi(1 + 2kh/\sinh 2kh)]^{-1} \log \sec (r\pi/2h) + 2 \sec^2 (r\pi/4h) \frac{1 - 2kh/\sinh 2kh}{1 + 2kh/\sinh 2kh}$$

$$= O(r/h)^2,$$

and, as $kh \to 0$

$$R = (2/\pi)kh \log \sec (r\pi/2h) + O(r/h)^4.$$

Proof. Without loss of generality we take $h = \pi$ so that the height of the reef is ϵ.
(i) The transformation function is obtained by Schwarz-Christoffel:

$$\frac{dz}{d\zeta} = \frac{e^{\zeta} + 1}{\{(e^{\zeta} + \rho)(e^{\zeta} + \rho^{-1})\}^{1/2}}.$$

(ii) ρ is determined from the height of the reef:
since $\log \rho - i\pi$ is the image of the foot, $-i\pi$ of the tip of the barrier

$$i\epsilon = \int_{\log \rho - i\pi}^{-i\pi} \frac{dz}{d\zeta}\, d\zeta = \int_{\log \rho}^{0} \frac{1 - e^{\xi}}{\{(e^{\xi} - \rho)(\rho^{-1} - e^{\xi})\}^{1/2}}\, d\xi = i2 \tan^{-1} \frac{1 - \rho}{2\rho^{1/2}}.$$

(iii) Since $0 < dz/d\zeta < 1$, on $\eta = 0$, $|1 - dz/d\zeta|_{\eta=0} = 1 - dz/d\zeta$

so that
$$\int_{-\infty}^{\infty} \left| 1 - \frac{dz}{d\zeta} \right| d\zeta = \int_{-\infty}^{\infty} \left(1 - \frac{dz}{d\zeta}\right) d\zeta = 4 \log \frac{1 + \rho}{2\rho^{1/2}}.$$

(iv) By (ii)
$$4 \log \frac{1 + \rho}{2\rho^{1/2}} = 4 \log \sec \frac{1}{2} \epsilon.$$

(v) It is easily verified that $dz/d\zeta$ attains its minimum at $\zeta = 0$ so that

$$\max_{\eta=0} \left| 1 - \frac{dz}{d\zeta} \right| = 2 \sin^2 \frac{1}{4} \epsilon.$$

Thus we get α, and R as in the discussion of the paper given above.

To compare the α for various domains of fluid we use the following lemma.

Lemma. Suppose $\zeta(z)$ maps a domain D, which is bounded above by $y = 0$ and is contained in the strip $0 > y > -h$, into the strip $0 > \eta > -h$ so that the infinities correspond. Then

$$1 \leq \left(\frac{d\zeta}{dz}\right)_{y=0} = \frac{\partial \xi}{\partial x} < \infty.$$

For, $\Im[\zeta(z) - z]$ is bounded and harmonic in D, zero on $y = 0$ and not positive on the lower boundary of D. Thus $\Im[\zeta(z) - z] \leq 0$ in D.

Since
$$\Im[\zeta(x, 0)] = 0, \qquad \frac{\Im[\zeta(x, -y) - \zeta(x, 0)] + y}{-y} \geq 0,$$

and therefore
$$\lim_{y \to 0} \frac{\Im[\zeta(x_1 - y) - \zeta(x, 0)]}{-y} - 1 \geq 0 \qquad \text{i.e.} \qquad \frac{\partial}{\partial y} \Im[\zeta(z)]_{y=0} \geq 1.$$

By Cauchy's relation $(\partial/\partial y)\Im = (\partial/\partial x)\Re$ so that $\partial\xi/\partial x \geq 1$ on $y = 0$, $dz/d\zeta \neq 0$ since the transformation can be continued across $y = 0$ and is conformal.

Theorem II (cf. Fig. 2). If two domains D and D_1 are both bounded above by $y = 0$ and are contained in the strip $0 > y > -h$, and if $D_1 \subset D$

then
$$\alpha \leq \alpha_1 .$$

Let $\zeta_1(z)$ map D_1 on $0 > \eta > -h$, and without loss of generality we may take $\zeta(0) = \zeta_1(0) = 0$.

