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OF

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The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter l and the prime ($'$), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

All subscripts and exponents should be clearly marked, and dots, bars, tildes, etc. over letters should be avoided.

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Complicated exponents and subscripts should be avoided. Any complicated expression that reoccurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp\,[(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2+b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos\,(\pi x/2b)}{\cos\,(\pi a/2b)} \text{ is preferable to } \frac{\cos\dfrac{\pi x}{2b}}{\cos\dfrac{\pi a}{2b}}$$

In many instances the use of negative exponents permits saving of space. Thus,

$$\int u^{-1} \sin u\, du \text{ is preferable to } \int \frac{\sin u}{u}\, du.$$

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

$$(a + bx) \cos t \text{ is preferable to } \cos t\,(a + bx).$$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

$$\{[a + (b + cx)^n] \cos ky\}^2 \text{ is preferable to } ((a + (b + cx)^n) \cos ky)^2.$$

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Obviously, the problem as described above in terms of flux of heat can be restated in terms of other physical quantities such as potential in an electrostatic field, velocity potential in fluid flow, gravitational potential, etc.

For example, consider a potential function $f(u, v) = c$ and a vector force function $\mathbf{F}(u, v)$. Let $f(u, v) = c$ and $f(u, v) = c + \Delta c$ be two neighboring equipotential curves. Then $\mathbf{F}(u, v)$ acts in a direction orthogonal to the equipotential curves and its magnitude is given by $F = \Delta c/\Delta n$, where Δn denotes the distance between the two curves. That is, the magnitude of $\mathbf{F}$ is inversely proportional to the distance between the two curves. Hence if $f_1, f_2, f_3, \cdots$ are a set of equipotential curves, and g_1, g_2 any two lines of force and if $\mathbf{F}_1$ acts along g_1, $\mathbf{F}_2$ along g_2 the property of proportional arc length leads to $F_1/F_2 = \text{const.}$

BOOK REVIEWS

Mathematics our great heritage. Selected and Edited by William L. Schaaf, Ph.D. Harper & Brothers, New York. xi + 291 pp. $3.50.

This book is a collection of essays on mathematics and its relationships to other fields of endeavor, including the fine arts, philosophy, experimental science and technology. The authors include both eminent mathematicians and those whose primary fields are not pure mathematics, but whose work brings them in close contact with mathematics, and has induced them to examine the relations between mathematics and other activities. The essays are on a high level, and should be of particular interest to applied mathematicians.

P. S. Symonds

The principles of quantum mechanics. By P. A. M. Dirac. Oxford, at the Clarendon Press, 1947. xii + 311 pp. $9.00.

This work, which has always been a cornerstone in the theoretical foundation of Quantum Mechanics, retains all of its stature in this new edition. Substantially the same subject matter is covered as in the two earlier editions. However, it is apparent that the new edition has been very largely rewritten. The author has changed his notation, but the new notation is almost self-explanatory and should cause little difficulty. The new "bra" and "ket" vector notation for states allows a more direct connection to be made between the abstract algebra of states and observables, and the theory of representations. At the same time, the author has put the formal algebra of states and observables into neater and more elegant form.

Other significant changes are in the presentation of the theory of systems containing identical particles, which is in a somewhat simpler form in this new edition, and in the treatment of quantum electrodynamics, which has been carried a little further than in the previous edition. However, the problem of divergent solutions remains, and the theory cannot be carried to a satisfactory conclusion.

D. F. Hornig

Mechanical behavior of high polymers. By Turner Alfrey, Jr. Interscience Publishers, Inc., New York, 1948. xiv + 580 pp. $9.50.

While the number of books dealing with deformation and strength of materials considered as an elastic continuum is growing like mushrooms, no book has so far been written in which the mechanical behavior of real materials (which, being formed by aggregation of particles into groups, are neither

homogeneous nor isotropic) is being discussed from both the phenomenological and the structural point of view, and the observable phenomenological behavior related to the underlying structural mechanism. Therefore the appearance of a treatise covering a very complex and a very important section of this general subject of mechanical behavior of real materials—the field of high polymers—is most welcome. The author's principal purpose is clear from the preface in which he states that "the analysis of the mechanical behavior has been made on two levels—a phenomenological description and a molecular interpretation."

The book is divided into six main chapters: A. Introduction; B. Plastoelastic Behavior of Amorphous Linear High Polymers; C. Three-Dimensional Cross-Linked Polymers; D. Crystallization of High Polymers; E. Systems Containing High Polymers and Materials of Low Molecular Weight; F. Ultimate Strength and Related Properties. The text is followed by four Appendici, the first of which gives a very short explanation of tensor representation, the next two deal with mathematical concepts used in the analysis of viscoelastic bodies, while the last contains a short discussion of mechanical testing.

