STUDIES ON TWO-DIMENSIONAL TRANSONIC FLOWS OF A COMPRESSIBLE FLUID.—PART II*

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7. An alternative method of treatment. In the foregoing sections, it has been shown that the fundamental non-linear equation in the ϕ, ψ plane governing the flow of our hypothetical gas has exact solutions of practical interest in several cases, and for two of these cases, the corresponding flow patterns have been discussed in detail. Unfortunately, however, it seems very difficult to proceed further along similar lines of attack, since it is unlikely that any more exact solutions of practical interest can be found.

In Part II, therefore, an alternative method of treatment is developed on the basis of the fundamental equation in the hodograph plane. Just as in the case of the isentropic flow of the real gas, the fundamental differential equation governing the flow of our hypothetical gas becomes linear in the hodograph plane, and by suitable linear combinations of appropriate exact solutions of this linear differential equation, various flow patterns of practical interest have been obtained.

If the independent and dependent variables be interchanged in Eqs. (3.5), we obtain the fundamental equations of motion for our hypothetical gas in the w, θ plane in the forms:

$$\phi_w = 2kw\psi_\theta , \qquad \phi_\theta = \psi_w , \tag{7.1}$$

where, as before,

$$w = \int_1^q \frac{\rho}{q}\, dq, \qquad k = \frac{\gamma + 1}{2} . \tag{7.2}$$

Eliminating ψ from these equations, we get a partial differential equation for determining the velocity potential ϕ of the form:

$$\phi_{ww} - \frac{1}{w}\phi_w - 2kw\phi_{\theta\theta} = 0, \tag{7.3}$$

while elimination of ϕ yields an equation for determining the stream function ψ of the form:

$$\psi_{ww} - 2kw\psi_{\theta\theta} = 0. \tag{7.4}$$

The flow of our hypothetical gas can be found by solving either of the above two equations. Since, however, the latter equation is more simple in form than the former, discussions on the flow of our hypothetical gas will be made on the basis of the latter.

We shall use in place of w a new variable z, defined as

$$z = \sqrt[3]{2k}\, w. \tag{7.5}$$

Equation (7.4) assumes the form

$$\psi_{zz} - z\psi_{\theta\theta} = 0. \tag{7.6}$$

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This is a simple typical partial differential equation of the "mixed type", changing from the elliptic to the hyperbolic type according as $z < 0$ (i.e. $w < 0$) or $z > 0$ (i.e. $w > 0$); in other words, according as the flow is subsonic or supersonic.

The equations of the characteristic curves for equation (7.6) are easily found to be

$$\theta - \theta_0 = \pm\tfrac{2}{3}z^{3/2}, \tag{7.7}$$

where θ_0 is an arbitrary parameter.

8. Two different families of particular solutions of Eq. (7.6). To solve our fundamental equation (7.6), we first assume ψ in the form:

$$\psi = Z(z)e^{\pm i\nu\theta}, \tag{8.1}$$

where ν is a constant. Then, putting this in (7.6), we get an ordinary differential equation for determining the function $Z(z)$ of the form:

$$\frac{d^2Z}{dz^2} + \nu^2 zZ = 0. \tag{8.2}$$

The solutions of this equation can be expressed in terms of Bessel functions of orders $\pm 1/3$ and thus we obtain the following family of particular solutions of equation (7.6), namely:

$$\psi = \begin{cases} z^{1/2}J_{1/3}(\tfrac{2}{3}\nu z^{3/2}) \\ z^{1/2}J_{-1/3}(\tfrac{2}{3}\nu z^{3/2}) \end{cases} \times \begin{cases} e^{i\nu\theta}, \\ e^{-i\nu\theta}. \end{cases} \tag{8.3}$$

On the other hand, another family of particular solutions of equation (7.6) can be obtained in the following manner.