It is sufficient to show that

$$\max_{\eta=0}\left|1-\frac{dz}{d\zeta}\right| \le \max_{\eta=0}\left|1-\frac{dz_1}{d\zeta}\right| \qquad \text{and} \qquad \int_{-\infty}^{\infty}\left|1-\frac{dz}{d\zeta}\right| d\zeta \le \int_{-\infty}^{\infty}\left|1-\frac{dz_1}{d\zeta}\right| d\zeta.$$

Suppose $\zeta(D_1) = D'$. Since $D_1 \subset D$, D' is contained in the strip $0 > y > -h$. If $\zeta^{(1)}(u)$ maps D' on the strip, $\zeta_1(z) = \zeta^{(1)}[\zeta(z)]$ and by the lemma $d\zeta^{(1)}/du \ge 1$ when $\mathfrak{I}(u) = 0$. Therefore

$$\frac{d\zeta_1}{dz} = \frac{d\zeta^{(1)}}{d\zeta}\left(\frac{d\zeta}{dz}\right)_{y=0} \ge \frac{d\zeta}{dz}$$

and

$$\max_{\eta=0}\left(1-\frac{dz}{d\zeta}\right) \le \max_{\eta=0}\left(1-\frac{dz_1}{d\zeta}\right). \tag{3.2}$$

So far we have used the condition that D_1 is contained in D, but not that D is contained in $0 > y > -h$. Then $1 - (dz/d\zeta)_{\eta=0} \ge 0$ and

$$\max_{\eta=0}\left|1-\frac{dz}{d\zeta}\right| \le \max_{\eta=0}\left|1-\frac{dz_1}{d\zeta}\right|.$$

Next consider the integrals

$$\int_{-\infty}^{\infty}\left|1-\frac{dz}{d\zeta}\right| d\zeta = \int_{-\infty}^{\infty}\left(1-\frac{dz}{d\zeta}\right) d\zeta = \int_{-\infty}^{\infty}\left(\frac{d\zeta}{dz}-1\right) dz$$

$$= \int_{-\infty}^{\infty}\left|\frac{d\zeta}{dz}-1\right| dz \le \int_{-\infty}^{\infty}\left|\frac{d\zeta_1}{dz}-1\right| dz = \int_{-\infty}^{\infty}\left|1-\frac{dz_1}{d\zeta}\right| d\zeta.$$

Hence the theorem.

Theorem III (cf. Fig. 3). Suppose a domain D is bounded above by $y = 0$. Denote its intersection with the strip $0 > y > -h$ by D_1, its join with the strip by D_2. Then

$$\max_{\eta=0}\left|1-\frac{dz}{d\zeta}\right| \le \max_{\eta=0}\left(1-\frac{dz_1}{d\zeta}, \frac{dz_2}{d\zeta}-1\right) \le \max_{\eta=0}\left|1-\frac{dz_1}{d\zeta}\right| + \max_{\eta=0}\left|\frac{dz_2}{d\zeta}-1\right|,$$

and $$\int_{-\infty}^{\infty}\left|1-\frac{dz}{d\zeta}\right| d\zeta \le \int_{-\infty}^{\infty}\left|1-\frac{dz_1}{d\zeta}\right| d\zeta + \int_{-\infty}^{\infty}\left|\frac{dz_2}{d\zeta}-1\right| d\zeta.$$

(i) By (3.2) $\max_{\eta=0}(1 - dz/d\zeta) \le \max_{\eta=0}(1 - dz_1/d\zeta)$, since $D_1 \subset D$ and $\max_{\eta=0}(dz/d\zeta - 1) \le \max_{\eta=0}(dz_2/d\zeta - 1)$ since $D \subset D_2$. Because on $\eta = 0$ $dz_1/d\zeta \le 1$, $dz_2/d\zeta \ge 1$ the first result follows.

(iii) Since $D_1 \subset D \subset D_2$, by (3.2) $d\zeta_2(x, 0)/dz \le d\zeta(x, 0)/dz \le d\zeta(x_1, 0)/d\zeta$.

Thus $\left|\frac{d\zeta}{dz}-1\right|_{y=0} \le \left|\frac{d\zeta_1}{dz}-1\right|$ when $\frac{d\zeta}{dz} \ge 1$

and $\left|\frac{d\zeta}{dz}-1\right|_{y=0} \le \left|\frac{d\zeta_2}{dz}-1\right|$ when $\frac{d\zeta}{dz} \le 1$ so that

$$\left|\frac{d\zeta}{dz}-1\right|_{y=0} \le \left|\frac{d\zeta_1}{dz}-1\right| + \left|\frac{d\zeta_2}{dz}-1\right|.$$

Hence

$$\int_{-\infty}^{\infty}\left|1-\frac{dz}{d\zeta}\right|d\zeta=\int_{-\infty}^{\infty}\left|\frac{d\zeta}{dz}-1\right|dz\le\int_{-\infty}^{\infty}\left|\frac{d\zeta_1}{dz}-1\right|dz+\int_{-\infty}^{\infty}\left|\frac{d\zeta_2}{dz}-1\right|dz,$$

and the second result follows.