The introductory chapter presents the basic concepts of the mechanics of deformable solid and of the viscous and the complex (non-Newtonian) liquid, limited to conditions of equilibrium and of stationary flow. This is followed by a discussion of the structural (molecular) mechanisms of flow and of viscoelastic deformation, illustrated by simple mechanical models consisting of springs and dashpots. The discussion of stress, strain and flow starts on a level which is rather elementary; very soon, however, tensor notation is introduced together with a somewhat erratic discussion of crystal geometry in relation to the multiple-constant theory of elasticity, and of the compatibility conditions for infinitesimal strain; these concepts are hardly very important for an understanding of the main subject. On the other hand, one of the major problems of the mechanics of highly deformable bodies, such as many high polymers, the arbitrariness of the definition of strain and the analysis of finite deformations and strains, is not considered worthy of any, but the most superficial treatment; while Murnaghan's analysis is given a few words, there is no mention of Hencky's or Seth's work, although their methods have been applied with success to the representation of the deformation of rubber. The discussion of flow, which follows, is mainly concerned with the presentation of the equations of the capillary and the rotating cylinder viscometers for the Newtonian fluid and the Bingham body, as developed by Buckingham and Reiner. The discussion of thixotropy lacks the fundamental approach to be found in Freundlich's work to which it refers, and is more a discussion of specific experimental procedure; there is no attempt to present yield-limit, thixotropy and non-Newtonian flow within an integrated structural picture. The rest of the chapter is taken up by a simple analysis of the Maxwell and the Voigt (Kelvin) model-unit, by a discussion of the bulk-modulus of single crystals in terms of atomic structure, and by a presentation of the structural mechanism of viscous liquid flow.

The second chapter treats the inelastic behavior of amorphous, linear high polymers, which are the thermoplastics and the unvulcanized rubbers. The molecular structure is broadly discussed and its response to load quantitatively reproduced with the aid of combined linear mechanical models. In selecting the type of model, the author expresses a definite preference for the combination in series of "retarded" Voigt (Kelvin)-units, as against the parallel combination of "relaxing" (Maxwell)-units. The discussion of the dynamics of viscoelastic behavior introduces methods of circuit analysis, which are simple and very effective in the analysis of complex models.

The analysis of the effect of non-homogeneous and combined stresses starts very promising; an attempt is made to derive the equations for the general conditions of stress for a Maxwell body, from which relations between the components of the velocity of relaxation and of the stress tensor could be derived, but is not followed up. The presentation of the elastic-viscoelastic analogy, however, is of importance; its understanding requires a study of Appendix III, reproduction of an original paper by the author.

The engineer interested in photoelasticity, who may turn to the book for fundamental information concerning the interrelation of photoelastic and mechanical behavior of the principal photoelastic materials will be disappointed. The discussion of the photoelastic effects in high polymers is more or less on the level found in most textbooks on photoelasticity.

The third chapter contains a discussion of the behavior of three-dimensional, cross-linked polymers, such as thermosetting plastics and vulcanized rubbers. Mechanical model analysis can no longer be applied, because of the cross-linking of molecular chains. The network analysis used instead in the correlation of phenomenological and structural response to load is based on concepts of statistical mechanics. Nearly one-half of this chapter is taken up by verbatim quotations of a few papers, which are easily accessible.

The fourth chapter deals with the crystallization of high polymers. The discussion of the complex phenomenon of "crystallization" and of the mechanical properties of "crystalline" (read "anisotropic") polymers, which comprise all fibrous materials, yarns and filaments, is thorough, although no serious attempt at a correlation of phenomenological behavior and structural response is made beyond the level of a purely descriptive experimental approach.

The fifth chapter brings an analysis of the behavior of polymers admixed with components of low molecular weight. Again, the high complexity of the structure makes any quantitative correlation between large-scale behavior and structure impracticable, and the discussion is largely a qualitative interpretation of experiments. A short discussion of the viscosity of suspensions is included.

In the comparatively short sixth chapter an attempt is made to explain fracture in terms of the structural mechanism involved. However, the discussion of the phenomenological aspect is oversimplified. Even the important aspect of the different significance, with regard to failure by excessive deformation and by fracture, of the tensors of volumetric and of deviatoric stress is omitted. Its consideration would immediately have shown up the untenability of the proposed concept of a "single limiting (failure) stress surface, containing a region of fracture and a region of plastic failure" (p. 480). No mention is made of the interrelation between fracture and the inelastic deformation preceding it. Since it is this interrelation which is at the root of the time-dependence of fracture-stress, the discussion of this dependence must necessarily remain on a purely empirical level. It is but in the sections dealing with the structural aspects, particularly the dependence of tensile strength on molecular weight and orientation, in which the principal object of the book is again brought into sharp focus. All chapters are followed by extensive lists of references.

Appendix II, a reproduction of a paper by the author and P. M. Doty is an excellent comparison of the different mathematical methods used to represent visco-elastic behavior.

The author, in attempting to write a comprehensive book on a subject which is not only very complex, but in the flux of rapid development, has taken upon himself an extremely difficult task for which the large number of workers in high polymer research owe him a debt of gratitude. The principal weakness of the book is the lack of balance in the presentation of the various aspects of the subject, particularly the phenomenological (mechanical) aspect. The omission of an adequate discussion of the thermodynamic foundation of mechanical behavior is a definite handicap for the reader with an engineering background. A second weakness is the sometimes rather loose terminology and the lack of definition of unconventional terms used, such as "inertial elasticity", "plastoelastic", "stress biased molecular diffusion" (which is used as a synonym for Brownian motion), etc.

Because of its unique character the book will easily find its way to the desk of most workers in polymer research.

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