We now put[1]

$$z = \xi, \qquad \theta = \eta z^{3/2}, \tag{8.4}$$

and transform the independent variables in (7.6) from z, θ to ξ, η. Then, equation (7.6) becomes

$$\xi^2\psi_{\xi\xi} - 3\xi\eta\psi_{\xi\eta} + \left(\frac{9}{4}\eta^2 - 1\right)\psi_{\eta\eta} + \frac{15}{4}\eta\psi_\eta = 0. \tag{8.5}$$

To solve this equation, if we assume ψ in the form:

$$\psi = \xi^r Y(\eta), \tag{8.6}$$

with an arbitrary constant r, we get the following ordinary differential equation for determining the function $Y(\eta)$, namely:

$$\left(\frac{9}{4}\eta^2 - 1\right)\frac{d^2Y}{d\eta^2} + \left(\frac{15}{4} - 3r\right)\eta\frac{dY}{d\eta} + r(r-1)Y = 0, \tag{8.7}$$

[1]This transformation is suggested by Eqs. (7.7) for the characteristic curves. It may be remarked that the curves defined by $\eta = \pm 2/3$ coincide with the two characteristics passing through the point ($z = 0$, $\theta = 0$).

and this equation is further reduced to the form:

$$\zeta(1-\zeta)\frac{d^2Y}{d\zeta^2}+\left\{\frac{1}{2}-\left(\frac{4}{3}-\frac{2}{3}r\right)\zeta\right\}\frac{dY}{d\zeta}-\frac{1}{9}r(r-1)Y=0, \tag{8.8}$$

when use is made of a substitution of the form:

$$\frac{9}{4}\eta^2=\zeta. \tag{8.9}$$

Equation (8.8) is a special case of the hypergeometric equation and therefore its solutions can be expressed in terms of hypergeometric functions. Thus, we find that in case when $|\zeta|>1$,

$$Y=\begin{cases}\zeta^{-(1/3)(1-r)}F\left(\frac{1}{3}(1-r),\frac{5}{6}-\frac{1}{3}r;\frac{4}{3};\frac{1}{\zeta}\right),\\[2ex] \zeta^{(1/3)r}F\left(-\frac{1}{3}r,\frac{1}{2}-\frac{1}{3}r;\frac{2}{3};\frac{1}{\zeta}\right),\end{cases} \tag{8.10}$$

while, in case when $|1-\zeta|<1$,

$$Y=\begin{cases}F\left(\frac{1}{3}(1-r),-\frac{1}{3}r;\frac{5}{6}-\frac{2}{3}r;1-\zeta\right),\\[2ex] (1-\zeta)^{(1/6)+(2/3)r}F\left(\frac{1}{2}+\frac{1}{3}r,\frac{1}{6}+\frac{1}{3}r;\frac{7}{6}+\frac{2}{3}r;1-\zeta\right).\end{cases} \tag{8.11}$$

Hence, we ultimately obtain another family of particular solutions of our fundamental equation (7.6) in the following forms: for $|\theta|>(2/3)z^{3/2}$,

$$\psi=\begin{cases}z\theta^{-(2/3)(1-r)}F\left(\frac{1}{3}(1-r),\frac{5}{6}-\frac{1}{3}r;\frac{4}{3};\frac{4z^3}{9\theta^2}\right),\\[2ex] \theta^{(2/3)r}F\left(-\frac{1}{3}r,\frac{1}{2}-\frac{1}{3}r;\frac{2}{3};\frac{4z^3}{9\theta^2}\right),\end{cases} \tag{8.12}$$

and for $|\theta|<[2(2)^{1/2}/3]z^{3/2}$,

$$\psi=\begin{cases}z^rF\left(\frac{1}{3}(1-r),-\frac{1}{3}r;\frac{5}{6}-\frac{2}{3}r;1-\frac{9\theta^2}{4z^3}\right),\\[2ex] z^r\left(1-\frac{9\theta^2}{4z^3}\right)^{(1/6)+(2/3)r}F\left(\frac{1}{2}+\frac{1}{3}r,\frac{1}{6}+\frac{1}{3}r;\frac{7}{6}+\frac{2}{3}r;1-\frac{9\theta^2}{4z^3}\right).\end{cases} \tag{8.13}$$[2]

Furthermore, these solutions can be extended beyond the respective limits by the use of the well-known formula for analytic continuation of the hypergeometric function.

In this place, the relation between the two families of particular solutions (8.3) and

[2] In the range $(2/3)z^{3/2}<|\theta|<[2(2)^{1/2}/3]z^{3/2}$, the second expression becomes complex. But, it is readily found that both its real and imaginary parts are still particular solutions of equation (7.6).