4. Reflection from obstacles in the surface. In the study of the reflection of waves by obstacles in the surface we consider first a special case: reflection by a flat plate lying in the surface. By change of scale we make the half width of the plate unity.

Theorem I. Provided

$$4\sin^2 k/(1+2kh/\sinh 2kh)+\frac{1-2kh/\sinh 2kh}{1+2kh/\sinh 2kh}<1 \qquad \text{or}$$

$$4\sin^2 k/(1+2kh/\sinh 2kh)+2[2^{-1/2}\exp(-2^{1/2}kh)+(kh)^{-1}+2\pi^{-1/2}k^{1/2})<1,$$

there is a unique potential ϕ, bounded in $0>y>-h$ which satisfies:

$$\text{(D)}\qquad \sigma^2\phi-g\,\partial\phi/\partial y=0 \qquad y=0 \qquad |x|>1,$$

$$\partial\phi/\partial y=0 \qquad y=0 \qquad |x|<1, \qquad \text{and on } y=-h,$$

$$\phi\to ae^{ikx}\cosh k(y+h)/\cosh kh \qquad \text{as} \qquad x\to+\infty, \qquad 0>y>-h.$$

Also

$$R\sim\frac{2k}{1+2kh/\sinh 2kh}+O(k^2),$$

and for $0<\alpha^2/(1-\alpha)<\sin 2k/2k$ we get precise bounds

$$\frac{2k[\sin 2k/2k-\alpha^2/(1-\alpha)]/(1+2kh/\sinh 2kh)}{\{1+4k^2[\sin 2k/2k-\alpha^2/(1-\alpha)]^2/(1+2kh/\sinh 2kh)^2\}^{1/2}}$$

$$<R<\frac{2k[\sin 2k/2k+\alpha^2/(1-\alpha)]/(1+2kh/\sinh 2kh)}{\{1+4k^2[\sin 2k/2k+\alpha^2/(1-\alpha)]^2/(1+2kh/\sinh 2kh)^2\}^{1/2}}. \qquad (4.1)$$

(i) Observe first that if $f(x)$ is the function of Lemma I, Sec. 2, and $\alpha=\sigma^2\int_{-1}^{1}|f(x-x')|\,dx'<1$ in $-1<x<1$, ϕ is unique.
By that lemma the difference Φ of any two potentials satisfying (D) also satisfies

$$\Phi(x,0)=\int_{-1}^{1}\left(\sigma^2\Phi-g\frac{\partial\Phi}{\partial y'}\right)f(x-x')\,dx'=\sigma^2\int_{-1}^{1}\Phi f(x-x')\,dx',$$

so that

$$|\Phi(x,0)|_{|x|<1}\le\alpha\max_{|x'|<1}|\Phi(x',0)|.$$

$$\text{Thus}\qquad \Phi(x',0)=0 \qquad \text{in} \qquad |x'|<|, \qquad \text{also} \qquad \partial\Phi/\partial y'=0$$

so that $\Phi\equiv 0$.

(ii) To construct a ϕ we use the method of Lemma II, Sec. 2.

If $\phi = \phi_1 + ae^{ikx} \cosh k(y + h)/\cosh kh$, then ϕ_1 satisfies conditions of form (B):
$g(x) = 0, |x| > 1; \quad g(x) = -\sigma^2, |x| < 1;$
$h(x) = 0, |x| > 1, h(x) = -a\sigma^2 e^{ikx}, |x| < 1.$

