but, aside from the simple case $v/\chi = 0$ discussed above, (9) seems to be of no help. It is equally easy to write out the Fredholm type of solution of (3),

$$l_\iota k_\iota = \frac{1}{\iota} + \frac{1}{\Delta} \sum_{\kappa=1}^{\infty} \frac{1}{\kappa} \Delta_{\iota\kappa} , \tag{10}$$

where

$$\Delta = 1 - \sum_{\iota=1}^{\infty} K_{\iota\iota} + \frac{1}{2!} \sum_{\iota=1}^{\infty} \sum_{\kappa=1}^{\infty} \begin{vmatrix} K_{\iota\iota} & K_{\iota\kappa} \\ K_{\kappa\iota} & K_{\kappa\kappa} \end{vmatrix} - \cdots ,$$

$$\Delta_{\iota\kappa} = K_{\iota\kappa} - \sum_{n=1}^{\infty} \begin{vmatrix} K_{\iota\kappa} & K_{\iota n} \\ K_{n\kappa} & K_{nn} \end{vmatrix} + \frac{1}{2!} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \begin{vmatrix} K_{\iota\kappa} & K_{\iota n} & K_{\iota m} \\ K_{n\kappa} & K_{nn} & K_{nm} \\ K_{m\kappa} & K_{mn} & K_{mm} \end{vmatrix} - \cdots ,$$

and

$$K_{\iota\kappa} = C_{\kappa-1}^{\iota+\kappa-1}/l_\kappa ,$$

but no use has been found for it.

3. Numerical example illustrating convergence.

$$s = 2\chi, \qquad v/\chi = .4, \qquad \epsilon/\epsilon_0 = 4.$$

If we assume that 1, 2, 3, 4 and 10 k terms are enough, we compute in turn, $k_1 = .2097$, $\sum_{n=1}^{2} k_n = .2544$, $\sum_{n=1}^{3} k_n = .2692$, $\sum_{n=1}^{4} k_n = .2751$ and $\sum_{n=1}^{10} k_n = .2807$. The sum of the first four of the k_n in $\sum_{n=1}^{10} k_n$ is .2797. The late terms in the sum are more important indirectly in computing the earlier terms than in the sum itself.

THE TEMPERATURE IN AN ACCRETING MEDIUM WITH HEAT GENERATION*

By A. E. BENFIELD (*Cruft Laboratory, Harvard University*)

The thermal problem of this note was solved with the hope of using it to try to test the theory that the earth was formed by accretion on the dust cloud hypothesis;[1] but there are many uncertain and unknown physical factors involved and, on reflection, it seems that the contemplated thermal considerations are unable at present to help in drawing conclusions. However, it is hoped that the following mathematical solution may be of interest and aid to others having related problems involving less uncertain physical conditions.

As the spherical case presents some difficulties, we shall merely consider here a

*Received March 25, 1949.

[1]See, for instance, F. L. Whipple, Scientific American **178**, 34 (1948).

homogeneous semi-infinite medium moving with constant velocity v in the positive x-direction, as shown in Fig. 1. Material is supplied both at a constant rate and at a