(8.12) will be considered. For this purpose, we remember a formula[3] for a definite integral containing Bessel function of the form:

$$\int_0^\infty e^{\pm ia\nu} J_\lambda(b\nu)\nu^{\mu-1}\, d\nu \propto \left(\frac{b}{2a}\right)^\lambda a^{-\mu} F\left(\frac{\lambda+\mu}{2}, \frac{\lambda+\mu+1}{2}; \lambda+1; \frac{b^2}{a^2}\right).$$

If, in this formula, we take

$$\left.\begin{aligned} \lambda &= \pm\frac{1}{3}, & \mu &= \frac{1}{3} - \frac{2}{3}r, \\ a &= \theta, & b &= \frac{2}{3}\,{}^{3/2}, \end{aligned}\right\}$$

and multiply both sides by $z^{1/2}$, we readily find that the right-hand side of the formula becomes, except for a certain constant, just equal to the expression on the right-hand side of (8.12).

Therefore, the expressions for ψ as given by (8.12) can also be put in the following forms:

$$\psi \propto z^{1/2} \int_0^\infty e^{\pm i\nu\theta} J_{\pm 1/3}(\tfrac{2}{3}\nu z^{3/2})\nu^{-(2/3)(1+r)}\, d\nu, \tag{8.14}$$

so far as the integrals on the right-hand side are convergent.

Comparing this expression with (8.3), we can easily understand in what manner the expressions for ψ as given by (8.12) are constructed by superposition of the former family of particular solutions (8.3).

9. Properties of singular points of the solutions. The solutions defined in the foregoing sections have been found independently by other investigators [1], [2]. However, we may now generalize these functions in such a way as to render them more useful. We note that the fundamental equation (7.6) retains its original form when the variable θ is replaced by $\pm(\theta - i\lambda)$, where λ is a real constant, and therefore, the expressions as obtained from (8.12) and (8.13) by replacing θ by $\pm(\theta - i\lambda)$ are also solutions of equation (7.6). As will be seen presently, this transformation of variables, though of a very simple nature, yields very important results.

If this transformation of variables is applied to the second expression in (8.13), we obtain the expression for ψ in the form:

$$\psi = z^r\left\{1 - \frac{9(\theta - i\lambda)^2}{4z^3}\right\}^{(1/6)+(2/3)r} F\left(\frac{1}{2} + \frac{1}{3}r, \frac{1}{6} + \frac{1}{3}r; \frac{7}{6} + \frac{2}{3}r; 1 - \frac{9(\theta - i\lambda)^2}{4z^3}\right). \tag{9.1}$$

Since this expression contains the factor:

$$\left\{1 - \frac{9(\theta - i\lambda)^2}{4z^3}\right\}^{(1/6)+(2/3)r}, \tag{9.2}$$

the function ψ will have a point of singularity at the place where this factor vanishes, the order of singularity being determined by the constant r. Since, however, the value

[3] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge Univ. Press, 1922, p. 385.

of r can be chosen arbitrarily, we can make the above expression (9.1) for ψ to have a point of singularity of any order at the place where the relation:

$$4z^3 - 9(\theta - i\lambda)^2 = 0 \tag{9.3}$$

holds.

Remembering that z and θ are both real, the above relation (9.3) can be split up into the following two relations:

$$z^3 + \frac{9}{4}(\lambda^2 - \theta^2) = 0, \qquad i\frac{9}{2}\lambda\theta = 0. \tag{9.4}$$

When $\lambda \neq 0$, these relations give

$$\theta = 0, \qquad z = -\left(\frac{9}{4}\lambda^2\right)^{1/3}. \tag{9.5}$$

Thus, in this case there is an isolated point of singularity on the axis $\theta = 0$ in the range $z < 0$, i.e. in the subsonic region.[4]

On the other hand, when $\lambda = 0$ the points of singularity of the function ψ as given by (9.1) are distributed continuously along the curves:

$$\theta = \pm\frac{2}{3}z^{3/2},$$

which are nothing else than the two characteristic curves passing through the point ($\theta = 0$, $z = 0$).