The iteration converges in $|x| < 1$ if $\sigma^2 \int_{-1}^{1} f(x - x')dx' < 1$ for $|x| < 1$ to give $\phi_1(x, 0), |x| < 1$; and since

$$g \frac{\partial \phi_1}{\partial y}(x, 0) = -agk \tanh kh\, e^{ikx} = -a\sigma^2 e^{ikx}$$

on the plate, $\sigma^2\phi_1 - g\, \partial\phi_1/\partial y$ is known on $y = 0$, and by Lemma I, Sec. 2, we get ϕ_1 in the whole strip.
(iii) Write

$$\phi_1 = \sum_0^\infty \phi_n^{(1)} \qquad \text{with} \qquad \phi_0^{(1)} = \sigma^2 a \int_{-1}^{1} e^{ikx'} f(x - x', y)\, dx',$$

$$\phi_{n+1} = \sigma^2 \int_{-1}^{1} \phi_n^{(1)}(x', 0) f(x - x', y)\, dx'.$$

The asymptotic form of $\phi_0^{(1)}(x, y)$, (at $-\infty$) is

$$\frac{aki}{1 + 2kh/\sinh 2kh}\left[e^{-ikx}\int_{-1}^{1} e^{2ikx'}\, dx' - e^{ikx}\int_{-1}^{1} dx'\right] \cosh k(y + h)/\cosh kh,$$

and the asymptotic form of $\phi_{n+1}^{(1)}$ is

$$\frac{ki}{1 + 2kh/\sinh 2kh}\left[e^{-ikx}\int_{-1}^{1} \phi_n^{(1)} e^{ikx'}\, dx'\right.$$
$$\left. - e^{ikx}\int_{-1}^{1} \phi_n^{(1)} e^{-ikx'}\, dx'\right] \cosh k(y + h)/\cosh kh.$$

The reflected wave [i.e. coefficient of $e^{-ikx} \cosh k(y + h)/\cosh kh$ in the asymptotic form] of

$$\sum_1^\infty \phi_n^{(1)}$$

is less than

$$\frac{k}{1 + 2kh/\sinh 2kh} \sum_1^\infty \int_{-1}^{1} |\phi_n^{(1)}|\, dx' < \frac{2ka}{1 + 2kh/\sinh 2kh} \sum_1^\infty \alpha^{n+1}$$

$$= \frac{2ka}{(1 + 2kh/\sinh 2kh)} \frac{\alpha^2}{(1 - \alpha)}.$$

The reflected wave in $\phi_0^{(1)}$ is $a \sin 2k/(1 + 2kh/\sinh 2kh)$.
The amplitude of the reflected wave of

$$\sum_0^\infty \phi_n^{(1)}$$

lies between

$$\frac{2ka}{1 + 2kh/\sinh 2kh}\left[\frac{\sin 2k}{2k} \pm \frac{\alpha^2}{1 - \alpha}\right].$$

Since the transmitted wave is $ae^{ikx} \cosh k(y + h)/\cosh kh$, we get a bound for the in coming wave, and find (4.1).

Evaluation of α.

(iv) First we get a bound suitable for small kh. Denote the infinite series part of $f(x)$ by $f_1(x)$. By Lemma I, Sec. 2,

$$\int_{-1}^{1} | f(x - x') | \, dx' < 2 \int_0^1 | f_1(x') | \, dx'$$

$$+ [g \tanh kh(1 + 2kh/\sinh 2kh)]^{-1} \int_x^1 | e^{ik(x-x')} - e^{-ik(x-x')} | \, dx'.$$

Now

$$\int_0^1 | f_1(x') | \, dx' < \int_0^\infty | f_1(x') | \, dx' = \frac{1}{2\sigma^2}\frac{1 - 2kh/\sinh 2kh}{1 + 2kh/\sinh 2kh}.$$

Also

$$\int_x^1 | e^{ik(x-x')} - e^{-ik(x-x')} | \, dx' = 2 \int_0^{1-x} | \sin ku | \, du \leq 2 \int_0^2 | \sin ku | \, du$$

$$= 4 \sin^2 k/k \qquad \text{if} \qquad k < \frac{1}{2}\pi.$$

Thus

$$\sigma^2 \int_{-1}^{1} | f(x - x') | \, dx' < 4 \sin^2 k/(1 + 2kh/\sinh 2kh) + \frac{1 - 2kh/\sinh 2kh}{1 + 2kh/\sinh 2kh}.$$