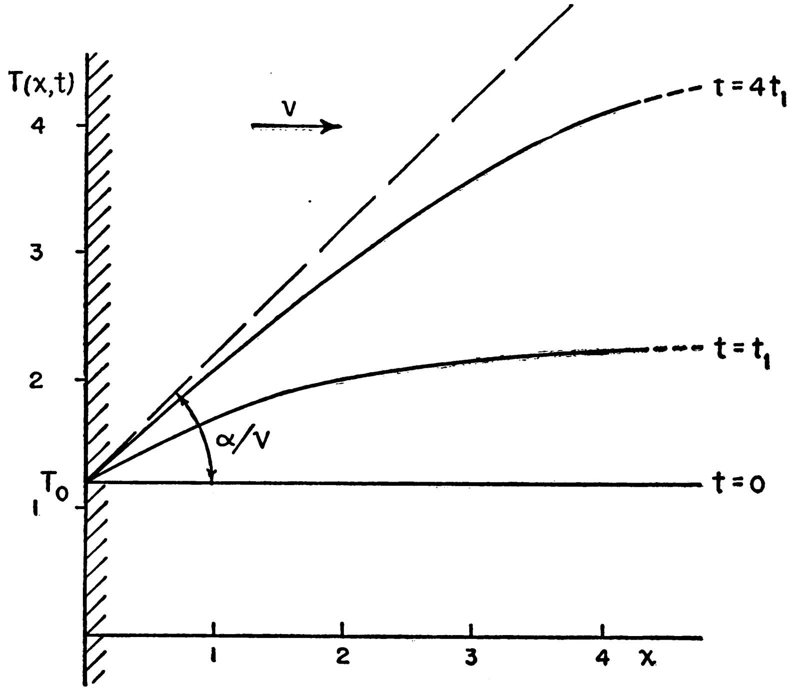

FIG. 1. Curves of $T(x, t)$ as given by (5) for $t = 0, t_1$, and $t_2 = 4t_1$, when α, v, K and t_1 all equal unity. The plane $x = 0$, which is held at a fixed temperature T_0, represents the free surface of the accreting semi-infinite medium at $t \geq 0$. The dashed line of slope α/v represents the asymptotic value of the temperature for large values of t.

steady temperature T_0 to the surface $x = 0$ of the medium, in such a way that the medium always exists continuously in the region $0 < x < \infty$, while for $x < 0$ there is a vacuum. In other words, x is measured to the right from the free surface of an accreting semi-infinite homogeneous medium.†

For $t < 0$ and $0 < x < \infty$ the temperature is everywhere a constant, $T = T_0$. At time $t = 0$, however, the generation of heat begins throughout the semi-infinite medium at the constant rate A cals/cm^3-sec and this heat generation continues indefinitely. For $t > 0$ the surface $x = 0$ continues to be held at the constant temperature $T = T_0$, and the material accreting at the plane $x = 0$ contains the same uniform distribution of heat sources, A cals/cm^3-sec, as the rest of the material in the region $0 < x < \infty$. The problem, then, is to find $T(x, t)$ for this situation when $t > 0$ and $0 < x < \infty$.

The appropriate differential equation is

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2} - v \frac{\partial T}{\partial x} + \frac{A}{c\rho}, \tag{1}$$

where c and ρ are, respectively, the uniform specific heat and the uniform density of

†The accreting material arrives at the plane $x = 0$ with zero velocity, so that it carries no kinetic energy, and hence no heat is evolved on its arrival.

the homogeneous medium, and K is its diffusivity (defined as the thermal conductivity divided by $c\rho$). Accordingly, K, v, A, c and ρ are all constants in this problem.

Using the method of the Laplace transform we now introduce the quantity $\overline{T} = L\{T\} \equiv \int_0^\infty e^{-pt}T(x, t)\,dt$, where p is a constant. Defining $\alpha \equiv A/c\rho$ we may follow the usual procedure of the Laplacian method and rewrite Eq. (1) as

$$K\frac{d^2\overline{T}}{dx^2} - v\frac{d\overline{T}}{dx} - p\overline{T} + T_0 + \alpha/p = 0, \tag{2}$$

which is a second order differential equation with constant coefficients. The solution of Eq. (2) may be written as

$$\begin{aligned}\overline{T} = \frac{T_0}{p} + \frac{\alpha}{p^2} + e^{vx/2K}[A_1 \exp\{x[(v^2/4K^2) + p/K]^{1/2}\}\\ + A_2 \exp\{-x[(v^2/4K^2) + p/K]^{1/2}\}],\end{aligned} \tag{3}$$

where A_1 and A_2 are functions of p, found by inserting the boundary conditions into Eq. (3). Proceeding now to do this, we may say that (i) when $x = 0$, $T = T_0$ and $\overline{T} = T_0/p$, from which it follows that $\alpha/p^2 = -(A_1 + A_2)$; and that (ii) as $x \to \infty$, $T \to T_0 + \alpha t$, so that $\overline{T} \to (T_0/p) + \alpha/p^2$, from which we have $A_1 = 0$.