Thus, summarizing the above results we may infer that the isolated point of singularity of any order in the subsonic region in the hodograph plane undergoes remarkable changes as it approaches the supersonic region, and as we pass from the subsonic to the supersonic region, this concentrated singularity is prolonged to the continuous distribution of singularities along the above two particular characteristic curves in the supersonic region.

It will naturally be expected that such an interesting behaviour of singularities would still be found in the case of solutions of an equation of the mixed type of the following form:

$$\psi_{zz} + (c_1 z + c_2 z^2 + \cdots)\psi_{\theta\theta} = 0,$$

where c_1, c_2, $\cdots$ are all constants, which is nothing but an extension of equation (7.6).

10. Special cases in which the solutions are expressible in terms of elementary functions. In this section, we shall consider a few special cases in which the hypergeometric functions involved in (8.12) degenerate into elementary functions.

First, we take, in (8.12),

$$r = \begin{cases} -\dfrac{3}{2}, \\ -\dfrac{1}{2}; \end{cases} \qquad \text{or} = \begin{cases} -3, \\ -2. \end{cases}$$

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[4]The position of the point of singularity is definitely determined by the value of λ. Physically speaking, the constant λ corresponds to a parameter for determining the Mach number of the uniform flow in the problem we shall consider.

Then, replacing θ by $\pm(\theta - i\lambda)$ as before and remembering the well-known relation:

$$F(n, \beta; \beta; x) = (1 - x)^{-n},$$

we obtain the results that

$$\psi = \begin{cases} z(\theta - i\lambda)^{-5/3} F\left(\frac{5}{6}, \frac{4}{3}; \frac{4}{3}; \frac{4z^3}{9(\theta - i\lambda)^2}\right) = \dfrac{z}{\left\{(\theta - i\lambda)^2 - \frac{4}{9} z^3\right\}^{5/6}}, \\ (\theta - i\lambda)^{-1/3} F\left(\frac{1}{6}, \frac{2}{3}; \frac{2}{3}; \frac{4z^3}{9(\theta - i\lambda)^2}\right) = \dfrac{1}{\left\{(\theta - i\lambda)^2 - \frac{4}{9} z^3\right\}^{1/6}} \end{cases} \tag{10.1}$$

$$\psi = \begin{cases} z(\theta - i\lambda)^{-8/3} F\left(\frac{4}{3}, \frac{11}{6}; \frac{4}{3}; \frac{4z^3}{9(\theta - i\lambda)^2}\right) = \dfrac{z(\theta - i\lambda)}{\left\{(\theta - i\lambda)^2 - \frac{4}{9} z^3\right\}^{11/6}}, \\ (\theta - i\lambda)^{-4/3} F\left(\frac{2}{3}, \frac{7}{6}; \frac{2}{3}; \frac{4z^3}{9(\theta - i\lambda)^2}\right) = \dfrac{\theta - i\lambda}{\left\{(\theta - i\lambda)^2 - \frac{4}{9} z^3\right\}^{7/6}}. \end{cases} \tag{10.2}$$

It is worth noticing here that the two expressions in (10.2) can evidently be derived by differentiating respectively the two expressions in (10.1) with respect to θ. Similar results hold good in general; namely, as is suggested by the form of the fundamental equation (7.6) as well as by the form of its general solution as given by (9.1), we can obtain, by differentiating or integrating any solution with respect to θ, another solution whose order of singularity differs by unity from that of the original solution.

In the next place, we take

$$r = 1/2$$

in (8.12). Then, replacing θ by $\pm(\theta - i\lambda)$ and remembering the general formula:

$$F\left(n, n + \frac{1}{2}; 2n + 1; x\right) = \left(\frac{2}{x}\right)^{2n} \{\sqrt{1 - x} - 1\}^{2n},$$

we have

$$\psi = \begin{cases} z(\theta - i\lambda)^{-1/3} F\left(\frac{1}{6}, \frac{1}{6} + \frac{1}{2}; \frac{2}{6} + 1; \frac{4z^3}{9(\theta - i\lambda)^2}\right) \\ \qquad = \left(\frac{9}{2}\right)^{1/3} \left\{\sqrt{(\theta - i\lambda)^2 - \frac{4}{9} z^3} \pm (\theta - i\lambda)\right\}^{1/3}, \\ (\theta - i\lambda)^{1/3} F\left(-\frac{1}{6}, -\frac{1}{6} + \frac{1}{2}; -\frac{2}{6} + 1; \frac{4z^3}{9(\theta - i\lambda)^2}\right) \\ \qquad = \dfrac{z}{\left(\frac{9}{2}\right)^{1/3} \left\{\sqrt{(\theta - i\lambda)^2 - \frac{4}{9} z^3} \pm (\theta - i\lambda)\right\}^{1/3}}, \end{cases} \tag{10.3}$$