(v) If $kh \to \infty$ the bound becomes useless since the second summand tends to 1. We use a refined estimate for $\int_0^1 | f_1(x') | \, dx'$. Recall that

$$- f_1(x) = \sum_1^\infty \frac{e^{-k_r x}}{g \tan k_r h(1 + 2k_r h/\sin 2k_r h)} < \frac{1}{g} \sum_1^\infty \frac{(hk_r)e^{-k_r x}}{(2r - 1)^2\pi^2/4 + \tau}$$

where $$\tau = (\sigma^2 h/g)(\sigma^2 h/g - 1).$$

Thus

$$\int_0^1 | f_1(x) | \, dx < \frac{h}{g} \sum_1^\infty \frac{(1 - e^{-k_r})}{(2r - 1)^2\pi^2/4 + \tau}$$

$$= \frac{h}{2g} \sum_{-\infty}^\infty \frac{1}{(2r - 1)^2\pi^2/4 + \tau} - \frac{h}{g} \sum_1^\infty \frac{e^{-k_r}}{(2r - 1)^2\pi^2/4 + \tau} > 0.$$

By contour integration of $\cot \pi z[(2z - 1)^2/4 + \tau]^{-1}$ around a large rectangle it is seen that

$$\sum_{-\infty}^{\infty} \frac{1}{(2r - 1)^2\pi^2/4 + \tau} = \frac{\tanh \tau^{1/2}}{\tau^{1/2}}.$$

$$\sum_{1}^{\infty} \frac{e^{-k_r}}{(2r - 1)^2\pi^2/4 + \tau} > \sum_{1}^{\infty} \frac{e^{-r\pi/h}}{(2r - 1)^2\pi^2/4 + \tau^2} > \int_1^{\infty} \frac{e^{-\pi x/h}}{(2r - 1)^2\pi^2/4 + \tau^2}\, dx,$$

since $k_r h < \pi r$, and the terms of the series are decreasing.

$$\int_1^{\infty} \frac{e^{-\pi x/h}}{(2x - 1)^2\pi^2/4 + \tau^2}\, dx = \frac{1}{\pi\tau^{1/2}} \int_{\pi/2\tau^{1/2}}^{\infty} \frac{\exp(-\tau^{1/2}u/h)}{u^2 + 1}\, du$$

$$> \frac{1}{\pi\tau^{1/2}} \int_0^{\infty} \frac{du}{u^2 + 1} - \frac{1}{\pi\tau^{1/2}} \int_0^{\pi/2\tau^{1/2}} du - \frac{1}{\pi\tau^{1/2}} \left\{ \int_0^N + \int_N^{\infty} \frac{1 - \exp(-\tau^{1/2}u/h)}{u^2 + 1}\, du \right.$$

$$\left. = \frac{1}{2\tau^{1/2}} - \frac{1}{2\tau} - \frac{1}{\pi\tau^{1/2}} \left\{ \int_0^N + \int_N^{\infty} \frac{1 - \exp(-\tau^{1/2}u/h)}{u^2 + 1}\, du \right\}.\right.$$

Since

$$1 - \exp(-\tau^{1/2}N/h) < \tau^{1/2}N/h, \qquad \frac{1}{\pi\tau^{1/2}} \int_0^N \frac{1 - \exp(-\tau^{1/2}u/h)}{u^2 + 1}\, du < \frac{1}{2} N/h, \quad \text{and}$$

$$\int_N^{\infty} \frac{1 - \exp(-\tau^{1/2}u/h)}{u^2 + 1}\, du < \int_N^{\infty} \frac{du}{u^2 + 1} = \frac{1}{2}\pi - \tan^{-1} N = \tan^{-1} \frac{1}{N} < \frac{1}{N}.$$

Therefore

$$\frac{1}{\pi\tau^{1/2}} \int_0^{\infty} \frac{1 - \exp(-\tau^{1/2}uh)}{u^2 + 1}\, du < \frac{1}{\pi N \tau^{1/2}} + \frac{1}{2}\frac{N}{h}.$$

If we choose

$$N = (2h/\pi)^{1/2}\tau^{-1/4}, \qquad \int_0^1 |\, f_1(x) \,|\, dx < \frac{h}{g}\left[\frac{\tanh \tau^{1/2} - 1}{2\tau^{1/2}} + \frac{1}{2\tau} + \left(\frac{2}{\pi h}\right)^{1/2} \tau^{1/4}\right].$$