Combining these values of A_1 and A_2 with Eq. (3) we find that

$$\overline{T} = \frac{T_0}{p} + \frac{\alpha}{p^2} - (\alpha/p^2)e^{vx/2K} \exp\{-x[(v^2/4K^2) + p/K]^{1/2}\}. \tag{4}$$

The inverse transforms of the first two terms are T_0 and αt, respectively, and, as shown in a recent paper,[2]

$$\begin{aligned}\frac{\exp\{-x[(v^2/4K^2) + p/K]^{1/2}\}}{p^2} = L\left\{\frac{1}{2v}\left[e^{vx/2K}(x + vt)\operatorname{erfc}\left(\frac{x + vt}{2(Kt)^{1/2}}\right)\right.\right.\\ \left.\left. - e^{-vx/2K}(x - vt)\operatorname{erfc}\left(\frac{x - vt}{2(Kt)^{1/2}}\right)\right]\right\},\end{aligned}$$

where

$$\operatorname{erfc} y \equiv 1 - \operatorname{erf} y = 1 - 2(\pi)^{-1/2}\int_0^y \exp(-u^2)\,du.$$

Our solution may now be written by adding the inverse transforms of Eq. (4); it is

$$T(x, t) = T_0 + \alpha t - \frac{\alpha}{2v}\left[e^{vx/K}(x + vt)\operatorname{erfc}\left(\frac{x + vt}{2(Kt)^{1/2}}\right) - (x - vt)\operatorname{erfc}\left(\frac{x - vt}{2(Kt)^{1/2}}\right)\right]. \tag{5}$$

Curves of this expression for three values of the time are shown in Fig. 1. Equation (5) is seen to reduce properly for the special cases of $x = 0$, $\alpha = 0$ and $t = 0$. Furthermore, it satisfies the differential equation (1), and it reduces as $v \to 0$ to the appropriate expression for a stationary medium.[3]

[2] A. E. Benfield, Q. Appl. Math. **6**, 439 (1949), in which Eq. (11) is equivalent to the transform given here.

[3] Cf. H. S. Carslaw and J. C. Jaeger, *Conduction of heat in solids*, Clarendon Press, Oxford, 1947, p. 60, where Eq. (2) is for a similar but not identical case with $v = 0$.

A point of some interest is the fact that, as time goes on, the slope $\partial T(x, t)/\partial x$ of the curve of Eq. (5) does not increase without limit, but instead approaches an asymptotic value α/v, shown in Fig. 1. This asymptotic behavior may be seen by differentiating Eq. (5) with respect to x, and then inserting the conditions $t \gg x/v$ and $t \gg K/v^2$.

It is a pleasure to thank Professor P. Le Corbeiller, Professor F. Birch and Professor F. Whipple for helpful discussions.

DEGENERATE TWO-DIMENSIONAL NON-STEADY IRROTATIONAL FLOWS OF A COMPRESSIBLE GAS*

By N. COBURN (*University of Michigan*)

1. Introduction. A class of non-steady, two-dimensional, irrotational, compressible flows which are very similar to steady, two-dimensional, irrotational, compressible flows will be studied. In order to do this, we introduce the well-known potential equation and Bernoulli relation for general non-steady flows. Our degenerate flows are defined by requiring that two families of cylindrical characteristic surfaces (with generators parallel to the time axis) exist in space-time. These flows have the following properties: (1) the wave fronts are stationary; (2) each of the velocity components and the speed of sound depends upon a single function of time multiplied by appropriate functions, which we shall call "reduced" velocities, of the space variables; (3) the single function of time is such that the motion decays as time increases. A canonical characteristic system, consisting of five equations with five dependent variables (the reduced velocities and the rectangular coordinates of the plane) and two independent variables, is obtained. It is shown that simple waves do not exist. Finally, it is shown that a degenerate non-steady flow, whose stream lines are logarithmic spirals, exists.

2. The system of flow equations and the potential equation. Let $x^\lambda (\lambda = 1, 2)$ denote a rectangular Euclidean coordinate system in the physical plane, and let t denote the time variable. If v^λ $(\lambda = 1, 2)$ denotes the components of the velocity vector in the x^λ-coordinate system, and ρ and c denote the density and local speed of sound, respectively, then the equations of motion and the equation of continuity may be written as

$$\frac{\partial v_\lambda}{\partial t} + v^\mu \frac{\partial v_\lambda}{\partial x^\mu} + \frac{c^2}{\rho} \frac{\partial \rho}{\partial x^\lambda} = 0, \tag{2.1}$$

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial v^\lambda}{\partial x^\lambda} + v^\lambda \frac{\partial \rho}{\partial x^\lambda} = 0. \tag{2.2}$$

In the above equations, the contravariant and covariant components of a vector are equal since the coordinate system is Euclidean orthogonal. However, we have introduced the notation of tensor analysis in order to use the summation convention.

*Received Jan. 31, 1949. This work was done under a University of Michigan contract with the U. S. Army Air Forces.