and by differentiation or integration of these solutions with respect to θ, we can also derive, as before, various solutions in closed forms having different orders of singularity. It will readily be seen that all the above solutions (10.1), (10.2) and (10.3) have, in the case when $\lambda \neq 0$, an isolated point of singularity at the point $\theta = 0$, $z = -(9/4\ \lambda^2)^{1/3}$ in the subsonic region, while in the case when $\lambda = 0$, their singularities are distributed continuously along the two characteristic curves $\theta = \pm 2/3\ z^{3/2}$.

11. Solutions having logarithmic singularity. When the parameter r takes the value of $-1/4$, the two solutions in (8.13) coincide with each other. In this case, there exists another independent solution which has logarithmic singularity and represents the flow due to a source or a vortex in the hodograph plane.

To obtain such a solution, we shall start again from the differential equation (8.8). When $|1 - \zeta| < 1$, the two independent solutions of this equation are given by (8.11), but in case when $r = -1/4$ these two solutions coincide with each other. Another independent solution can be obtained in the usual way. Thus, making use of the fundamental solutions of the form (8.11), namely:

$$Y_a = F\left(\frac{1}{3}(1 - r), -\frac{1}{3}r; \frac{5}{6} - \frac{2}{3}r; 1 - \zeta\right),$$

$$Y_b = (1 - \zeta)^{(1/6)+(2/3)r} F\left(\frac{1}{2} + \frac{1}{3}r, \frac{1}{6} + \frac{1}{3}r; \frac{7}{6} + \frac{2}{3}r; 1 - \zeta\right),$$

which are valid in case when $r \neq -1/4$, the required solution Y_2 can be obtained as:

$$Y_2 = \frac{3}{2} \lim_{r \to -1/4} \frac{Y_b - Y_a}{r + 1/4},$$

where the factor 3/2 has been multiplied in order to regulate the form of the solution.

Calculating the limiting value, we ultimately obtain, in case when $r = -1/4$, the following independent solutions valid when $|1 - \zeta| < 1$:

$$Y_1(\zeta) = F\left(\frac{5}{12}, \frac{1}{12}; 1; 1 - \zeta\right) = 1 + \sum_{n=1}^{\infty} a_n(1 - \zeta)^n,$$

$$Y_2(\zeta) = Y_1 \log(1 - \zeta) + \sum_{n=1}^{\infty} a_n \sum_{m=0}^{n-1} \left\{\frac{1}{m + \frac{5}{12}} + \frac{1}{m + \frac{1}{12}} - \frac{2}{m + 1}\right\}(1 - \zeta)^n, \quad (11.1)$$

$$a_n = \frac{1}{(n!)^2} \frac{5}{12}\left(\frac{5}{12} + 1\right) \cdots \left(\frac{5}{12} + n - 1\right) \frac{1}{12}\left(\frac{1}{12} + 1\right) \cdots \left(\frac{1}{12} + n - 1\right).$$

Hence, in case when $r = -1/4$, the stream function ψ for the flow due to a source in the hodograph plane can be obtained by the use of the second solution $Y_2(\zeta)$ in (11.1), taking (8.4), (8.6) and (8.9) into account. We thus have

$$\psi = z^{-1/4}\left\{F\left(\frac{5}{12}, \frac{1}{12}; 1; 1 - \frac{9(\theta - i\lambda)^2}{4z^3}\right) \log\left(1 - \frac{9(\theta - i\lambda)^2}{4z^3}\right)\right.$$

$$\left. + \sum_{n=1}^{\infty} a_n \sum_{m=0}^{n-1} \left\{\frac{1}{m + \frac{5}{12}} + \frac{1}{m + \frac{1}{12}} - \frac{2}{m + 1}\right\}\left(1 - \frac{9(\theta - i\lambda)^2}{4z^3}\right)^n\right\}, \quad (11.2)$$

where a_n assumes the same value as that given in (11.1).