Since $(\tanh \tau^{1/2} - 1)/2 = \exp(-2\tau^{1/2})/[1 + \exp(-2\tau^{1/2})] < \exp(-2^{1/2}kh)$, and $(1/h\tau^{1/2})^{1/2} < 2^{1/2}h^{-1}k^{-1/2}$ we get after some simplification that

$$2\sigma^2 \int_0^1 |\, f_1(x) \,|\, dx < 2[2^{-1/2} \exp(-2^{1/2}kh) + (kh)^{-1} + 2\pi^{-1/2}k^{1/2}].$$

Since at the edges of the plate in Th. I the horizontal velocity gets infinite while in the derivation of the boundary conditions the velocity is assumed to be everywhere small, it is not clear that the calculation gives a sensible approximation to the reflection of waves by shallow obstacles. We shall therefore show separately that the reflection of an arbitrary (cylindrical) obstacle of beam equal to the width of the plate tends to the reflection calculated in Th. I provided only that the draught tends to zero.

Theorem II. We map the domain of the fluid on $0 > \eta > -h$ so that the edges of the obstacle are at ($\xi = \pm 1$, $\eta = 0$). If

(D′) ϕ is bounded and harmonic in $0 > \eta > -h$,

$$\partial\phi/\partial\eta = 0 \quad \text{on} \quad \eta = -h, \quad \text{and on} \quad \eta = 0, \quad |\xi| < 1,$$
$$\sigma^2\phi - g\,\partial\phi/\partial\eta + \sigma^2(dz/d\zeta - 1)\phi = 0 \quad \text{on} \quad \eta = 0 \quad |\xi| > 1,$$
$$\phi \to ae^{ik\xi}\cosh k(\eta + h)/\cosh kh \quad \text{as} \quad \xi \to \infty,$$

ϕ can be constructed by iteration provided

$$0 < \delta = \frac{\beta}{1 - \alpha} < 1 \qquad \text{where}$$

$$\sigma^2 \int_{-1}^{1} |\, f(\xi - \xi') \,|\, d\xi' \le \alpha$$

$$\sigma^2\left(\int_{-\infty}^{-1} + \int_{1}^{\infty} \left|\frac{dz}{d\zeta'} - 1\right| \left| f(\xi - \xi') \right| d\xi'\right) \le \beta \qquad \text{for all } \xi.$$

Further if the draught is small, β is small.

(i) Let ϕ_0 be the potential of Th. I. We write then

$$\phi = \phi_0 + \sum_{1}^{\infty} \phi_n ,$$

and split up ϕ_{n+1} into $\phi_{n+1}^{(1)} + \phi_{n+1}^{(2)}$, which satisfy the following conditions:

(D″) $\phi_{n+1}^{(1)}$ is bounded and harmonic in $0 > \eta > -h$,

$$\partial\phi_{n+1}^{(1)}/\partial\eta = 0 \quad \text{on} \quad \eta = -h,$$

$$\sigma^2\phi_{n+1}^{(1)} - g\,\partial\phi_{n+1}^{(1)}/\partial\eta = \begin{cases} -\sigma^2(dz/d\zeta - 1)\phi_n & \text{on} \quad \eta = 0, \quad |\xi| > 1, \\ 0 & \text{on} \quad \eta = 0, \quad |\xi| < 1, \end{cases}$$

$$\phi_{n+1}^{(1)} \to 0 \quad \text{as} \quad \xi \to +\infty.$$

(D‴) $\phi_{n+1}^{(2)}$ is bounded and harmonic in $0 > \eta > -h$,

$$\partial\phi_{n+1}^{(2)}/\partial\eta = 0 \quad \text{on} \quad \eta = -h, \quad \partial\phi_{n+1}^{(2)}/\partial\eta = -\partial\phi_{n+1}^{(1)}/\partial\eta = -(\sigma^2/g)\phi_{n+1}^{(1)}$$

$$\text{on} \quad \eta = 0 \quad |\xi| < 1,$$

$$\sigma^2\phi_{n+1}^{(2)} - g\,\partial\phi_{n+1}^{(2)}/\partial\eta = 0 \quad \text{on} \quad \eta = 0 \quad |\xi| > 1,$$

$$\phi_{n+1}^{(2)} \to 0 \quad \text{as} \quad \xi \to +\infty.$$

Let $|\phi_{n+1}^{(1)}| < M_{n+1}^{(1)}$, $|\phi_{n+1}^{(2)}| < M_{n+1}^{(2)}$ and $|\phi_{n+1}| < M_{n+1}$.

(ii) By Lemma II, Sec. 2 (we may take) $M_{n+1}^{(1)} < \beta M_n$.

$$\phi_{n+1}^{(2)} = \sigma^2 \int_{-1}^{1} \phi_{n+1}^{(2)} f(\xi - \xi') \, d\xi' - g \int_{-1}^{1} \frac{\partial \phi_{n+1}^{(2)}}{\partial \eta'} f(\xi - \xi') \, d\xi'$$

$$= \sigma^2 \int_{-1}^{1} \phi_{n+1}^{(2)} f(\xi - \xi') \, d\xi' + \sigma^2 \int_{-1}^{1} \phi_{n+1}^{(1)} f(\xi - \xi') \, d\xi'.$$

Therefore

$$M_{n+1}^{(2)} < \alpha M_{n+1}^{(2)} + \alpha M_{n+1}^{(1)},$$

so that

$$M_{n+1}^{(2)} < \frac{\alpha\beta M_n}{1 - \alpha} \qquad \text{and} \qquad M_{n+1} < \frac{\beta M_n}{1 - \alpha} = \delta M_n .$$

(iii) If $\delta < 1$, $\alpha < 1$ so that ϕ_0 exists and the iteration converges to a solution of (D′). If δ is small compared with α, the solution is nearly equal to ϕ_0.
(iv) $\delta \to 0$ if the draught is small, provided at its edges A, B the obstacle has tangents which make angles greater than ϵ with the mean free surface (Fig. 5). We take A at $(-1, 0)$.

We enclose the obstacle in a trapezoid $ABCD$, and denote by D_1 the domain obtained by removing the trapezoid from the strip $0 > y > -h$. Let $z_1(\zeta)$ map $0 > \eta > -h$ on D_1, keeping A fixed, and let its inverse be $\zeta_1(z)$. Assume for the moment that
(i) $1 \geq d\zeta/dz \geq d\zeta_1/dz > 0$ on $z < \underline{-1}$; (ii) $\underline{\text{in } z} \leq 1 - \rho$, $\rho > 0$, $1 - d\zeta_1/dz$ tends to 0 uniformly as the distance d between $\overline{AB}$ and $\overline{CD} \to 0$, ϵ fixed;
(iii) $d\zeta_1/dz$ decreases monotonically to zero in $-\infty < z < -1$ as $z \to -1$ so that if z, z', $z' + h$ are less than -1 and $z < z'$

$$| \zeta_1(z' + h) - \zeta_1(z') | < | \zeta_1(z + h) - \zeta_1(z) | .$$

Now

$$\int_{-\infty}^{-1} | f(\xi - \xi') | \left| \frac{dz}{d\zeta'} - 1 \right| d\zeta'$$

$$\leq [g \tanh kh(1 + 2kh/\sinh 2kh)]^{-1} \int_{-\infty}^{-1} \left| \frac{dz}{d\zeta'} - 1 \right| | e^{ik(\xi-\xi')} - e^{-ik(\xi-\xi')} | \, d\xi'$$

$$+ \int_{-\infty}^{-1} | f_1(\xi - \xi') | \left| \frac{dz}{d\zeta'} - 1 \right| d\xi' = I_1 + I_2(\text{p.def.})$$

$$I_1 < 2[g \tanh kh(1 + 2kh/\sinh 2kh)]^{-1} \int_{-\infty}^{-1} \left| \frac{dz}{d\zeta'} - 1 \right| d\zeta'$$

$$= 2[g \tanh kh(1 + 2kh/\sinh 2kh)]^{-1} \int_{-\infty}^{-1} \left| \frac{d\zeta}{dz} - 1 \right| dz$$

$$< 2[g \tanh kh(1 + 2kh/\sinh 2kh)]^{-1} \int_{-\infty}^{-1} \left| \frac{d\zeta_1}{dz} - 1 \right| dz, \text{ by (i)}, = J_1(\text{p.def.}).$$

A simple calculation shows that $J_1 \to 0$ as $d \to 0$.

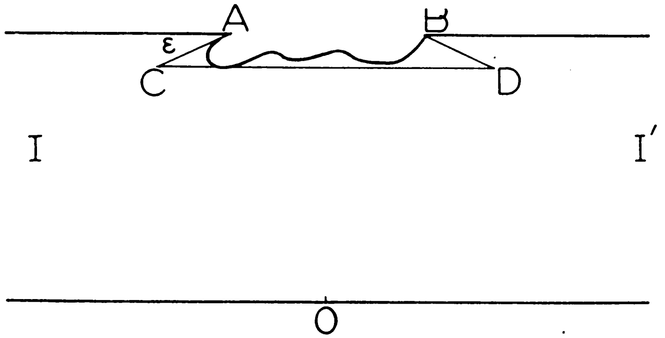

FIG. 5.