Further, by differentiating this expression with respect to θ once or several times, we can obtain, as before, the stream function for the flows due to a doublet and a multiplet having logarithmic singularity of any order in the hodograph plane.

It is worth mentioning here that logarithmic singularities of these solutions experience also the characteristic change as mentioned in §9 when we pass from the subsonic to the supersonic region.

12. An example of flow. We shall now construct a field of flow by employing various solutions obtained in the foregoing sections.

In the first place, by superposing the first two solutions in (10.3), we have

$$\psi = \left\{\sqrt{(\theta - i\lambda)^2 - \frac{8k}{9} w^3} - (\theta - i\lambda)\right\}^{1/3} - \left\{\sqrt{(\theta - i\lambda)^2 - \frac{8k}{9} w^3} + (\theta - i\lambda)\right\}^{1/3}, \tag{12.1}$$

where the variable z has been replaced again by the variable w with the aid of (7.5) for the sake of comparison with the analysis in Part I.

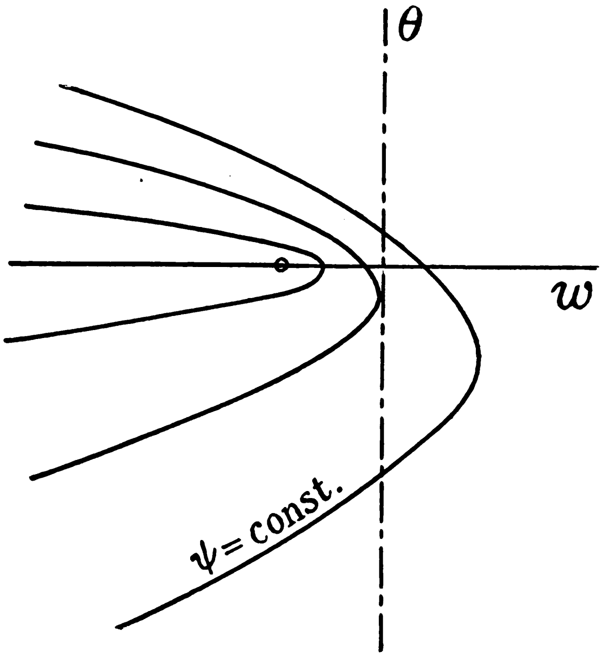


FIG. 17. The flow represented by the stream function ψ_1.

Taking the real part of the above expression (12.1) for ψ, we define a solution:

$$\psi_1 = R(\psi). \tag{12.2}$$

Then, this solution has evidently a branch-point of order 1/2 at the point $\theta = 0$, $w = -(9/8\ k)^{1/3}\lambda^{2/3}$ and becomes always zero along the axis $\theta = 0$ in the range $w < -(9/8\ k)^{1/3}\lambda^{2/3}$.

Next, differentiating ψ in (12.1) partially with respect to θ, we have

$$\psi_\theta = -\frac{1}{3}\frac{\left\{\sqrt{(\theta-i\lambda)^2-\frac{8k}{9}w^3}-(\theta-i\lambda)\right\}^{1/3}+\left\{\sqrt{(\theta-i\lambda)^2-\frac{8k}{9}w^3}+(\theta-i\lambda)\right\}^{1/3}}{\sqrt{(\theta-i\lambda)^2-\frac{8k}{9}w^3}} \tag{12.3}$$

and, as mentioned already, this becomes also a particular solution of equation (7.6). Thus, by taking the imaginary part of this expression, we define a solution:

$$\psi_2 = I(\psi_\theta), \tag{12.4}$$

which has a branch-point of order $-1/2$ at the point $\theta = 0$, $w = -(9/8\,k)^{1/3}\lambda^{2/3}$, just as in the case of the previous solution ψ_1, and becomes always zero along the axis $\theta = 0$ in the range $w < -(9/8\,k)^{1/3}\lambda^{2/3}$.

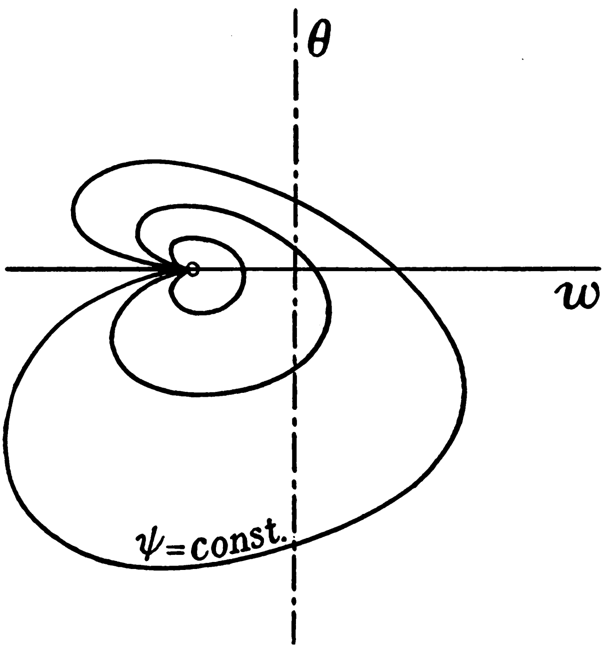


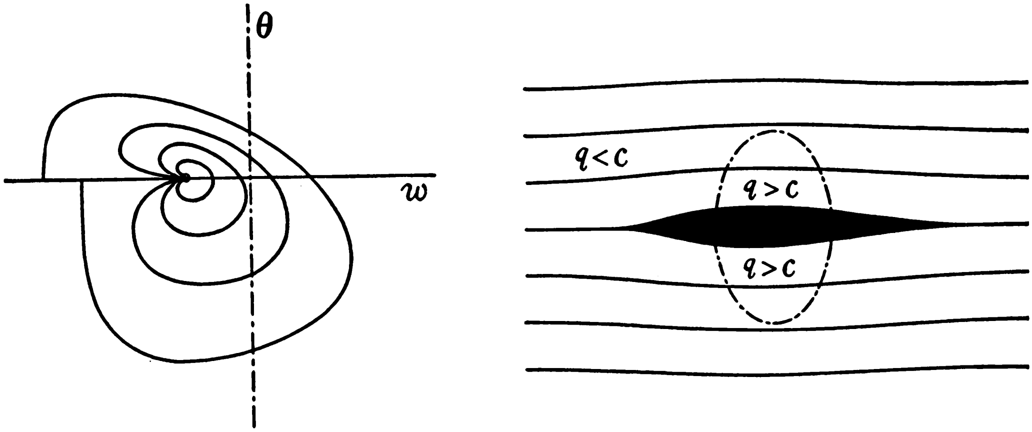
FIG. 18. The flow represented by the stream function ψ_2.

Figs. 17 and 18 show schematically the flow patterns in the w, θ plane as represented by the above two stream functions ψ_1 and ψ_2 respectively.

These figures suggest that if we superpose these two solutions in the form $\psi_1 + K\psi_2$ where K is a constant, we can obtain the flow pattern in the w, θ plane as shown in Fig. 19 (a), which corresponds, in the physical plane, to the field of flow past a symmetrical body having sharp leading and trailing edges as shown in Fig. 19 (b).

In such a flow, the condition of nearly uniform velocity is considered to be fulfilled, and therefore, the flow of our hypothetical gas is expected to approximate satisfactorily the corresponding flow of the real gas subject to the adiabatic law.

Detailed investigations will make it clear that the flow can still be continuous and irrotational in spite of the existence of limited supersonic regions in the vicinity of the surface of the body (Fig. 19 (b)), but at a certain critical Mach number there appear two points of singularity $J = 0$, one each on the upper and lower surfaces, which grow up into the curves of singularity $J = 0$ for still higher Mach number, thus violating the one-valuedness of the physical field of flow. However, we shall not enter into the



(a) Hodograph plane. (b) Physical plane.

Fig. 19.

numerical computations for the present example of flow, since it has been found that more full discussions concerning the similar field of flow can be made by introducing a second new hypothetical gas which represents the real gas to a higher degree of approximation than our first hypothetical gas used in Parts I and II. The flow of our second hypothetical gas will be discussed in detail in Part III.

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