Let $\xi = \xi(z)$. Then, since $|f_1(x)|$ is decreasing, and, by (i), $|\xi_1(z) - \xi_1(z')| \le |\xi(z) - \xi(z')|$

$$I_2 = \int_{-\infty}^{-1} |f_1[\xi(z) - \xi(z')]| \left| \frac{d\zeta}{dz'} - 1 \right| dz' < \int_{-\infty}^{-1} |f_1[\xi_1(z) - \xi_1(z')]| \left| \frac{d\zeta_1}{dz'} - 1 \right| dz',$$

$$< \left(\int_{-\infty}^{-1+\rho} |f_1[\xi_1(z) - \xi_1(z')]|\, dz' \right) \left| \frac{d\zeta_1(-1+\rho, 0)}{dz} - 1 \right|$$

$$+ \int_{-1+\rho}^{-1} |f_1[\xi_1(z) - \xi_1(z')]|\, dz', \qquad \text{by (i) and (ii)},$$

$$< 2\left(\int_{-\infty}^{-1} |f_1[-1 - \xi_1(z')]|\, dz' \right) \left| \frac{d\zeta_1(-1+\rho, 0)}{dz} - 1 \right|$$

$$+ 2 \int_{-1+1/2\rho}^{-1} |f_1[-1 - \xi_1(z')]|\, dz', \qquad \text{by (iii),} \qquad \text{independently of } z.$$

Note that both integrals decrease if ρ is fixed and d decreases, since by (i) $\xi_1(z)$ decreases with d. Choose ρ small to make the second integral small. Then choose d small to make $|d\zeta_1(1 - \rho, 0)/dz - 1|$ small; this makes the first term small since the integral converges uniformly as $d \to 0$: at $-\infty$ since $f_1(x)$ is exponentially small, and at -1 since $\zeta_1(z) + 1 = O|z + 1|^{1+\epsilon/\pi}$, and $f_1[1 - \xi_1(z)]$ has only a logarithmic singularity. Thus $I_2 \to 0$ as $d \to 0$.

To estimate the integral

$$\int_1^{\infty} |f(\xi - \xi')| \left| \frac{dz}{d\zeta'} - 1 \right| d\zeta'$$

map $0 > \eta > -h$ on D_1, keeping B fixed. We find $\delta \to 0$ as $d \to 0$.

It remains to prove the assumptions:

(i) follows by an argument essentially similar to that of the Lemma of Sec. 3.

(ii) follows from Th. I(i) of Sec. 3 where it is shown that

$$\left| 1 - \frac{d\zeta_1}{dz} \right| < \frac{2d}{h} \frac{e^{\pi(x+1)/h}}{(1 - e^{\pi(x+1)/h})^2}.$$

(iii) follows from the fact that the boundary of the domain of log $d\zeta_1/dz$

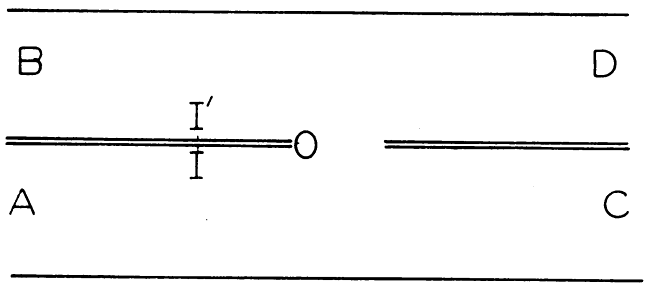

FIG. 6

is simple (since the trapezoid is convex) so that log $(d\zeta_1/dz)$ and $d\zeta_1/dz$ are schlicht in D_1 and $d^2\zeta_1/dz^2 \neq 0$. It follows that $\partial^2\zeta/\partial x^2 \neq 0$ on $z < -1$, i.e. $\partial^2\zeta/\partial x^2 < 0$. Since $d\zeta/dz = \partial\zeta/\partial x$ on $z < -1$, $d\zeta/dz$ decreases monotonically in $-\infty < z < -1$.